The Milne Quadrics of the Trinodal Cubic Surface, and the Contact Conics of the Harmonic Envelope of the Plane Trinodal Quartic

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There are two planes, called "parabolic planes," through each line on the cubic surface cutting the surface in residual conics touching the line in question at two points called the "parabolic points" of that line.

If the cubic surface be taken in the form $UVW = U'V'W'$, where U, V, W, U', V', W' are six planes defining nine lines on the cubic surface, then the remaining eighteen lines on the surface form "three associated double-sixes," each line belonging to two double-sixes.

The following results were obtained by W. P. Milne and A. L. Dixon.

[1] *The twenty-four tangent planes to a cubic surface at the twenty-four parabolic points on the twelve lines of a double-six of the surface touch the same quadric, and there are thirty-six such quadrics corresponding to the thirty-six double-sixes of the surface. These thirty-six quadrics are contact quadrics of the harmonic envelope T of the surface (i.e. they touch T wherever they meet it), and have been termed "Milne Quadrics" by A. L. Dixon.*

[2] *The eighteen lines of "three associated double-sixes" meet the parabolic curve in thirty-six points, and the tangent planes to the cubic surface at each of these thirty-six points touch the same cubic envelope Γ, which also touches the common tangent planes of the harmonic envelope T and each of the "triad" of Milne quadrics corresponding to the three associated double-sixes.*

[2] *If M_1, M_2, M_3 be a triad of Milne quadrics corresponding to three associated double-sixes, then $M_1 + M_2 + M_3 \equiv 3P^2$ where P is a point.*

[1] $T \equiv 3\,\Gamma^2 - M_1\, M_2\, M_3$ *for all "triads" of Milne quadrics.*

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[1] W. P. Milne, *Proc. London Math. Soc.* (2), **26** (1927), 377-394.

[2] A. L. Dixon, *Proc. London Math. Soc.* (2), **26** (1927), 351-362.

It was suggested to me by Professor W. P. Milne that I should investigate in this connection the case of the trinodal cubic surface; I have obtained the following results.

§ 2. *The Trinodal Cubic Surface.*

Take the three nodes of the cubic surface at A, B, and C, and let D be the point of intersection of the three constant tangent planes along BC, CA, AB, where $ABCD$ is the tetrahedron of reference. Then the equation of the trinodal cubic surface can be written

$$U \equiv dt^3 + 3d_1 t^2 x + 3d_2 t^2 y + 3d_3 t^2 z + 6s\, xyz = 0. \qquad (1)$$

We can map this surface U on a plane π by means of a web of plane cubics passing through the six points $A_1, A_2 \ldots . A_6$, where A_1 coincides with A_2, A_3 with A_4, and A_5 with A_6. Taking $A_1A_3A_5$ as the triangle of reference in π, and ξ, η, ζ as coordinates, we obtain the following mapping equations:—

$$x = \xi^2 (p_1 \eta + \zeta), \quad y = \eta^2 (q_1 \zeta + \xi), \quad z = \zeta^2 (r_1 \xi + \eta), \quad t = s_1 \xi \eta \zeta.$$

By comparison with equation (1) we see that

$$p_1q_1r_1 + 1 = -dk, \; s_1r_1 = -3d_1k, \; s_1p = -3d_2k, \; s_1q_1 = -3d_3k, \; s_1^3 = 6ks,$$

whence k satisfies the equation $9d_1d_2d_3k^2 - 2dsk - 2s = 0$.

Take $\Delta = d^2 s + 18d_1 d_2 d_3, \quad \sigma = \Delta^{\frac{1}{2}} s^{-\frac{1}{2}}, \quad k_1 = \dfrac{s(d+\sigma)}{9\, d_1d_2d_3}, \quad k_2 = \dfrac{s(d-\sigma)}{9\, d_1d_2d_3}$

To the points $A_1 \equiv A_2$ etc., correspond the lines $a_1 \equiv a_2$ etc., on the cubic surface which we will denote by α, β, and γ. To the lines $A_1A_2 \equiv p_1\eta + \zeta = 0$, A_3A_4 etc., correspond the lines c_{12}, c_{34}, c_{56}, which we shall denote by q, r, s respectively. To the lines $A_3A_5 \equiv A_3A_6 \equiv A_4A_5 \equiv A_4A_6$ etc., correspond the lines $c_{35} \equiv c_{36} \equiv c_{45} \equiv c_{46}$ etc., which we shall denote by x, y, z. To the conic touching the lines A_3A_4, A_5A_6, at A_3 and A_5 and passing through A_1 corresponds the line $b_1 \equiv b_2$ which we denote by α', and similarly for β' and γ'.

The equations of the lines on the cubic surface together with the two parabolic planes corresponding to each are given below.

Line on the Surface.	*Corresponding Parabolic Planes.*
$x \equiv \{x = 0,\, t = 0\}$.	$x = 0$ taken twice over,

and similarly for y and z.

$\alpha \equiv \{dt + 3d_2y + 3d_3z = 0,\; t = 3d_3k_1 z\}$.	$(d+\sigma)t + 3d_2y + 3d_3(1-\sigma k_1)z = 0$ taken twice over.

