CERTAIN EXPANSIONS INVOLVING E-FUNCTIONS

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1. Introduction. The E-functions were defined by MacRobert [3] in 1937; they are denoted by $E(p\,;\ \alpha_r : q\,;\ \rho_s : z)$.

In § 3 of this paper, I prove a new expansion for $E(p\,;\ \alpha_r : q\,;\ \rho_s : z)$ which is similar to an expansion due to MacRobert [2], viz.,

$$(\alpha_2\,;\ n)\,E\,(p\,;\ \alpha_r : q\,;\ \rho_s : z)$$
$$= \sum_{r=0}^{n} \binom{n}{r} (\alpha_1 - n + r\,;\ n - r)(\alpha_1 - \alpha_2 - n\,;\ r)\, z^{-r} E\begin{pmatrix} \alpha_1 - n + r,\, \alpha_2 + n,\, \alpha_3 + r,\, \ldots,\, \alpha_p + r : z \\ q\,;\ \rho_s + r \end{pmatrix}, \quad \ldots(1)$$

where $\qquad (a\,;\ r) = \Gamma(a + r)/\Gamma(a), \quad (a\,;\ 0) = 1.$

In § 4 I give a simple direct proof of the formula

$$F\,[\alpha + m\,;\ 2m + 1\,;\ a]\, E\,(\alpha + m,\, \alpha - m : : b)$$
$$= \left(\frac{b}{a + b}\right)^{\alpha + m} \sum_{r=0}^{\infty} \frac{(1 - \alpha + m\,;\ r)}{(2m + 1\,;\ r)\, r!} \left(\frac{-ab}{a + b}\right)^{r} E\,(\alpha + m + r,\, \alpha - m - r : : a + b), \quad \ldots\ldots\ldots(2)$$

which was deduced by MacRobert [2] from a certain integral. Later in the same section a very general expansion involving a product of a generalized confluent hypergeometric series and a general E-function in terms of a multiple series of E-functions is given.

2. The following known formulae ([1], pp. 203–206) are used.

$$\alpha_1 E\,(p\,;\ \alpha_r : q\,;\ \rho_s : z)$$
$$= E\,(\alpha_1 + 1,\, \alpha_2,\, \alpha_3,\, \ldots,\, \alpha_p : q\,;\ \rho_s : z) + z^{-1} E\,(p\,;\ \alpha_r + 1 : q\,;\ \rho_s + 1 : z), \quad \ldots\ldots\ldots(3)$$

$$(\rho_1 - 1)\,E\,(p\,;\ \alpha_r : q\,;\ \rho_s : z)$$
$$= E\,(p\,;\ a_r : \rho_1 - 1,\, \rho_2,\, \rho_3,\, \ldots,\, \rho_q : z) + z^{-1} E\,(p\,;\ \alpha_r + 1 : q\,;\ \rho_s + 1 : z), \quad \ldots\ldots\ldots\ldots(4)$$

$$\frac{d^m}{dz^m}\{z^{-\alpha_1} E\,(p\,;\ \alpha_r : q\,;\ \rho_s : z)\} = (-1)^m z^{-\alpha_1 - m}\, E\,(\alpha_1 + m,\, \alpha_2,\, \alpha_3,\, \ldots,\, \alpha_p : q\,;\ \rho_s : z). \quad \ldots.(5)$$

3. We now prove that

$$(\alpha_1\,;\ n)\,E\,(p\,;\ \alpha_r : q\,;\ \rho_s : z)$$
$$= \sum_{r=0}^{n} \binom{n}{r} (\rho_1 + r\,;\ n - r)(\rho_1 - \alpha_1\,;\ r)\, z^{-r} E\begin{pmatrix} \alpha_1 + n,\, \alpha_2 + r,\, \alpha_3 + r,\, \ldots,\, \alpha_p + r : z \\ \rho_1 + n + r,\, \rho_2 + r,\, \rho_3 + r,\, \ldots,\, \rho_q + r \end{pmatrix}. \quad \ldots\ldots.(6)$$

Eliminating the E-function on the left of (3) and (4) and finally replacing ρ_1 by $\rho_1 + 1$, we have

$$\alpha_1 E\,(p\,;\ \alpha_r : q\,;\ \rho_s : z) = \rho_1 E\,(\alpha_1 + 1,\, \alpha_2,\, \alpha_3,\, \ldots,\, \alpha_p : \rho_1 + 1,\, \rho_2,\, \rho_3,\, \ldots,\, \rho_q : z)$$
$$+ (\rho_1 - \alpha_1)\, z^{-1}\, E\,(p\,;\ \alpha_r + 1 : \rho_1 + 2,\, \rho_2 + 1,\, \rho_3 + 1,\, \ldots,\, \rho_q + 1 : z). \quad \ldots\ldots\ldots\ldots(7)$$

