Canad. Math. Bull. Vol. **55** (4), 2012 pp. 842–849
http://dx.doi.org/10.4153/CMB-2011-140-5

The Rank of Jacobian Varieties over the Maximal Abelian Extensions of Number Fields: Towards the Frey–Jarden Conjecture

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Abstract. Frey and Jarden asked if any abelian variety over a number field K has the infinite Mordell–Weil rank over the maximal abelian extension K^{ab}. In this paper, we give an affirmative answer to their conjecture for the Jacobian variety of any smooth projective curve C over K such that $\sharp C(K^{\mathrm{ab}}) = \infty$ and for any abelian variety of GL_2-type with trivial character.

1 Introduction

Let A be an abelian variety over a number field K and let L be a finite extension of K. Then it is interesting to study the difference between the Mordell–Weil rank of $A(K)$ and that of $A(L)$. This prompts the following conjecture which was proposed by Frey and Jarden.

Conjecture 1.1 (Frey–Jarden Conjecture [4]) *For any abelian variety A over a number field K, the module $\mathbb{Q} \otimes_{\mathbb{Z}} A(K^{\mathrm{ab}})$ has infinite rank.*

We briefly introduce a history of Conjecture 1.1. The first attempt was made by Billing in 1938 [2]. He proved that for the elliptic curve E defined by $y^2 = x^3 - x$ and for any m in $\mathbb{N}$, there exist squarefree integers $d_1, \dots, d_m$ such that

$$\mathrm{rank}_{\mathbb{Q}}\, \mathbb{Q} \otimes_{\mathbb{Z}} E(\mathbb{Q}(\sqrt{d_1}, \dots, \sqrt{d_m})) \geq m.$$

Fourteen years later, Néron proved Conjecture 1.1 for the Jacobian variety of any hyperelliptic curve over a number field K with a K-rational point P by using the specialization argument [13, Corollaire, p. 157].

On the other hand, Imai [5], Top [23], and Murabayashi [11] gave a simple proof for Néron's theorem. Murabayashi's result included the case of superelliptic curves defined by $y^p = f(x)$ with $(p, \deg f(x)) = 1$ for any prime p, but where the sequence of extension fields over which the Mordell–Weil rank increase is no longer abelian if $p \geq 3$. Rosen and Wong [20] proved Conjecture 1.1 for the Jacobian variety of any covering of $\mathbb{P}^1$ that factors through some cyclic cover of $\mathbb{P}^1$, but their method is not constructive and rather complicated.

Received by the editors October 21, 2009; revised December 15, 2009.
Published electronically August 3, 2011.
The authors were partially supported by JSPS Core-to-Core Program No.18005. The first and second authors were also partially supported by JSPS Grant-in-Aid for Scientific research No.18740021 and No.19740017, respectively.
AMS subject classification: **11G05**, 11D25, 14G25, 14K07.
Keywords: Mordell–Weil rank, Jacobian varieties, Frey-Jarden conjecture, abelian points.

As noted above, many people have attacked Conjecture 1.1. However, they have not mentioned any evidence for Conjecture 1.1. The purpose of this paper is to find what relates to Conjecture 1.1 for the case of Jacobian varieties. We further give a different approach to proving the Frey–Jarden conjecture for abelian varieties of GL_2-type with trivial character.

Let C be a smooth projective curve over a number field K and let J be Jacobian variety of C. Then we prove the following theorem.

Theorem 1.2 *If C has infinitely many K^{ab}-rational points, Conjecture* 1.1 *is true for J.*

If C is hyperelliptic, then Theorem 1.2 is included in the result by Néron which we mentioned above.

Recently, Petersen treated the more general setting [15]. His results also help us to make the relation between abelian points on a curve C and $\mathrm{Jac}(C)$ clear. As a result, Theorem 1.2 is a special case of [15, Theorem 1.1], but our method is completely different and elementary. Furthermore, our results include a new case for which Conjecture 1.1 holds.

Next, we give another approach to our problem, applying the modularity problem of abelian varieties and a non-vanishing result of the L-functions of automorphic forms. Let A be an abelian variety over $\mathbb{Q}$ of GL_2-type (see [18] for the definition of GL_2-type). Then for any prime l, we have the two-dimensional Galois representation ρ_l attached to A such that $\det \rho_l = \varepsilon\chi_l$, where ε is a finite character that is independent of l and χ_l is the l-adic cyclotomic character (see [18, Lemma 3.1]). We say that A is an abelian variety of GL_2-type with trivial character if the character ε is trivial. For such a class of abelian varieties, we prove the following.