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Similarly for β and γ; in the case of α', β', γ', k_1 is replaced by k_2 in the equations of α, β, γ respectively; in the corresponding parabolic planes the sign of σ is also changed, besides replacing k_1 by k_2 in the parabolic planes corresponding to α, β, γ respectively.

$$q \equiv \{dt + 3d_1 x + 3d_2 y + 3d_3 z = 0, \; x = 0\}.$$
$$x' \equiv 6dt + d_2^{-1} d_3^{-1} \Delta x + 18 d_2 y + 18 d_3 z = 0, \; x = 0.$$

And similarly for r and s.

Corresponding to the curve of section of the cubic surface and the plane $lx + my + nz + pt = 0$, we have in π the curve

$$p_1 l\xi^2 \eta + l\xi^2 \zeta + q_1 m\eta^2 \zeta + m\eta^2 \xi + r_1 n\zeta^2 \xi + n\zeta^2 \eta + s_1 p\xi\eta\zeta = 0. \tag{3}$$

Using Salmon's conditions that the curve (3) be harmonic and equianharmonic and the fact that there is a 1 : 1 correspondence between the two curves we find that if T_4 and S_6 be the harmonic and equianharmonic envelopes of the cubic surface, and if we transform to the lettering $d, \ldots, s$ we have

$$S = 16 s^2 s_1^{-4} S_4 \equiv \Theta'^2 - \Theta\,\Delta' = 0 \tag{4}$$
$$T = 16 s^3 s_1^{-6} T_6 \equiv 2\,\Theta'^3 - 3\,\Theta\,\Theta' - 3\,\Delta\,\Delta' = 0, \tag{5}$$

where $\Delta' = 2lmn$,

$$\Theta' = sp^2 + 2\,(d_1 mn + d_2 nl + d_3 lm),$$
$$\Theta \;= 2\,\{dsp + 3\,(d_2 d_3 l + d_3 d_1 m + d_1 d_2 n)\}.$$

The quadric $\Theta' = 0$ is the quadric inscribed in the three nodal cones of the cubic surface, and the point $\Theta = 0$ is the intersection of the six parabolic planes through α, α', β, β', γ, γ'.

Since α and α' are generators of the nodal cone at A of the cubic surface, and the parabolic planes through these lines touch the nodal cone at A along the lines, it follows that $\Theta = 0$ is the pole of the plane $\alpha\alpha'$ with respect to the nodal cone at A, and similarly with respect to the nodal cones at B and C.

We see from its equation that the equianharmonic envelope S has four nodes at the points A, B, C, and $\Theta = 0$, and the nodal cones of S at A, B, and C are the nodal cones of the surface.

From (5) we see that

$$T \equiv 3\,\Delta^{-1}(\Delta\,\Delta' - \tfrac{1}{2}\,\Theta\,\Theta') - \Theta'^2\,(\tfrac{3}{4}\Theta'^3/\Delta - 2\Theta') = 0 \tag{6}$$

which leads to the following important results:—

The harmonic envelope T, the quadric $\Theta' = 0$, and the cubic envelope $\Theta\,\Theta' - 2\,\Delta\,\Delta' = 0$ have a common developable touching all the three *surfaces, each plane of which touches T at two points, and every tangent plane of each of the three nodal cones of the surface U are bitangent planes of T.*

Now W. P. Milne proved, in the paper referred to, that a Milne quadric is the envelope of planes cutting the cubic surface in sections U' such that the pairs of points in which U' is met by conjugate lines of a double-six are "apolar pairs" of points with respect to the Hessian of U' (*i.e.* the polar line of one point of the pair with respect to the Hessian of U' passes through the other point, and *vice-versa*). By the $1:1$ correspondence between points in π and points on the cubic surface, we see that we can obtain the Milne quadrics by stating that the pairs of points in π, corresponding to the pairs of points in which conjugate pairs of lines of a double-six cut U', are "apolar pairs" with respect to the Hessian of (3) *i.e.* H_1' (corresponding to U'). The points in π corresponding to the points in which the lines on the cubic surface cut $lx + my + nz + pt = 0$ are readily obtained; taking

$$H_1' \equiv a_1'\,\xi^3 + b_2'\,\eta^3 + c_3'\,\zeta^3 + 3a_2'\,\xi^2\eta + 3a_3'\,\xi^2\zeta + 3b_1'\,\eta^2\xi + 3b_3'\,\eta^2\zeta$$
$$+ 3c_1'\,\zeta^2 + 3c_2'\,\zeta^2\eta + 6q'\,\xi\eta\zeta = 0,$$

we find that, if $\phi = r_1 mn + p_1 nl + q_1 lm - (s_1^2/4)\,p^2$, then

$$\begin{array}{lll} a_1' = l^2(m + p_1^2 r_1 n - p_1 s_1 p) & b_2' = m^2\{n + p_1 q_1^2 l - q_1 s_1 p\}, & c_3' = n^2\{l + q_1 r_1^2 m - r_1 s_1 p\} \\ 3a_2' = p_1 l\{\phi - 3q_1 lm\}, & 3a_3' = l\ \{\phi - 3p_1 nl\} & \\ 3b_3' = q_1 m\{\phi - 3q_1 lm\}, & 3b_1' = m\ \{\phi - 3r_1 mn\} & \\ 3c_1' = r_1 n\{\phi - 3p_1 nl\}, & 3c_2' = n\ \{\phi - 3r_1 mn\}, & 6q' = \{s_1 p\phi - 3(p_1 q_1 r_1 + 1)\, lmn\}. \end{array}$$

