Bull. Austral. Math. Soc. 16A76
Vol. 43 (1991) [115–122]

LEFT IDEALS IN THE NEAR-RING OF AFFINE TRANSFORMATIONS

Wolfgang Mutter

In this paper we determine the left ideals in the near-ring Aff(V) of all affine transformations of a vector space V. It is shown that there is a Galois correspondence between the filters of affine subspaces of V and those left ideals of Aff(V) which are not left invariant. In particular, the not left invariant finitely generated left ideals of Aff(V) are precisely the annihilators of the affine subspaces of V. A similar correspondence exists between the filters of linear subspaces of V and the left invariant left ideals of Aff(V). If V is finite-dimensional, then all left ideals of Aff(V) are finitely generated.

1. Introduction

Let V be a vector space and let Aff(V) denote the collection of all affine transformations of V. Under pointwise addition and under composition of mappings Aff(V) is a near-ring. In [**2**] Blackett showed that the set C of all constant transformations forms an ideal of Aff(V). If V is finite dimensional, then C is the only non-trivial ideal of Aff(V). Wolfson [**5**] determined all ideals of Aff(V) for an arbitrary vector space V. He observed that C is contained in all non-trivial ideals of Aff(V) and that Aff(V)/C is isomorphic to the ring Hom(V, V) of all linear transformations of V. Thus the ideals of Aff(V) are the sets $T_\nu + C$ with $T_\nu = \{f \in \mathrm{Hom}(V, V) \mid \mathrm{Range}\, f < \aleph_\nu\}$, where $\aleph_\nu$ is a cardinal number.

In this paper we investigate the structure of the left ideals of Aff(V). We use the results of Baer on the left ideals of the ring Hom(V, V) in [**1**, p.172 following], where he showed that the finitely generated left ideals of Hom(V, V) are precisely the annihilators of the linear subspaces of the vector space V. In particular, Baer established a Galois correspondence between the left ideals of Hom(V, V) and the filters of linear subspaces of V. Thus, by the second isomorphism theorem for near-rings (see for example Theorem 1.31 in [**3**]), the left invariant left ideals of Aff(V) are completely determined, since a left ideal of Aff(V) is left invariant if and only if it contains the ideal C of all constant transformations of V.

The purpose of this paper is to show that there is a similar correspondence between the left ideals of Aff(V) which are not left invariant and the affine subspaces of V, as

Received 21 February 1990

in the case of $\mathrm{Hom}(V, V)$. If V is finite dimensional, then all left ideals of $\mathrm{Aff}(V)$ are finitely generated. In this case the left ideals of $\mathrm{Aff}(V)$ which are not left invariant are precisely the annihilators of the affine subspaces of V. The left invariant left ideals of $\mathrm{Aff}(V)$ are the sets $L + C$, where L is the annihilator of a linear subspace of V.

2. Basic definitions and results

For details on near-rings and N-groups we refer the reader to [4]. According to [4] we consider right near-rings.

Definition 2.1: Let $(N, +, \cdot)$ be a near-ring. A subset L of N is called a left ideal of N provided that

1. $(L, +)$ is a normal subgroup of $(N, +)$, and
2. $m(n+i) - mn \in L$ for all $i \in L$ and $m, n \in N$.

If S is a subset of a near-ring N, let $\langle S\rangle_\ell$ denote the left ideal generated by S. In particular, $\langle n_1, \ldots, n_k\rangle_\ell$ denotes the left ideal generated by $n_1, \ldots, n_k \in N$. If a near-ring N is regarded as a N-group in the usual way, the left ideals of N are precisely the kernels of N-homomorphisms with domain N.