Now multiply (6) by $(\alpha_1 + n)$ and apply (7) to every term on its right-hand side. Then, since

$$\binom{n}{r}(\rho_1+r\,;\ n-r)(\rho_1+n+r)+\binom{n}{r-1}(\rho_1+r-1\,;\ n-r+1)=\binom{n+1}{r}(\rho_1+r\,;\ n+1-r),$$

we get

$$(\alpha_1\,;\ n+1)\,E(p\,;\ \alpha_r:q\,;\ \rho_s:z)$$
$$=\sum_{r=0}^{n+1}\binom{n+1}{r}(\rho_1+r\,;\ n+1-r)(\rho_1-\alpha_1\,;\ r)z^{-r}E\left(\begin{matrix}\alpha_1+n+1,\ \alpha_2+r,\ \alpha_3+r,\ \dots,\ \alpha_p+r:z\\ \rho_1+n+1+r,\ \rho_2+r,\ \rho_3+r,\ \dots\ \rho_q+r\end{matrix}\right), \quad \text{.........(8)}$$

which is (6) with n replaced by $n+1$. Putting $n=1$ in (6) we get (7), the truth of which has been established directly. For $n=2$ one can easily verify (6) by multiplying (7) by (α_1+1) and applying (7) itself to both the terms on the right-hand side of the equation thus obtained. Hence, by virtue of (8), (6) is true by induction.

4. In this section I give a generalization of (2). We first note a Taylor's expansion for the E-function, as it will be needed later. Taylor's multiplication theorem

$$f(\lambda x)=\sum_{n=0}^{\infty}\frac{(\lambda-1)^n}{n!}\,x^n\frac{d^n}{dx^n}f(x)$$

gives, with the help of (5), that

$$E(p\,;\ \alpha_r:q\,;\ \rho_s:z+h)=\left(\frac{z+h}{z}\right)^{\alpha_1}\sum_{n=0}^{\infty}\frac{(-h)^n}{n!}\,z^{-n}\,E\left(\begin{matrix}\alpha_1+n,\ \alpha_2,\ \alpha_3,\ \dots,\ \alpha_p:z\\ q\,;\ \rho_s\end{matrix}\right). \quad \text{......(9)}$$

Now we prove that

$$F\,[\alpha\,;\ 1+\alpha-\beta\,;\ x]\,E(\alpha,\beta::y)$$
$$=\left(\frac{y}{x+y}\right)^{\alpha}\sum_{r=0}^{\infty}\frac{(1-\beta\,;\ r)}{(1+\alpha-\beta\,;\ r)\,r!}\left(\frac{-xy}{x+y}\right)^{r}E(\alpha+r,\ \beta-r::x+y). \quad \text{...............(10)}$$

We deduce, from (1), that

$$(\alpha\,;\ n)\,E(\alpha,\beta::y)$$
$$=(-1)^n\sum_{r=0}^{n}\binom{n}{r}(1-\beta\,;\ r)(1+\alpha-\beta+r\,;\ n-r)\,y^{-n+r}E(\alpha+n,\ \beta-r::y). \quad \text{.........(11)}$$

Multiply both sides of (11) by $\dfrac{x^n}{(1+\alpha-\beta\,;\ n)\,n!}$ and sum from $n=0$ to ∞. Then we get

$$F\,[\alpha\,;\ 1+\alpha-\beta\,;\ x]\,E(\alpha,\beta::y)$$
$$=\sum_{n=0}^{\infty}\frac{(-x)^n}{(1+\alpha-\beta\,;\ n)\,n!}\sum_{r=0}^{n}\binom{n}{r}(1-\beta\,;\ r)(1+\alpha-\beta+r\,;\ n-r)\,y^{-n+r}E(\beta-r,\ \alpha+n::y).$$

Put $n=r+t$ on the right and re-arrange. The repeated series then becomes

$$\sum_{r=0}^{\infty}\frac{(1-\beta\,;\ r)(-x)^r}{(1+\alpha-\beta\,;\ r)\,r!}\sum_{t=0}^{\infty}\frac{(-x)^t}{t!}\,y^{-t}E(\alpha+r+t,\ \beta-r::y)$$
$$=\sum_{r=0}^{\infty}\frac{(1-\beta\,;\ r)(-x)^r}{(1+\alpha-\beta\,;\ r)\,r!}\left(\frac{y}{x+y}\right)^{\alpha+r}E(\beta-r,\ \alpha+r::x+y),$$

by (9); whence (10) follows. (2) can be seen to be equivalent to (10).