The condition that two points (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) be an apolar pair of points with respect to $H_1' = 0$ is

$$\xi_1^2(a_1'\,\xi_2 + a_2'\,\eta_2 + a_3'\,\zeta_2) + \ldots\ldots + 2\eta_1\zeta_1(q'\,\xi_2 + b_3'\,\eta_2 + c_3'\,\zeta_2) = 0.$$

The thirty-six double-sixes of the general cubic surface reduce to fourteen in the trinodal case, which fall into types as follows, each giving a Milne quadric obtained by the foregoing method. (The lettering has been converted from p_1, q_1 etc., to d, d_1 etc., and the types of the double-sixes are made to correspond with Jolliffe's types of Steinerian Complexes in his paper "The Inflexional Tangents of the Plane Quartic,"[1] there being no type corresponding to his Type II.)

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[1] A. E. Jolliffe, *Proc. London Math. Soc.* (2), 23 (1924), 250-278.

	Double-Six.	*Corresponding Milne Quadric.*
Type I.	$\beta\,\beta'\,y\,y\,x\,s$ $\gamma\,\gamma'\,z\,z\,r\,x$	$L \equiv \Theta' - 6d_1\,mn$, and similarly for M and N.
Type III.	$\alpha\,\beta\,\gamma\,x\,y\,z$ $x\,y\,z\,\alpha'\,\beta'\,\gamma'$	$\Theta' = 0$
Type IV.	$\alpha\,\alpha\,\beta\,\beta\,\gamma\,\gamma$ $\alpha'\alpha'\beta'\beta'\gamma'\gamma'$	$\Theta_0 \equiv 3\,\Theta^2/4\Delta - 2\,\Theta' = 0$
Type V.	$\alpha\,r\,\gamma\,\gamma\,z\,z$ $s\,\alpha'\,y\,y\,\beta'\,\beta'$	$U_1 \equiv \Theta' - 6l\,d_1^{-1}\{-d_2\,d_3\,(d\,k_1+1)\,l$ $+ d_2\,d_1\,m + d_1\,d_2\,n + 3d_1\,d_2\,d_3\,k_1\,p\} = 0.$

U_2 has k_2 in place of k_1 in U_1, and so on for V_1, V_2; W_1, W_2.

The point $\{-d_2\,d_3\,(dk_1+1)\,l + d_3\,d_1\,m + d_1\,d_2\,n + 3d_1\,d_2\,d_3\,k_1\,p\} = 0$ *is the point of intersection of* γ *and* p'. The remaining three double-sixes need not be considered as they give rise to the degenerate quadrics $l^2 = 0$, $m^2 = 0$, $n^2 = 0$.

Professor Milne also proved in his paper that the cubic envelope Γ is the envelope of planes cutting the cubic surface in sections U' such that the triad of points, in which any set of three non-intersecting lines on the cubic surface cut U', is an apolar triad with respect to H' the Hessian of U' (*i.e.* the mixed polar line of one pair of points with respect to the Hessian of U' passes through the third point). Hence as before by $1:1$ correspondence, we see that, if our three points in π are (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2), (ξ_3, η_3, ζ_3), Γ is given by

$$\xi_3\,(A\xi_2 + H\eta_2 + G\zeta_2) + \eta_3\,(H\xi_2 + B\eta_2 + F\zeta_2) + \zeta_3\,(G\xi_2 + F\eta_2 + C\zeta_2) = 0,$$

where $A = (a_1'\,\xi_1 + a_2'\,\eta_1 + a_3'\,\zeta_1)$ etc., $F = (q'\,\xi_1 + b_3'\,\eta_1 + c_2'\,\zeta_1)$ etc., since this is the condition that the three points (ξ_1, η_1, ζ_1) etc., are an apolar triad with respect to H_1'.

The point $P = 0$, where $M_1 + M_2 + M_3 = 3P^2$, was proved by W. P. Milne to be the fixed point through which π_1 passes, if the plane π_1 intersects the cubic surface in a curve U' such that the three non-intersecting lines (lines which would not intersect on the general cubic surface, and give a distinct Γ) on the surface, corresponding to the triad M_1, M_2, M_3, cut π_1 in three points forming an apolar triad on U'.

The sets of three non-intersecting lines, their corresponding triads of Milne quadrics, points $P = 0$, and cubic envelopes Γ are found to be as follows, and are divided into five classes.