In general, a left ideal of a near-ring is not invariant under multiplication from the left. Therefore, we call a left ideal L of a near-ring N *left invariant*, if for all $n \in N$ and $i \in L$ the element $n \cdot i$ is in L. The left invariant left ideals of a near-ring can be characterised as follows:

Lemma 2.2. *Let N be a near-ring with constant part N_c and let L be a left ideal of N. Then L is left invariant if and only if $N_c \subseteq L$.*

Proof: If L is left invariant and n_c is in N_c, then $n_c = n_c \cdot i \in L$ for all $i \in L$. Conversely, if $N_c \subseteq L$, then for all $n \in N$ and $i \in L$ the element $n \cdot i = n \cdot i - n \cdot 0 + n \cdot 0$ is in L, since $n \cdot 0$ is in N_c. ∎

If V is a vector space and S is a subset of V, then $\mathrm{Ann}(S)$ denotes the annihilator $\{f \in \mathrm{Aff}(V) \mid f(S) = 0\}$. If p is an element of V, let $\langle p\rangle$ denote the constant transformation of V which carries all of V onto p. Any affine transformation $f \in \mathrm{Aff}(V)$ can be decomposed as $f = f - \langle f(0)\rangle + \langle f(0)\rangle$ with $f - \langle f(0)\rangle \in \mathrm{Hom}(V, V)$ and $\langle f(0)\rangle \in C$. $\mathrm{Hom}(V, V)$ is a subnear-ring of $\mathrm{Aff}(V)$ and

$$\varphi\colon \mathrm{Aff}(V) \to \mathrm{Hom}(V, V)\colon f \mapsto f - \langle f(0)\rangle$$

is a surjective near-ring homomorphism with $\ker\varphi = C$. By Lemma 2.2 and by the second isomorphism theorem for near-rings ([**3**, Theorem 1.31]) φ induces a bijective correspondence between the left invariant left ideals of $\mathrm{Aff}(V)$ and the left ideals of $\mathrm{Hom}(V, V)$ by $L \to \varphi(L)$.

A left ideal L of $\mathrm{Aff}(V)$ which is not left invariant does not contain many constant transformations, for we have

LEMMA 2.3. *If L is a not left invariant left ideal of* $\mathrm{Aff}(V)$, *then* $L \cap C = \{0\}$.

PROOF: It is easy to show that $L \cap C$ is isomorphic to a submodule of the simple $\mathrm{Hom}(V, V)$-module V. Hence, by Lemma 2.2, the assertion of the lemma is obvious. $\square$

For an affine transformation f let $Z(f)$ denote the zero-set of f, that is $Z(f) = \{p \in V \mid f(p) = 0\}$. If $Z(f)$ is not empty, then it is an affine subspace of V. Conversely, every affine subspace of a vector space is the zero-set of an affine transformation. More precisely:

LEMMA 2.4. *Let $A = p + U$ be an affine subspace of a vector space V, where U is a linear subspace of V and $p \in V$. Then there exists $f \in \mathrm{Aff}(V)$ with $Z(f) = A$. In particular, if W is a linear complement of U in V, there exists $f \in \mathrm{Aff}(V)$ with $Z(f) = A$ and $f(V) = W$.*

PROOF: By the Complementation Theorem in [**1**, p.12], there exists a linear subspace W of V with $V = U \oplus W$. If τ_{-p} denotes the translation by $-p$ and pr_W is the projection map from V onto W, then $f = pr_W \circ \tau_{-p}$ is an affine transformation of V with the required properties. $\square$

3. The not left invariant left ideals

In this section we determine the left ideals of $\mathrm{Aff}(V)$ which are not left invariant.

LEMMA 3.1. *Let L be a left ideal of* $\mathrm{Aff}(V)$ *and let $f_1, \dots, f_n$ be in L with $Z(f_1) \cap \dots \cap Z(f_n) \neq \emptyset$. If g is an affine transformation of V with $Z(g) \supseteq Z(f_1) \cap \dots \cap Z(f_n)$, then $g \in L$.*