Next we prove

$$F\begin{bmatrix}\alpha, 1+\alpha-\rho\,;\, x \\ 1+\alpha-\beta, 1+\alpha-\gamma\end{bmatrix} E\begin{pmatrix}\alpha, \beta, \gamma : y \\ \rho\end{pmatrix} = \left(\frac{y}{x+y}\right)^{\alpha} \sum_{r=0}^{\infty} \frac{(1+\alpha-\rho\,;\, r)(1-\beta\,;\, r)}{(1+\alpha-\beta\,;\, r)(1+\alpha-\gamma\,;\, r)\, r!}$$
$$\times\left(\frac{-xy}{x+y}\right)^{r} \sum_{s=0}^{\infty} \frac{(\rho-\gamma\,;\, s)}{(1+\alpha-\gamma+r\,;\, s)\, s!} \frac{(xy)^{s}}{(x+y)^{2s}} E\begin{pmatrix}\alpha+r+2s, \beta-r, \gamma : x+y \\ \rho+s\end{pmatrix}, \quad \ldots\ldots(12)$$

which reduces to (10) when $\gamma=\rho$. For this we need the following expansion :

$$(1+\alpha-\rho\,;\, t)\, E(\alpha+t, \beta, \gamma+t : \rho+t : z)$$
$$= \sum_{s=0}^{t} \binom{t}{s} (-1)^{s} (1+\alpha-\gamma+s\,;\, t-s)(\rho-\gamma\,;\, s)\, E(\alpha+t+s, \beta, \gamma : \rho+s : z). \quad \ldots\ldots.(13)$$

With $p=3$, $q=1$, we can easily derive from (3) and (4) that

$$(1+\gamma-\rho)\, E(\alpha, \beta, \gamma : \rho : z) = E(\alpha, \beta, \gamma+1 : \rho : z) - E(\alpha, \beta, \gamma : \rho-1 : z).$$

And, from (3), we have

$$(\gamma-\alpha)\, E(\alpha, \beta, \gamma : \rho : z) = E(\alpha, \beta, \gamma+1 : \rho : z) - E(\alpha+1, \beta, \gamma : \rho : z).$$

Eliminating the E-function on the left from the above two relations and finally replacing α, ρ by $\alpha+1$, $\rho+1$ respectively, we get

$$(1+\alpha-\rho)\, E(\alpha+1, \beta, \gamma+1 : \rho+1 : z)$$
$$= (\gamma-\rho)\, E(\alpha+2, \beta, \gamma : \rho+1 : z) + (1+\alpha-\gamma)\, E(\alpha+1, \beta, \gamma : \rho : z)\,;$$

(13) then follows by induction by an argument similar to that used in (6).

Now, from (1),

$$F\begin{bmatrix}\alpha, 1+\alpha-\rho\,;\, x \\ 1+\alpha-\beta, 1+\alpha-\gamma\end{bmatrix} E\begin{pmatrix}\alpha, \beta, \gamma : y \\ \rho\end{pmatrix}$$
$$= \sum_{n=0}^{\infty} \frac{(1+\alpha-\rho\,;\, n)(-x)^{n}}{(1+\alpha-\beta\,;\, n)(1+\alpha-\gamma\,;\, n)\, n!} \sum_{r=0}^{n} \binom{n}{r} (1-\beta\,;\, r)(1+\alpha-\beta+r\,;\, n-r)\, y^{-n+r}$$
$$\times E(\alpha+n, \beta-r, \gamma+n-r\,;\, \rho+n-r : y)$$
$$= \sum_{r=0}^{\infty} \frac{(1+\alpha-\rho\,;\, r)(1-\beta\,;\, r)(-x)^{r}}{(1+\alpha-\beta\,;\, r)(1+\alpha-\gamma\,;\, r)\, r!}$$
$$\times \sum_{u=0}^{\infty} \left(\frac{-x}{y}\right)^{u} \frac{(1+\alpha-\rho+r\,;\, u)}{(1+\alpha-\gamma+r\,;\, u)\, u!} E\begin{pmatrix}\alpha+r+u, \beta-r, \gamma+u : y \\ \rho+u\end{pmatrix}, \quad \ldots\ldots\ldots(14)$$

where $n=u+r$. And, from (13),

$$(1+\alpha-\rho+r\,;\, u)\, E(\alpha+r+u, \beta-r, \gamma+u : \rho+u : y)$$
$$= \sum_{s=0}^{u} \binom{u}{s} (-1)^{s} (1+\alpha+r-\gamma+s\,;\, u-s)(\rho-\gamma\,;\, s)\, E(\alpha+r+u+s, \beta-r, \gamma : \rho+s : y).$$