	Three non-intersecting lines on the surface.	*Triad of Milne quadrics.*	*Point P.*	*Cubic Envelope* Γ.
Class I.	$\alpha\beta\gamma$	LMN	$p=0$	$s^{\frac{1}{2}}(\Theta' p - d\,\Delta')=0$
Class II.	$\beta\gamma y$	$L\Theta' U_1$	$A_1' \equiv sp - 3d_2 d_3 k_1 l = 0$	$s^{-\frac{1}{2}}[\Theta' A_1' + \Delta^{\frac{1}{2}} s^{\frac{1}{2}} \Delta'] = 0.$

For the triad $L\,\Theta'\,U_2$, corresponding to $\beta\gamma z$, k_2 is exchanged for k_1 in the above, and similar results are obtained for the triads $M\,\Theta' V_1$, $M\,\Theta' V_2$; $N\,\Theta' W_1$, $N\,\Theta' W_2$.

Class III.	$\beta\gamma q$	LV_1W_2	$B_1' \equiv sp - 3d_3 d_1 k_1 m - 3d_1 d_2 k_2 n = 0$	$s^{-\frac{1}{2}}\left[\begin{array}{l}\Theta' B_1' \\ +2mn\left\{\begin{array}{l} als + 9d_1^2 d_3 k_1 m \\ +9d_1^2 d_2 k_2 n - 6d_1 sp \end{array}\right\}\end{array}\right]$

For the triad $LV_2W_1\,(\beta'\gamma' q)$ k_1 is exchanged for k_2, and *vice versa* in the above, similar results being obtained from the triads MW_1U_2, MW_2U_1; NU_1V_2, NU_2V_1.

Class IV.	$\alpha\alpha' x$	$\Theta'^2\,\Theta_0$	$\Theta = 0$	$\Delta^{-\frac{1}{2}}(\Delta\,\Delta' - \Theta\,\Theta'/2) = 0.$
Class V.	$\beta\beta' z$	$\Theta_0 U_1 U_2$	$3d_1\,\Theta = 4\,\Delta\, l$	$\Delta^{-\frac{1}{2}}\left[\begin{array}{l}\Theta'(\frac{1}{2}\Theta + \frac{4}{3}\Delta\, l d_1^{-1} \\ -l d_1^{-1}(\frac{1}{2}\Theta^2 + 2d_1\Delta mn)\end{array}\right] = 0$

The results for $\Theta_0 V_1 V_2$, $\Theta_0 W_1 W_2$ are obtained by symmetry.

The properties of the Milne quadrics, Γ's, and the points P can be easily obtained from their equations, but will not be discussed here.

§ 3. *The Plane Trinodal Quartic Curve.*

In the paper, previously referred to, A. E. Jolliffe proved that *if I_4 and J_6 denote the harmonic and equianharmonic envelopes of a plane quartic curve, then a conic K can be so defined that the sextic envelope $J_6 + KI_4 = 0$ shall touch the twenty-four inflexional tangents of the quartic curve, and every one of the twelve bitangents of a Steinerian complex.*

The properties of the conics K, which are contact conics of J_6 were further discussed by W. P. Milne in two papers "Contravariant envelopes of the plane quartic curve[1]," and "Contravariant envelopes of the cubic surface."

[1] W. P. Milne, *Proc. London Math. Soc.* (2) **24** (1925), 335-338.

It was shown in my own paper "Note on a property of the plane quartic curve" that

(i) *J_6 can be expressed in the form $J_6 \equiv \Phi_3^2 - K_1 K_2 K_3$, where $K_1 K_2 K_3$ is a "triad of contact conics" of J_6.*

(ii) *$K_1 + K_2 + K_3 \equiv P'^2$, where $P' = 0$ is a point.*

I now proceed to obtain these triads of contact conics of J_6 in the case of the trinodal plane quartic.

§ 4. *The Trinodal Plane Quartic.*

Take the equation of the trinodal plane quartic to be (as in Jolliffe's paper)

$$Q \equiv ay^2 z^2 + bz^2 x^2 + cx^2 y^2 + 2xyz\,(fx + gy + hr) = 0,$$

where A, B, and C, the three vertices of the triangle of reference are taken at the three nodes of the quartic. We also use the notation

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2,$

$\Theta = 2\,(Fl + Gm + Hn)$,

where $F = gh - cf$ etc., and l, m, n are tangential coordinates,

$\Theta' = -\,(al^2 + bm^2 + cn^2 - 2fmn - 2gnl - 2hlm),$

$\Delta' = 2lmn, \qquad F_1 = F - ia^{\frac{1}{2}}\,\Delta^{\frac{1}{2}}, \qquad F_2 = F + ia^{\frac{1}{2}}\,\Delta^{\frac{1}{2}}$ etc.

$\alpha \equiv Cz - F_2\,y,\ \ \alpha' \equiv Cz - F_1\,y = 0;$

similarly for the tangents from B and C to Q we use β, β'; γ, γ'.

$x \equiv x = 0$, and similarly for y and z

$$\left.\begin{aligned} p &\equiv f_1 x + g_1 y + h_1 z = 0\\ q &\equiv f_1 x + g_2 y + h_2 z = 0\\ r &\equiv f_2 x + g_1 y + h_2 z = 0\\ s &\equiv f_2 x + g_2 y + h_1 z = 0\end{aligned}\right\}$$

Bitangents of Q (where $f_1 = f - \sqrt{bc}$, $f_2 = f + \sqrt{bc}$, etc.)