Theorem 1.3 *Conjecture* 1.1 *is true for any abelian variety over $\mathbb{Q}$ of GL_2-type with trivial character.*

Remark 1.4 Let f be a newform in $S_2(\Gamma_0(M))$ for some level M. Then Shimura's abelian variety A_f attached to f is of GL_2-type with trivial character (see [22]).

E. Kobayashi considered when K^{ab} could be $K\mathbb{Q}^{\mathrm{ab}}$ in Conjecture 1.1 for any elliptic curve over an odd dimensional abelian extension K of $\mathbb{Q}$. In fact, she proved that under some (unsolved) conjectures [9]. The idea is similar to that in Section 3, but we use the non-vanishing theorem by Murty–Murty [12] and our proof is unconditional.

The proof of [4, Theorem 2.2] actually proves Conjecture 1.1 for elliptic curves over $\mathbb{Q}$. This fact is also mentioned in the formulation of the conjecture [4, p. 127].

Theorem 1.2 gives a relation between abelian points on a curve and those on its Jacobian variety. It is also interesting to consider the case of an algebraic curve having only finitely many abelian points and how to construct infinitely many abelian points on its Jacobian varieties. This is also related to the *largeness* of $\mathbb{Q}^{\mathrm{ab}}$ in the sense of Pop [16].

In Section 2, we study abelian points on any abelian variety and give a proof of Theorem 1.2. Then we see that sufficiently many abelian points yield points that are independent of each other. The case of an abelian variety of GL_2-type is treated in Section 3.

2 The Case of Jacobian Varieties

2.1 Abelian Points on A

Let A be an abelian variety over a number field K. We fix an algebraic closure $\overline{K}$ of K. Let G_K be the absolute Galois group of K and let K^{ab} be the maximal abelian extension of K in $\overline{K}$. We call a point P on $A(K^{\mathrm{ab}})$ an abelian point on A.

Definition 2.1 Let M be a finite extension of K contained in K^{ab} and let us take P in $A(K^{\mathrm{ab}})$. Then we define the minimal field of definition of the class $P + A_{\mathrm{tor}}(K^{\mathrm{ab}})$ in $A(K^{\mathrm{ab}})/A_{\mathrm{tor}}(K^{\mathrm{ab}})$ to be equal to M if the following condition is satisfied: ${}^{\sigma}P + A_{\mathrm{tor}}(K^{\mathrm{ab}}) = P + A_{\mathrm{tor}}(K^{\mathrm{ab}})$ if and only if $\sigma \in G_M$. We remark that the minimal field of definition of a class $P + A_{\mathrm{tor}}(K^{\mathrm{ab}})$ is unique.

Lemma 2.2 *Assume that the minimal field of definition of the class $P + A_{\mathrm{tor}}(K^{\mathrm{ab}})$ in $A(K^{\mathrm{ab}})/A_{\mathrm{tor}}(K^{\mathrm{ab}})$ is equal to $K(P)$. Then for an arbitrary proper subfield K' of $K(P)$ containing K, we have $na \notin A(K')$, $(\forall n \geq 1)$.*

Proof Suppose that there exists a natural number n such that na is in $A(K')$. Then for a given σ in $G_{K'}$, we have

$$P - {}^{\sigma}P \in A[n] \cap A(K(P)) \subset A_{\mathrm{tor}}(K^{\mathrm{ab}}).$$

By the assumption of the field of definition of the class $P + A_{\mathrm{tor}}(K^{\mathrm{ab}})$, we have $\sigma \in G_{K(P)}$. Thus, we have $K' = K(P)$. This leads to a contradiction. ∎

Theorem 2.3 *If there exist infinitely many K^{ab}-rational points P_j $(j \geq 1)$ on A satisfying the following two conditions, then* $\mathrm{rank}_{\mathbb{Q}}\, \mathbb{Q} \otimes_{\mathbb{Z}} A(K^{\mathrm{ab}}) = \infty$.