Therefore, on writing $u=s+t$, the u-series in (14) is equal to

$$\sum_{s=0}^{\infty} \frac{(\rho-\gamma\,;\, s)}{(1+\alpha-\gamma+r\,;\, s)\, s!} \left(\frac{x}{y}\right)^{s} \sum_{t=0}^{\infty} \left(\frac{-x}{y}\right)^{t} \frac{1}{t!}\; E\begin{pmatrix}\alpha+r+2s+t, \beta-r, \gamma : y \\ \rho+s\end{pmatrix}$$
$$= \sum_{s=0}^{\infty} \frac{(\rho-\gamma\,;\, s)}{(1+\alpha-\gamma+r\,;\, s)\, s!} \left(\frac{x}{y}\right)^{s} \left(\frac{y}{x+y}\right)^{\alpha+2s+r} E\begin{pmatrix}\alpha+r+2s, \beta-r, \gamma : x+y \\ \rho+s\end{pmatrix},$$

by (9) ; whence, on simplification, we get (12).

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Now, generalizing (13), we have, by an exactly similar argument,

$$(1+\alpha_1-\rho_1\,;\ n)\,E(\alpha_1+n,\alpha_2+n,\alpha_3,\alpha_4+n,\alpha_5+n,\ldots,\alpha_p+n:q\,;\ \rho_s+n:z)$$
$$=\sum_{r=0}^{n}(-1)^r\binom{n}{r}(1+\alpha_1-\alpha_2+r\,;\ n-r)(\rho_1-\alpha_2\,;\ r)$$
$$\times E\begin{pmatrix}\alpha_1+n+r,\alpha_2,\alpha_3,\alpha_4+n,\alpha_5+n,\ldots,\alpha_p+n:z\\ \rho_1+r,\rho_2+n,\rho_3+n,\ldots.\rho_q+n\end{pmatrix}.$$

Using this and proceeding just as for (12), we get the general expansion

$${}_qF_q\begin{bmatrix}\alpha_1,1+\alpha_1-\rho_1,1+\alpha_1-\rho_2,\ldots,1+\alpha_1-\rho_{q-1}\,;\ x\\ 1+\alpha_1-\alpha_2,1+\alpha_1-\alpha_3,\ldots,1+\alpha_1-\alpha_{q+1}\end{bmatrix}E\,(q+1\,;\ \alpha_r:q-1\,;\ \rho_s:y)$$
$$=\left(\frac{y}{x+y}\right)^{\alpha_1}\sum_{r_1,r_2,\ldots,r_q=0}^{\infty}\frac{(-1)^{r_1}(1-\alpha_2\,;\ r_1)(x+y)^{r_1}}{(1+\alpha_1-\alpha_{q+1}\,;\ s_q)\,r_q!}\left\{\frac{xy}{(x+y)^2}\right\}^{s_q}$$
$$\times\prod_{n=1}^{q-1}\frac{(1+\alpha_1-\rho_n\,;\ s_n)(\rho_n-\alpha_{n+2}\,;\ r_{n+1})}{(1+\alpha_1-\alpha_{n+1}\,;\ s_n)\,r_n!}$$
$$\times E\begin{pmatrix}\alpha_1+r_1+2\sigma_q,\alpha_2-r_1,\alpha_3,\alpha_4+\sigma_2,\ldots,\alpha_{q+1}+\sigma_{q-1}:x+y\\ \rho_1+\sigma_2,\rho_2+\sigma_3,\ldots,\rho_{q-1}+\sigma_q\end{pmatrix},$$

where

$$s_n=r_1+r_2+\ldots+r_n,\quad \sigma_n=s_n-r_1,\quad (n=1,2,\ldots q)$$

and the empty product (for $q=1$) is unity.

It may be noted that the rearrangement of the foregoing double series are valid since the E-functions involved are integral functions and, consequently, the series are absolutely convergent.

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REFERENCES

1. Erdélyi, A., *Higher transcendental functions* (Bateman Manuscript Project), Vol. I (New York, 1953).

2. MacRobert, T. M., On an identity involving E-functions, *Phil. Mag.* (7), **39** (1948), 466–471.

3. MacRobert, T. M., Induction proofs of the relations between certain asymptotic expansions and the corresponding generalized hypergeometric functions, *Proc. Roy. Soc. Edinburgh*, **58** (1937), 1–13.

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