Then we have $I \equiv 12 I_4 \ \equiv \Theta'^2 - 3\Delta'\,\Theta$

$$J \equiv 432 J_6 \equiv 2\Theta'^3 - 9\Theta\Theta' + 27\Delta\Delta' = 0,$$

$$i.e.,\ \Delta J \equiv 3\,(\tfrac{1}{2}\Theta\Theta' - 3\Delta\Delta')^2 + \Theta'^2\,(2\Theta' - \tfrac{3}{4}\Theta^2\,/\,\Delta) = 0.$$

Since $\Theta' = 0$ is the equation of the conic touching the nodal tangents of the quartic, we see from the last equation that the harmonic envelope of the trinodal quartic curve has six bitangents, which are the six nodal *tangents of the quartic, and also the conics* $\Theta' = 0$, *and* $3\Theta^2 - 8\Delta\,\Theta' = 0$ *are contact conics of* J.

From $I = 0$ *we see that the equianharmonic envelope has four nodes at the points* A, B, C, *and* $\Theta = 0$, *and has as its four pairs of nodal tangents the nodal tangents of the quartic, and the pair of tangents from* $\Theta = 0$ *to the conic* $\Theta' = 0$.

Dividing the conics K into types as in Jolliffe's paper, and using his method for obtaining them, we obtain the following table (taking K = twice Jolliffe's K)

	Conic K.	*Corresponding Steinerian Complex.*
Type I.	$L \equiv \Theta' - 6f_2\,mn = 0$	yz twice, $\beta\gamma$, $\beta'\gamma'$, xr, xs,
	$L_1 \equiv \Theta' - 6f_1\,mn = 0$	yz twice, $\beta\gamma'$, $\beta'\gamma$, xp, xq,
	with similar results for M, M_1; N, N_1.	
Type II.	$P \equiv \Theta' + 3a^{-1}(Cm^2 + 2Fmn + Bn^2) = 0$	x^2 twice, $\alpha\alpha'$ twice, pq, rs,
	and similarly for Q and R.	
Type III.	$\Theta' = 0$	$x\alpha$, $x\alpha'$, $y\beta$, $y\beta'$, $z\gamma$, $z\gamma'$.
Type IV.	$\Theta_0 \equiv \frac{3}{4}\Theta^2/\Delta - 2\,\Theta' = 0$	$\alpha\alpha'$ twice, $\beta\beta'$ twice, $\gamma\gamma'$ twice.
Type V.	$U_1 \equiv \Theta' + 6A^{-1}f_1[G_1H_1A^{-1}l + G_1m + H_1n] = 0$	$y\gamma$ twice, $z\beta'$ twice, $\alpha'r$, αs.
	$U_2 \equiv \Theta' + 6A^{-1}f_1[G_2H_2A^{-1}l + G_2m + H_2n] = 0$	$y\gamma'$ twice, $z\beta$ twice, αr, $\alpha' s$.
	$U_3 \equiv \Theta' + 6A^{-1}f_2[G_2H_1A^{-1}l + G_2m + H_1n] = 0$	$y\gamma$ twice, $z\beta$ twice, $\alpha' p$, αq.
	$U_4 \equiv \Theta' + 6A^{-1}f_2[G_1H_2A^{-1}l + G_1m + H_2n] = 0$	$y\gamma'$ twice, $z\beta'$ twice, αp, $\alpha' q$.
	Similarly for V_1, V_2, V_3, V_4; W_1, W_2, W_3, W_4.	

The pair of points $Cm^2 + 2Fmn + Bn^2 = 0$ *are the points where* x *intersects* α *and* α' *etc.*

The point $G_1\,H_1\,A^{-1}\,l + G_1\,m + H_1\,n = 0$ *is the point of intersection of* β' *and* γ *etc.*

We know by Jolliffe's paper that, corresponding to each conic K, we have $Q \equiv k_1'\,S_1\,S_2 + S_3^2$, while S_1, S_2, S_3 are conics or line pairs, and k_1' is a constant. If Ψ_3 be the cubic envelope, which is the envelope of lines cutting S_1, S_2, S_3 in involution, then we obtain the following Ψ_3's corresponding to each K.

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	Conic K.	Cubic Envelope Ψ_3.
Type I.	L	$\frac{1}{2}mn\,[f_2\sqrt{b}m+f_2\sqrt{c}n-(h\sqrt{c}+g\sqrt{b})\,l]=0$
	L_1	$\frac{1}{2}mn\,[f_1\sqrt{b}m-f_1\sqrt{c}n+(h\sqrt{c}-g\sqrt{b})\,l]=0$, and so on.
Type II.	P	$a^{-\frac{3}{2}}[(hm-gn)(Cm^2+2Fmn+Bn^2)+al(Cm^2+Bn^2)]=0$ etc.
Type III.	$\Theta'=0$	$-i\Delta^{\frac{1}{2}}\Delta'/4=0$
Type IV.	Θ_0	$\Delta^{-\frac{3}{2}}[BCFl^3+\ldots+C(2HF+BG)\,l^2m+\ldots+(ABC+2FGH)\,lmn]=0$
Type V.	U_1	$\frac{1}{2}lA^{-1}[\sqrt{b}G_1m^2+\sqrt{c}H_1n^2-p_1H_1G_1A^{-1}l^2-p_1Amn+H_1\{\sqrt{c}G_1A^{-1}-p_1\}\,nl+G_1\{\sqrt{b}H_1A^{-1}-p_1\}\,lm]=0$