(i) *$K(P_i) = K(P_j)$ if and only if $i = j$.*

(ii) *The minimal field of definition of $P_j + A_{\mathrm{tor}}(K^{\mathrm{ab}})$ is $K(P_j)$ for each j.*

Proof We write $K_j := K(P_j)$ for simplicity. Since the number of subfields of K_j over K is finite, we may assume by taking subsequence of $\{K_j\}$ and by renumbering, that

$$K_1 \cdots K_{j-1} \cap K_j \neq K_j \quad (\forall j \geq 2). \tag{2.1}$$

It is enough to show $P_1, \dots, P_n$ are independent.

Suppose that $c_1P_1 + \cdots + c_nP_n = 0$. Since $c_1P_1 + \cdots + c_{n-1}P_{n-1} = -c_nP_n$, we have

$$c_nP_n \in A(K_1 \cdots K_{n-1} \cap K_n). \tag{2.2}$$

By Lemma 2.2, it follows from (2.1) and (2.2) that $c_n = 0$. Repeating, we obtain $c_{n-1} = c_{n-2} = \cdots = c_1 = 0$. ∎

Lemma 2.4 ([23, Lemma 1]) *Let A be an abelian variety over a number field M and let v be a prime of M such that*

(i) *$e_v < p - 1$, where e_v is the ramification index of $M/\mathbb{Q}$ at v and p is a rational prime under v.*

https://doi.org/10.4153/CMB-2011-140-5 Published online by Cambridge University Press

(ii) *A has good reduction at v.*

Then the reduction modulo v defines an injection $A_{\mathrm{tor}}(M) \to A_v(\mathbb{F}_v)$, where A_v is the reduction of A modulo v and $\mathbb{F}_v$ is the residue field.

Lemma 2.5 *If the minimal field of definition of $P + A_{\mathrm{tor}}(K^{\mathrm{ab}})$ is a proper subfield K' of $K(P)$ containing K, then each ramified prime v of $K(P)$ over K' satisfies one of the following two conditions.*

(i) *v is a bad prime of A.*
(ii) *$p - 1 \le [K(P):\mathbb{Q}]$.*

Proof Suppose that v is a good prime of A and $p - 1 > [K(P):\mathbb{Q}]$. Since $e_v \le [K(P):\mathbb{Q}]$, by using Lemma 2.4 we see that $A_{\mathrm{tor}}(K(P)) \to A_v(\mathbb{F}_v)$ is injective.

Let I_v be the inertia group for v of G_K. By the assumption, there exists a nontrivial element σ in $I_v \cap (G_{K'} \setminus G_{K(P)})$. Since σ is in I_v, by the above reduction, the mapping $P - {}^{\sigma}P$ is specialized to zero in $A_v(\mathbb{F}_v)$. Thus, we have $P - {}^{\sigma}P = 0$.

Then we obtain σ in $G_{K(P)}$, which implies a contradiction to the choice of σ. Thus the assertion follows. ■

Theorem 2.6 *Let p be a prime integer. Then there exist only finitely many cyclic extensions M of K of degree p satisfying the following condition:*

- *M is equal to $K(P)$ for some point P in $A(K^{\mathrm{ab}})$ such that the minimal field of definition of $P + A_{\mathrm{tor}}(K^{\mathrm{ab}})$ is not equal to $K(P)$.*

Proof Since $[M:K]$ is prime, we see that the minimal field of definition of $P + A_{\mathrm{tor}}(K^{\mathrm{ab}})$ is equal to K. Each ramified prime v of M over K, lying above a prime number q, either A has bad reduction at v or $q \le [M:\mathbb{Q}] + 1 = p[K:\mathbb{Q}] + 1$ from Lemma 2.5. Hence the number of the ramified primes of any such abelian extension M over K of degree p is bounded uniformly [14]. ■

2.2 Solvable Coverings of $\mathbb{P}^1$

Let $K^{p-\mathrm{cyclic}}$ be the composite field of all cyclic extension of degree p over a number field K. In this subsection, we discuss a version of the Frey–Jarden conjecture on Jacobian varieties of algebraic curves having infinitely many points defined over $K^{p-\mathrm{cyclic}}$ for some prime p.

Let C be a smooth projective curve of positive genus g over K. In this subsection, we always assume that C has a K-rational point O.