PROOF: Since $Z(f_1) \cap \dots \cap Z(f_n)$ is not empty, there exist an element $p \in V$ and a linear subspace U of V with $p + U = Z(f_1) \cap \dots \cap Z(f_n)$. Let $\tau_p \in \mathrm{Aff}(V)$ be given by $\tau_p(x) = x + p$. Then τ_p defines an $\mathrm{Aff}(V)$-automorphism of $\mathrm{Aff}(V)$ by $h \mapsto h \circ \tau_p$. Hence

$$U = Z(f_1 \circ \tau_p) \cap \dots \cap Z(f_n \circ \tau_p)$$

and $U \subseteq Z(g \circ \tau_p)$. In particular, $f_1 \circ \tau_p, \dots, f_n \circ \tau_p$ and $g \circ \tau_p$ are linear transformations of V. Since $\mathrm{Hom}(V, V)$ is a left ideal of $\mathrm{Aff}(V)$, the left ideal $\langle f_1 \circ \tau_p, \dots, f_n \circ \tau_p \rangle_\ell$ generated by $f_1 \circ \tau_p, \dots, f_n \circ \tau_p$ is obviously the smallest left ideal of the ring $\mathrm{Hom}(V, V)$ which contains $f_1 \circ \tau_p, \dots, f_n \circ \tau_p$. Hence $g \circ \tau_p \in \langle f_1 \circ \tau_p, \dots, f_n \circ \tau_p \rangle_\ell$ by [**1**, p.173, Theorem A, and p.177, Theorem 1]. The second isomorphism theorem 1.30 for N-groups in [4] implies $g \in \langle f_1, \dots, f_n \rangle_\ell \subseteq L$. $\square$

In order to prove the next lemma, we need the following two propositions:

Proposition 3.2. *Let V be a vector space and let A_1 and A_2 be affine subspaces of V with $A_1 \cap A_2 = \emptyset$. Then there exist maximal affine subspaces M_1 and M_2 of V such that $A_1 \subseteq M_1$, $A_2 \subseteq M_2$ and $M_1 \cap M_2 = \emptyset$.*

Proof: Let p_1, p_2 be in V and let U_1, U_2 be linear subspaces of V with $A_1 = p_1 + U_1$ and $A_2 = p_2 + U_2$. Since $A_1 \cap A_2 = \emptyset$, $p_1 - p_2$ is not in $U_1 + U_2$. By the Complementation Theorem in [**1**, p.12], there exists a linear subspace U of V such that V can be decomposed as

$$V = \operatorname{span}(p_1 - p_2) \oplus (U_1 + U_2) \oplus U.$$

Then $M_1 = p_1 + (U_1 + U_2 + U)$ and $M_2 = p_2 + (U_1 + U_2 + U)$ are maximal affine subspaces of V with the required properties. □

Proposition 3.3. *If L is a left ideal of $\operatorname{Aff}(V)$ and $f \in L$ with $Z(f) = \emptyset$, then L is left invariant.*

Proof: $f(V)$ is an affine subspace of V. Thus by Lemma 2.4 there exists $g \in \operatorname{Aff}(V)$ with $Z(g) = f(V)$. Furthermore the constant transformation

$$\langle -g(0) \rangle = g \circ f - g \circ \langle 0 \rangle$$

is in L. Moreover, $g(0)$ is not zero, since 0 is not in $f(V)$. Hence the assertion of the lemma follows by Lemmas 2.2 and 2.3. □

Lemma 3.4. *Let L be a left ideal of $\operatorname{Aff}(V)$ and suppose there are $f, g \in L$ with $Z(f) \cap Z(g) = \emptyset$. Then L is left invariant.*

Proof: By Proposition 3.3 it suffices to show that there exists an affine transformation $h \in L$ with $Z(h) = \emptyset$. Therefore we may assume that $Z(f)$ and $Z(g)$ are not empty. By Proposition 3.2 there exist maximal subspaces M_1 and M_2 of V such that $Z(f) \subseteq M_1$, $Z(g) \subseteq M_2$ and $M_1 \cap M_2 = \emptyset$. By Lemma 2.4 there exist nonzero elements p_1 and p_2 in V and transformations $f_1, f_2 \in \operatorname{Aff}(V)$ with $M_1 = Z(f_1)$, $M_2 = Z(f_2)$, $f_1(V) = \operatorname{span}(p_1)$ and $f_2(V) = \operatorname{span}(p_2)$. Lemma 3.1 implies $f_1, f_2 \in L$, since $Z(f_1) \supseteq Z(f)$ and $Z(f_2) \supseteq Z(g)$. Now we distinguish two cases:

Suppose $\dim V > 1$. Then there exist nonzero elements $q_1, q_2 \in V$ with $\operatorname{span}(q_1) \cap \operatorname{span}(q_2) = \{0\}$. Let h_1 and h_2 be invertible linear transformations of V with $h_1(p_1) = q_1$ and $h_2(p_2) = q_2$. Then $h_1 \circ f_1(V) = \operatorname{span}(q_1)$ and $h_2 \circ f_2(V) = \operatorname{span}(q_2)$. Furthermore the transformation $h = h_1 \circ f_1 - h_2 \circ f_2$ is in L. If $x \in V$, then

$$h(x) = 0 \Leftrightarrow h_1 \circ f_1(x) = h_2 \circ f_2(x) \Leftrightarrow h_1 \circ f_1(x) = h_2 \circ f_2(x) = 0 \Leftrightarrow f_1(x) = f_2(x) = 0.$$

Hence $Z(h) = \emptyset$, since $Z(f_1) \cap Z(f_2) = \emptyset$. This proves the assertion of the lemma for $\dim V > 1$.

https://doi.org/10.1017/S0004972700028823 Published online by Cambridge University Press

If $\dim V = 1$, then there exist distinct elements q_1 and q_2 in V with $Z(f_1) = \{q_1\}$ and $Z(f_2) = \{q_2\}$. An easy check shows that in this case f_1 and f_2 are injective. Hence there exist affine transformations h_1 and h_2 with $h_1 \circ f_1 = h_2 \circ f_2 = id$. The constant transformation

$$h = \langle g_2(0) - g_1(0) \rangle = (g_1 \circ f_1 - g_1 \circ \langle 0 \rangle) - (g_2 \circ f_2 - g_2 \circ \langle 0 \rangle)$$

is in L and is not zero, since $h_1(0) = h_1(f_1(q_1)) = q_1$ and $h_2(0) = h_2(f_2(q_2)) = q_2$. This completes the proof of the lemma. ∎

Now we are in a position to establish a bijective correspondence between the left ideals of $\mathrm{Aff}(V)$, which are not left invariant, and the filters of affine subspaces of V. First we need

DEFINITION 3.5: A nonempty family $\mathcal{F}$ of affine subspaces of a vector space V is called an $\mathcal{A}$-filter on V provided that

1. $\emptyset \notin \mathcal{F}$,
2. if $A_1, A_2 \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$, and
3. if $A \in \mathcal{F}$ and A' is an affine subspace of V with $A' \supseteq A$, then $A' \in \mathcal{F}$.

For example, if A is an affine subspace of V, the family $\mathcal{F}_A$ of all affine subspaces of V which contain A is an $\mathcal{A}$-filter on V. Obviously $\mathcal{F}_A$ is the smallest $\mathcal{A}$-filter containing A, hence we call $\mathcal{F}_A$ the *$\mathcal{A}$-filter generated by A*.

THEOREM 3.6. *Let V be a vector space.*

1. *If L is a left ideal of $\mathrm{Aff}(V)$ which is not left invariant, then*

$$Z[L] = \{Z(f) \mid f \in L\}$$

is an $\mathcal{A}$-filter on V.

2. *If $\mathcal{F}$ is an $\mathcal{A}$-filter on V, then*

$$Z \leftarrow [\mathcal{F}] = \bigcup \{\mathrm{Ann}\,(A) \mid A \in \mathcal{F}\}$$

is a not left invariant left ideal of $\mathrm{Aff}(V)$.

Moreover, the mapping Z is one-one between the set of all not left invariant left ideals of $\mathrm{Aff}(V)$ and the $\mathcal{A}$-filters on V.