For U_2 the suffix 1 is changed to 2 in G, H and p in above.

$$U_3 \qquad \tfrac{1}{2}lA^{-1}\,[\sqrt{b}G_2m^2-\sqrt{c}H_1n^2-p_3H_1G_2A^{-1}l^2-p_3Amn$$
$$-H_1\{\sqrt{c}G_2A^{-1}+p_3\}\,nl+G_2\{\sqrt{b}H_1A^{-1}-p_3\}lm]=0$$

For U_4, G_2 and H_1 are replaced by G_1 and H_2 respectively, and p_3 by p_4 in the above, where

$$p_1=(h\sqrt{c}+g\sqrt{b}+i\,\Delta^{\frac{1}{2}})f_2^{-1}, \qquad p_2=(h\sqrt{c}+g\sqrt{b}-i\,\Delta^{\frac{1}{2}})f_2^{-1},$$

$$p_3=(g\sqrt{b}-h\sqrt{c}-i\,\Delta^{\frac{1}{2}})f_1^{-1}, \qquad p_4=(g\sqrt{b}-h\sqrt{c}+i\,\Delta^{\frac{1}{2}})f_1^{-1} \text{ etc.}$$

It was proved by A. E. Jolliffe in a note on his paper that for any conic K we have $J\equiv 108k_1'^2\Psi_3^2+8K(9I_4-K^2)=0$.

Hence if we take any two conics K_1, K_2, and can find Φ_3 such that $\Phi_3\equiv k_1''(\Psi_3'+A_1K_1)\equiv k_2''(\Psi_3''+A_2K_2)$ where Ψ_3' refers to K_1, and Ψ_3'' to K_2, and A_1, A_2 are the tangential equations to two points, then K_1, K_2 are two conics of a triad, and the third conic K_3 of the triad can be obtained similarly (k_1'', k_2'' are constants).

We find that $J\equiv-3\,\Phi_3^2-K_1K_2K_3$, *and in each case*

$$A_1=A_2=A_3=P'.$$

Since the trinodal quartic curve is a degenerate case of the general quartic, the method used in my paper, referred to previously, of obtaining the "triad of conics" by means of three bitangents through whose six points of contact with the quartic no conic passes, does not give the complete set of triads, but the above method is applicable in all cases, and by its use we obtain the following triads, their points P', and their corresponding cubic envelopes Φ_3.

	Triad of Conics	Point P'
Class I.	$L\ M\ N$	$A_1' \equiv \sqrt{al} + \sqrt{bm} + \sqrt{cn} = 0$
	$L\ M_1 N_1$	$A_2' \equiv -\sqrt{al} + \sqrt{bm} + \sqrt{cn} = 0$
	$L_1 M\ N_1$	$A_3' \equiv \sqrt{al} - \sqrt{bm} + \sqrt{cn} = 0$
	$L_1 M_1 N$	$A_4' \equiv \sqrt{al} + \sqrt{bm} - \sqrt{cn} = 0$

Cubic Envelope Φ_3

$$\Phi_1' \equiv \Theta' A_1' - 3\,\Delta'\,(\ \ f\sqrt{a} + g\sqrt{b} + h\sqrt{c} + \sqrt{abc}) = 0$$
$$\Phi_2' \equiv \Theta' A_2' + 3\,\Delta'\,(\ \ f\sqrt{a} - g\sqrt{b} - h\sqrt{c} + \sqrt{abc}) = 0$$
$$\Phi_3' \equiv \Theta' A_3' + 3\,\Delta'\,(-f\sqrt{a} + g\sqrt{b} - h\sqrt{c} + \sqrt{abc}) = 0$$
$$\Phi_4' \equiv \Theta' A_4' + 3\,\Delta'\,(-f\sqrt{a} - g\sqrt{b} + h\sqrt{c} + \sqrt{abc}) = 0$$

Triad of Conics	Point P'	Cubic Envelope Φ_3
Class II.		
$L\Theta' U_1$	$B_1' \equiv -p_1 l + \sqrt{bm} - \sqrt{cn} = 0$	$\Theta' B_1' + 3i\,\Delta^{\frac{1}{2}}\,\Delta' = 0$
$L\Theta' U_2$	$B_2' \equiv -p_2 l + \sqrt{bm} + \sqrt{cn} = 0$	$\Theta' B_2' - 3i\,\Delta^{\frac{1}{2}}\,\Delta' = 0$
$L\Theta' U_3$	$B_3' \equiv p_3 l - \sqrt{bm} + \sqrt{cn} = 0$	$\Theta' B_3' + 3i\,\Delta^{\frac{1}{2}}\,\Delta' = 0$
$L\Theta' U_4$	$B_4' \equiv p_4 l - \sqrt{bm} + \sqrt{cn} = 0$	$\Theta' B_4' - 3i\,\Delta^{\frac{1}{2}}\,\Delta' = 0,$

and similarly for the triads $\begin{cases} M\Theta' V_1,\ M\Theta' V_2,\ M\Theta' V_3,\ M\Theta' V_4; \\ N\Theta' W_1,\ N\Theta' W_2,\ N\Theta' W_3,\ N\Theta' W_4. \end{cases}$

Class IIa.		
LL_1Q	$C_1' \equiv -hl + bm + fn = 0$	$\Phi_1'' \equiv b^{-\frac{1}{2}}\{\Theta' C_1' + 6mn(An + Gl)\} = 0$
LL_1R	$C_2' \equiv -gl + fm + cn = 0$	$\Phi_2'' \equiv c^{-\frac{1}{2}}\{\Theta' C_2' + 6mn(Hl + Am)\} = 0,$

and similarly for the triads MM_1R, MM_1P; NN_1Q, NN_1P.