Let J be the Jacobian variety of C which is also defined over K. We define a canonical embedding Λ, which is also defined over K from C to J such that $\Lambda(O)$ is the zero of J. The morphism Λ induces the isomorphism

$$\mathrm{Pic}^0(C) \to J\colon \sum m_i P_i \mapsto \sum m_i \Lambda(P_i).$$

Lemma 2.7 *Let C_i be a smooth projective curve over K with a K-rational point for $i = 1, 2$. Let J_i be its Jacobian variety. Assume that C_1 is a covering curve of C_2 over K. Then if J_2 satisfies Conjecture* 1.1, *so does J_1.*

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Proof The covering $C_1 \to C_2$ induces a homomorphism $J_1 \to J_2$ of Abelian varieties defined over K. This homomorphism is surjective because C_i (or rather its image in J_i) generates J_i (over $\bar{K}$). It follows that J_1 is isogenous to a direct product $J_2 \times B$ for some Abelian variety B defined over K. This yields an isomorphism (up to a finite kernel) $J_1(K^{\mathrm{ab}}) \simeq J_2(K^{\mathrm{ab}}) \times B(K^{\mathrm{ab}})$. Hence, if the rational rank of $J_2(K^{\mathrm{ab}})$ is infinite, so is the rational rank of $J_1(K^{\mathrm{ab}})$. ■

Theorem 2.8 *If C is a solvable covering over K of $\mathbb{P}^1$, then Conjecture* 1.1 *is true for J.*

Proof Since C is a solvable covering over K of $\mathbb{P}^1$, there exists a sequence of cyclic coverings of prime degree over K $C_0 := C \to C_1 \to \cdots \to C_n := \mathbb{P}^1$. There exists j such that $C_j \ncong \mathbb{P}^1$ and $C_{j+1} \cong \mathbb{P}^1$. Thus, by Lemma 2.7, we may assume that C is a cyclic covering of $\mathbb{P}^1$ of a prime degree p defined over K.

We denote the function fields of C and $\mathbb{P}^1$ by $K(x, y)$ and $K(x)$, respectively. The Hilbert irreducibility theorem asserts that specializations $(x, y) \to (\alpha, \beta) \in K \times K^{\mathrm{ab}}$ give infinitely many cyclic extensions of degree p of K. Now we can apply Theorems 2.3 and 2.6, and we have the assertion. ■

If C is a hyperelliptic curve over K, C is a cyclic covering over K of degree two. Thus Theorem 2.8 contains the results of Billing, Néron, Imai, and Petersen [15, Theorem 1.1]).

We may generalize Theorem 2.8 to the case where C is a covering over K of $\mathbb{P}^1$ that factors through a solvable covering of $\mathbb{P}^1$. Thus Theorem 2.8 is essentially the same as a main result of Rosen–Wong [20].

2.3 The Case of $\sharp C(K^{\mathrm{ab}}) = \infty$

In this subsection, we give a proof of Theorem 1.2. We first introduce the terminology of torsion packet according to Baker and Poonen [1].

We defined an equivalence relation on $C(\overline{K})$ by

$$P \sim Q \text{ if and only if } \Lambda(P) - \Lambda(Q) \in J_{\mathrm{tor}}(\overline{K}).$$

We call an equivalence class under $\sim$ a *torsion packet* on C. A torsion packet is said to be *trivial* if it has only one point. The Manin–Mumford Conjecture (proved by Raynaud [17]) states that every torsion packet on C consists of a finite number of points if $g \geq 2$.

Theorem 2.9 ([1, Theorem 2]) *Suppose $g \geq 2$. There are infinitely many nontrivial torsion packets on C if and only if either $g = 2$, or $g = 3$ and C is both hyperelliptic and bielliptic. Here we say that C is bielliptic if it admits a covering map of degree two to an elliptic curve.*

Let P be an abelian point on C. If the minimal field of definition of the class $\Lambda(P) + J_{\mathrm{tor}}(K^{\mathrm{ab}})$ in $J(K^{\mathrm{ab}})/J_{\mathrm{tor}}(K^{\mathrm{ab}})$ is not equal to $K(P)$, then the torsion packet represented by P is nontrivial. Indeed, there exists σ in $G_K \setminus G_{K(P)}$ such that $\Lambda(P) - {}^{\sigma}\Lambda(P)$ is in $J_{\mathrm{tor}}(K^{\mathrm{ab}})$, and we have $P \sim {}^{\sigma}P$.

Corollary 2.10 *Assume that C is a non-hyperelliptic curve over K. Then the minimal field of definition of the class $\Lambda(P) + J_{\rm tor}(K^{\rm ab})$ in $J(K^{\rm ab})/J_{\rm tor}(K^{\rm ab})$ is equal to $K(P)$ for almost all $K^{\rm ab}$-rational points P on C.*

Proof The assertion follows directly from Theorem 2.9. ∎

Now we can prove Theorem 1.2.