PROOF: 1. Let L be a left ideal of $\mathrm{Aff}(V)$ which is not left invariant. We have to show that $Z[L]$ satisfies the properties 1 – 3 of Definition 3.5. Proposition 3.3 implies $\emptyset \notin Z[L]$. Suppose now that $A_1, A_2 \in Z[L]$. If $A_1 \cap A_2 = \emptyset$, then by Lemma 3.4 L is left invariant, which contradicts the hypothesis. If $A_1 \cap A_2 \neq \emptyset$, then according to Lemma 2.4 there exists $f \in \mathrm{Aff}(V)$ with $Z(f) = A_1 \cap A_2$. Lemma 3.1 implies $f \in L$, hence $A_1 \cap A_2 \in Z[L]$. Finally, let $A \in Z[L]$ and let A' be an affine subspace of V with $A' \supseteq A$. By Lemma 2.4 there exists $f' \in \mathrm{Aff}(V)$ with $A' = Z(f')$. Since $A \neq \emptyset$ by Lemma 3.4, Lemma 3.1 implies $f' \in L$. Therefore A' is in $Z[L]$. Altogether, we have shown that $Z[L]$ is an $\mathcal{A}$-filter on V.

2. The proof of the second assertion of the theorem is straightforward and therefore omitted.

3. In order to verify that the mapping Z is one-one, we prove that $Z^{\leftarrow}$ is the inverse mapping of Z. If $\mathcal{F}$ is an $\mathcal{A}$-filter on V then clearly $Z[Z^{\leftarrow}[\mathcal{F}]] = \mathcal{F}$. Furthermore it is obvious that any left ideal L of $\mathrm{Aff}(V)$ satisfies $L \subseteq Z^{\leftarrow}[Z[L]]$. If, in addition, L is not left invariant, we have seen that $Z[L]$ is an $\mathcal{A}$-filter on V. Therefore, if f is an affine transformation with $Z(f) \in Z[L]$, then $Z(f) \neq \emptyset$, and hence $f \in L$ by Lemma 3.1. This proves the converse inclusion $Z^{\leftarrow}[Z[L]] \subseteq L$. □

As a consequence of Theorem 3.6 we note that for an affine transformation f with nonempty zero-set $Z(f)$ the left ideal $\langle f \rangle_\ell$ generated by f and the annihilator $\mathrm{Ann}(Z(f))$ of $Z(f)$ coincide. Furthermore, we get the following

Corollary 3.7. *The not invariant left invariant ideals L of* $\mathrm{Aff}(V)$ *are precisely the sets*

$$Z^{\leftarrow}[\mathcal{F}] = \bigcup \{\mathrm{Ann}(A) \mid A \in \mathcal{F}\}$$

where $\mathcal{F}$ is a filter of affine subspaces of V.

4. The finitely generated left ideals

Now we are in a position to determine the finitely generated left ideals of $\mathrm{Aff}(V)$.

Theorem 4.1. *Let V be a vector space.*

1. *The finitely generated left invariant left ideals of* $\mathrm{Aff}(V)$ *are precisely the sets* $\mathrm{Ann}(U) + C$, *where U is a linear subspace of V.*
2. *The finitely generated left ideals of* $\mathrm{Aff}(V)$*, which are not left invariant, are precisely the annihilators* $\mathrm{Ann}(A)$*, where A is an affine subspace of V.*

Proof: The first assertion of the theorem follows by Theorem A in [**1**, p.173], Theorem 1 in [**1**, p.177], the second isomorphism theorem for near-rings and Lemma 2.2. To show 2, suppose first that $L = \langle f_1, \ldots, f_n \rangle_\ell$ is a finitely generated left ideal of $\mathrm{Aff}(V)$ which is not left invariant. By Theorem 3.6 the family $Z[L]$ is an $\mathcal{A}$-filter on V. Hence there exists $f \in L$ with $Z(f) = Z(f_1) \cap \cdots \cap Z(f_n)$. Moreover, $Z(f)$ is not empty. By the remarks following Theorem 3.6 the left ideal $\langle f \rangle_\ell$ generated by f agrees with the annihilator $\mathrm{Ann}(Z(f))$. Therefore $\mathrm{Ann}(Z(f)) \subseteq L$. Since $\mathrm{Ann}(Z(f))$ is a left ideal of $\mathrm{Aff}(V)$ containing $f_1, \ldots, f_n$, it follows that $\mathrm{Ann}(Z(f)) = L$.