Class III.		
$L\,V_1 W_2$	$-\sqrt{al} + q_1 m + r_2 n = 0$	$\Phi_2' + L\{(q_1 - \sqrt{b})m + (r_2 - \sqrt{c})n\} = 0$
$L\,V_2 W_1$	$-\sqrt{al} + q_2 m + r_1 n = 0$	$\Phi_2' + L\{(q_2 - \sqrt{b})m + (r_1 - \sqrt{c})n\} = 0$
$L\,V_3 W_3$	$\sqrt{al} + q_3 m + r_3 n = 0$	$\Phi_1' + L\{(q_3 - \sqrt{b})m + (r_3 - \sqrt{c})n\} = 0$
$L\,V_4 W_4$	$\sqrt{al} + q_4 m + r_4 n = 0$	$\Phi_1' + L\{(q_4 - \sqrt{b})m + (r_4 - \sqrt{c})n\} = 0$
$L_1 V_1 W_3$	$\sqrt{al} - q_1 m + r_3 n = 0$	$\Phi_3' + L_1\{(q_1 + \sqrt{b})m + (r_3 + \sqrt{c})n\} = 0$
$L_1 V_2 W_4$	$\sqrt{al} - q_2 m + r_4 n = 0$	$\Phi_3' + L_1\{(q_2 + \sqrt{b})m - (r_4 + \sqrt{c})n\} = 0$
$L_1 V_3 W_2$	$\sqrt{al} + q_3 m - r_2 n = 0$	$\Phi_4' + L_1\{(r_2 + \sqrt{c})n - (q_3 + \sqrt{b})m\} = 0$
$L_1 V_4 W_1$	$\sqrt{al} + q_4 m - r_1 n = 0$	$\Phi_4' + L_1\{(r_1 + \sqrt{c})n - (q_4 + \sqrt{b})m\} = 0.$

Similar results are obtained for the triads:—

$\begin{cases} MW_1U_2,\ MW_2U_1, \text{ etc.;} \\ NU_1V_2,\ NU_2V_1, \text{ etc.} \end{cases}$

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Class IV.

$$\Theta'\Theta'\Theta_0 \qquad \Theta = 0 \qquad -i\Delta^{\frac{1}{2}}(3\Delta\Delta' - \tfrac{1}{2}\Theta\Theta') = 0.$$

Class V.

$$\Theta_0 U_1 U_2 \qquad \Theta - 4\Delta f_2^{-1} l = 0 \qquad \Delta^{-\frac{1}{2}}[(\Theta\Theta' - 3\Delta\Delta') - 2l\Delta\Theta_0 f_2^{-1}] = 0$$

$$\Theta_0 U_3 U_4 \qquad \Theta - 4\Delta f_1^{-1} l = 0 \qquad \Delta^{-\frac{1}{2}}[(\Theta\Theta' - 3\Delta\Delta') - 2l\Delta\Theta_0 f_1^{-1}] = 0,$$

and similarly for the triads $\Theta_0 V_1 V_2$, $\Theta_0 V_3 V_4$; $\Theta_0 W_1 W_2$, $\Theta_0 W_3 W_4$.

Class VI.

$$P V_1 V_4 \qquad al + \{h + 2aH_1 B^{-1}\}m - gn = 0 \qquad \Phi_5'' + 2H_1 a^{\frac{1}{2}} B^{-1} mP = 0$$

$$P V_2 V_3 \qquad al + \{h + 2aH_2 B^{-1}\}m - gn = 0 \qquad \Phi_5'' + 2H_2 a^{\frac{1}{2}} B^{-1} mP = 0$$

$$P W_1 W_3 \qquad al - hm + \{g + 2aG_1 C^{-1}\} n = 0 \qquad \Phi_4'' + 2G_1 a^{\frac{1}{2}} C^{-1} n P = 0$$

$$P W_2 W_4 \qquad al - hm + \{g + 2aG_2 C^{-1}\} n = 0 \qquad \Phi_4'' + 2G_2 a^{\frac{1}{2}} C^{-1} n P = 0.$$

Including the cases of degenerate bitangents (*i.e.* lines joining two nodes, and tangents from a node), we find that the only sets of triads which do not correspond to three bitangents, through whose points of contact with the quartic no conic passes, are those of Classes IV and V which both contain the conic Θ_0. The conic Θ_0 is defined by the Steinerian complex $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ (all taken twice), and hence there is no third bitangent such that no conic passes through its points of contact with the quartic Q, and those of any pair $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$. By considering all triads of bitangents it can be shown that Classes IV and V are the only two sets of triads of conics into which the conic Θ_0 enters, and hence that the triads given in the above tables are the only ones possible.