Proof of Theorem 1.2 We may assume that C is non-hyperelliptic. By the assumption there exists a set S of infinitely many $K^{\rm ab}$-rational points on C.

It follows from Corollary 2.10 that we may assume that all points in S satisfy Theorem 2.3(ii) by removing finite exceptional points. By the theorem of Faltings [3], there exist only finite number of M-rational points of C for each finite extension M of K. Thus we may inductively choose P_j in S satisfying Theorem 2.3(i). ∎

Corollary 2.11 *If $\mathbb{Q}^{\rm ab}$ is large, then Conjecture* 1.1 *is true for J.*

Proof By our assumption, C has at least one rational point. If $\mathbb{Q}^{\rm ab}$ is large, so is K^{ab}. Thus C has infinitely many $K^{\rm ab}$-rational points on C. Theorem 1.2 implies the assertion. ∎

Remark 2.12 Moon proved that $A(K^{\rm ab})/A_{\rm tor}(K^{\rm ab})$ is a free module when $A_{\rm tor}(K^{\rm ab})$ is finite [10]. On the other hand, Zarhin [24] and Ruppert [21] showed that $A_{\rm tor}(K^{\rm ab})$ is finite if and only if A has no abelian subvarieties with complex multiplication over K. Hence, under the additional assumption that J has no abelian subvarieties with complex multiplication over K, Theorem 1.2 also follows from Faltings' theorem (see [10, §2]).

3 The Case of Abelian Varieties of GL_2-type

In this section, we prove the Frey–Jarden conjecture for any abelian variety of GL_2-type. This follows easily from the celebrated works of recent developments in number theory, but it is important for understanding the Frey–Jarden conjecture. We remark that even if an algebraic curve whose Jacobian variety is of GL_2-type does not have infinitely many abelian points, its Jacobian variety does. A reason why such a phenomenon occurs is that a general algebraic curve does not have a twist by automorphisms, though any abelian variety always has the twist induced by the inversion -1.

We refer to [22] for the theory of elliptic modular forms. We first recall the facts needed here.

Theorem 3.1 ([6–8]) *Any abelian variety A of GL_2-type is modular; namely, there exists a new form f in $S_2(\Gamma_1(N))$, where N is the conductor of A such that A is isogenous to A_f over $\mathbb{Q}$. Here A_f is Shimura's abelian variety attached to f (see* [22]*).*

Theorem 3.2 ([12]) *Let f be a new form in $S_2(\Gamma_0(N))$. If the root number of f is $+1$, then there exist infinitely many quadratic characters χ such that $L(f\otimes\chi, s)$ has analytic rank one.*

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Theorem 3.3 ([25]) *Let r be* 0 *or* 1*. If the analytic rank of $L(f, s)$ is r, then weak Birch–Swinnerton–Dyer conjecture is true for A_f. In this case, the analytic rank of $L(A_f, s)$ and the Mordell–Weil rank of A_f over $\mathbb{Q}$ are both equal to the dimension of A_f times r.*

Now we prove Theorem 1.3.

Proof of Theorem 1.3 Let A be an abelian variety of GL_2-type with the trivial character. By Theorem 3.1, A is modular, namely, there exists a new form f in $S_2(\Gamma_0(M))$ such that A is isogenous over $\mathbb{Q}$ to A_f. We may assume that the root number of f is +1 by the twisting of some quadratic character χ such that $\chi(M) = -1$. By Theorems 3.2 and 3.3 for $r = 1$, there exist infinitely many quadratic characters χ of $G_{\mathbb{Q}}$ such that $\mathrm{rank}_{\mathbb{Q}}\, \mathbb{Q} \otimes_{\mathbb{Z}} A_\chi(\mathbb{Q}) = \dim A$, where A_χ is the quadratic twist of A by χ. Then we get a point P_χ on A which is strictly defined over the quadratic field K_χ fixed by the kernel of χ, since $A_\chi(\mathbb{Q})$ is isomorphic to the χ-subspace of $A(K_\chi)$. Then we see that the set of all P_χ (varying χ as above) satisfies the condition in Theorem 2.3 by using Theorem 2.6 for $p = 2$. ■

Acknowledgements The authors gratefully acknowledge helpful conversations with Professor H. Moon and Professor K. Matsuno.

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