Conversely, if A is an affine subspace of V, by Lemma 2.4 there exists $f \in \mathrm{Aff}(V)$ with $A = Z(f)$. The remarks following Theorem 3.6 imply $\mathrm{Ann}(A) = \langle f \rangle_\ell$, hence $\mathrm{Ann}(A)$ is a finitely generated and obviously not left invariant left ideal of $\mathrm{Aff}(V)$. ∎

For the proof of the next theorem it will be convenient to have

LEMMA 4.2. *Let V be a vector space. Then the following statements are equivalent:*

1. $\dim V < \infty$.
2. *Every $\mathcal{A}$-filter on V is generated by an affine subspace of V.*

PROOF: Let $\dim V < \infty$ and let $\mathcal{F}$ be an $\mathcal{A}$-filter on V. Let $A \in \mathcal{F}$ such that $\dim A \leqslant \dim A'$ for all $A' \in \mathcal{F}$. If $A' \in \mathcal{F}$, then $A \cap A' \in \mathcal{F}$ and so $\dim A \leqslant \dim(A \cap A')$. This implies $A \subseteq A'$. Hence $\mathcal{F}$ is contained in the $\mathcal{A}$-filter $\mathcal{F}_A$ generated by A. Since $A \in \mathcal{F}$, it follows that $\mathcal{F} = \mathcal{F}_A$.

To show the converse, suppose that $\dim V = \infty$. Then the family of all finite dimensional linear subspaces of V is an $\mathcal{A}$-filter on V which is not generated by an affine subspace of V. ∎

THEOREM 4.3. *Let V be a vector space. Then the following statements are equivalent:*

1. $\dim V < \infty$.
2. *All left ideals of* $\mathrm{Aff}(V)$ *are finitely generated.*

PROOF: Let V be a finite dimensional vector space and let L be a left ideal of $\mathrm{Aff}(V)$. If L is not left invariant, then according to Corollary 3.7 and Lemma 4.2 there exists an affine subspace A of V with $L = \bigcup\{\mathrm{Ann}(A') \mid A' \in \mathcal{F}_A\} = \mathrm{Ann}(A)$. Therefore Theorem 4.1 implies that L is finitely generated.

If L is a left invariant left ideal of $\mathrm{Aff}(V)$, then L can be decomposed as $L = L_0 + C$, where L_0 is a left ideal of $\mathrm{Hom}(V, V)$. In particular, L_0 is a left ideal of $\mathrm{Aff}(V)$ which is not left invariant. Hence L_0 is finitely generated. Furthermore, Lemma 2.3 implies that C is a finitely generated left ideal of $\mathrm{Aff}(V)$. Therefore L is finitely generated.

If conversely all left ideals of $\mathrm{Aff}(V)$ are finitely generated, then $\dim V < \infty$ by Theorem 4.1, Lemma 4.2 and Corollary 3.7. ∎

In particular, Theorem 4.1 and Theorem 4.3 show that for a finite dimensional vector space V there is a Galois correspondence between the left invariant left ideals of $\mathrm{Aff}(V)$ and the linear subspaces of V and a similar correspondence between the not left invariant left ideals of $\mathrm{Aff}(V)$ and the affine subspaces of V.

References

[1] R. Baer, *Linear Algebra and Projective Geometry* (Academic Press Inc., New York, 1952).

[2] D.W. Blackett, 'The near-ring of affine transformations', *Proc. Amer. Math. Soc.* **7** (1956), 517–519.

[3] J.D.P. Meldrum, *Near-rings and their links with groups: Research Notes in Mathematics* 134 (Pitman, Boston, London, Melbourne, 1985).

[4] G. Pilz, *Near-rings: Mathematics Studies* 23 (North Holland, New York, 1983).

[5] K.G. Wolfson, 'Two-sided ideals of the affine near-ring', *Amer. Math. Monthly* **65** (1958), 29–30.

Mathematisches Institut
Universität Erlangen-Nürnberg
Bismarckstr. $1\frac{1}{2}$
D-8520 Erlangen,
Federal Republic of Germany