The properties of the conics K, the cubic envelopes Φ_3, and the points P will not be considered in this paper, as they are easily obtainable from their equations.

§ 5. *Summary.*

By adopting Geiser's[1] method of regarding the quartic curve as the section of the enveloping cone to a cubic surface from a point on itself, we obtain analogous results in the case of the trinodal quartic curve as for the trinodal cubic surface, for a double-six projects by this method into a Steinerian complex, and hence the section of the enveloping cone of a Milne quadric by the plane of projection is a contact conic K.

[1] G. F. Geiser, *Math. Ann.* 1 (1868), 129-138.

Since the tangent plane to the cubic surface at the vertex of the enveloping cone projects into a bitangent (p) of the trinodal quartic curve, we find that this gives rise to more contact conics K than there are Milne quadrics of the trinodal cubic surface. Thus we find that there are no Milne quadrics corresponding to Jolliffe's Type II, and also that there are twice as many conics K in each of Jolliffe's Types I and V as there are Milne quadrics in the corresponding Types I and V. Conics K which are the sections of enveloping cones of Milne quadrics are lettered the same as their corresponding Milne quadrics, and all others are due to the bitangent p.

The Milne quadrics can be represented according to the number of the nodal cones they can be inscribed in as follows:—

Type	*Milne Quadrics*	*No. of nodal cones quadrics can be inscribed in*
IV.	Θ_0	0.
V.	$U_1, U_2;\ V_1, V_2;\ W_1, W_2$	1.
I.	L, M, N	2.
III.	Θ'	3.

The triads of Milne quadrics obtained are:—

Class I. LMN.

Class II. $\Theta' LU_1, \Theta' LU_2;\ \Theta' MV_1, \Theta' MV_2;\ \Theta' NW_1, \Theta' NW_2$.

Class III. $LV_1W_2;\ LV_2W_1;\ MW_1U_2, MW_2U_1;\ NU_1V_2, NU_2V_1$.

Class IV. $\Theta'\Theta'\Theta_0$.

Class V. $\Theta_0U_1U_2,\ \Theta_0V_1V_2,\ \Theta_0W_1W_2$.

We have the following classification for the conics K according to the number of nodal tangent pairs that they touch.

Type	*Conics K*	*Number of pairs of nodal tangents K touches*
IV. *II.*	Θ_0 P, Q, R	0
V.	$U_1, U_2;\ V_1, V_2;\ W_1, W_2$ $U_3, U_4;\ V_3, V_4;\ W_3, W_4$	1
I.	L, M, N L_1, M_1, N_1	2
III.	Θ'	3

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The triads of contact conics K are as follows:—

Class I. $\begin{cases} L\ M\ N. \\ LM_1N_1,\quad L_1MN_1,\quad L_1M_1N. \end{cases}$

Class IIa. $PMM_1,\ PNN_1;\ QNN_1;\ RLL_1,\ RMM_1.$

Class II. $\begin{cases} \Theta' L U_1,\ \Theta' L U_2;\ \Theta' M V_1,\ \Theta' M V_2;\ \Theta' N W_1,\ \Theta' N W_2. \\ \Theta' L_1 U_3,\ \Theta' L_1 U_4;\ \Theta' M_1 V_3,\ \Theta' M_1 V_4;\ \Theta' N_1 W_3,\ \Theta' N_1 W_4. \end{cases}$

Class III. $\begin{cases} L V_1 W_2,\ L V_2 W_1;\ M W_1 U_2,\ M W_2 U_1;\ N U_1 V_2,\ N U_2 V_1. \\ L V_3 W_3,\ L V_4 W_4;\ M W_3 U_3,\ M W_4 U_4;\ N U_3 V_3,\ N U_4 V_4. \\ L_1 V_1 W_3,\ L_1 V_2 W_4,\ L_1 V_3 W_2,\ L_1 V_4 W_1;\ M W_1 U_3,\ M_1 W_2 U_4, \\ \quad M_1 W_3 U_2,\ M W_4 U_1;\ N_1 U_1 V_3,\ N_1 U_2 V_4,\ N_1 U_3 V_2,\ N_1 U_4 V_1. \end{cases}$

Class IV. $\Theta' \Theta' \Theta_0.$ *Class V.* $\begin{cases} \Theta_0 U_1 U_2,\ \Theta_0 V_1 V_2,\ \Theta_0 W_1 W_2. \\ \Theta_0 U_3 U_4,\ \Theta_0 V_3 V_4,\ \Theta_0 W_3 W_4. \end{cases}$

Class VI. $PV_2V_3,\ PV_1V_4,\ PW_1W_3,\ PW_2W_4;\ QW_1W_4,\ QW_2W_3,$
$QU_1U_3,\ QU_2U_4;\ RU_1U_4,\ RU_2U_3,\ RV_1V_3,\ RV_2V_4.$

The triads in the first lines of Classes I-V constitute the triads of contact conics K corresponding to the total number of Milne quadric triads, and the classes in each case correspond.