




Some remarks on approximation in several complex variables

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Abstract. In Gauthier, Manolaki, and Nestoridis (2021, *Advances in Mathematics* 381, 107649), in order to correct a false Mergelyan-type statement given in Gamelin and Garnett (1969, *Transactions of the American Mathematical Society* 143, 187–200) on uniform approximation on compact sets K in \mathbb{C}^d , the authors introduced a natural function algebra $A_D(K)$ which is smaller than the classical one $A(K)$. In the present paper, we investigate when these two algebras coincide and compare them with the classes of all plausibly approximable functions by polynomials or rational functions or functions holomorphic on open sets containing the compact set K . Finally, we introduce a notion of O -hull of K and strengthen known results.

1 Introduction

Let K be a compact subset of \mathbb{C}^d , $d \geq 1$. The classical function algebra $A(K)$ is defined to be $C(K) \cap \mathcal{O}(K^\circ)$, where $\mathcal{O}(K^\circ)$ denotes the set of all holomorphic functions in the interior of K in \mathbb{C}^d . Note that if $K^\circ = \emptyset$, then $A(K) = C(K)$. Denote by $P(K)$ the class of functions on K which are uniform limits of polynomials. In complex dimension $d = 1$, Mergelyan [8] (see also [3, p. 97]) gave the following complete characterization of compact sets on which polynomial approximation is possible.

Theorem 1.1 (Mergelyan theorem) *Let K be a compact subset of \mathbb{C} . Then $P(K) = A(K)$ if and only if $\mathbb{C} \setminus K$ is connected.*

In [2], we constructed a counterexample to a Mergelyan-type statement given in [4] concerning uniform approximation on compact sets K in \mathbb{C}^d , $d \geq 1$. In order to correct the statement in [4], we introduced a natural function algebra $A_D(K)$, which is contained in the classical one $A(K)$. The algebra $A_D(K)$ is the algebra of all functions $f \in A(K)$ which are holomorphic on every analytic disk contained in K , even meeting the boundary ∂K of K . More precisely, a function $f : K \rightarrow \mathbb{C}$ belongs to $A_D(K)$ if it

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is continuous on K and for every open disk $D \subseteq \mathbb{C}$ and every injective holomorphic mapping $\phi : D \rightarrow K$ the composition $f \circ \phi$ is holomorphic on D .

In this paper, in addition to the family $P(K)$ of uniform limits on K of polynomials, we investigate the family $R(K)$ of uniform limits on K of rational functions which are holomorphic on a neighborhood of K . We also denote by $\overline{O}(K)$ the set of all functions $f : K \rightarrow \mathbb{C}$ such that there exists a sequence of open sets $(V_n)_{n=1}^{\infty}$ in \mathbb{C}^d with $K = \bigcap_{n=1}^{\infty} V_n$ and functions $f_n : V_n \rightarrow \mathbb{C}$ holomorphic on V_n such that $\sup_{z \in K} |f_n(z) - f(z)| \rightarrow 0$ as n goes to infinity. It is easy to see that $P(K) \subseteq R(K) \subseteq \overline{O}(K) \subseteq A_D(K) \subseteq A(K)$. Moreover, when $d = 1$, we have that $A_D(K) = A(K)$. One of the goals of this paper is to investigate when these classes coincide in higher dimensions, where the situation is far from being understood.

However, there are some known cases. For example, it is shown in [2] that if K_1, K_2, \dots, K_d are planar compact sets and $K = K_1 \times \dots \times K_d$, then a function $f : K \rightarrow \mathbb{C}$ belongs to $A_D(K)$ if and only if the slice functions belong to $A_D(K_i)$ for all $i = 1, \dots, d$; that is, for every $i_0 \in \{1, \dots, d\}$ and every choice of points $w_i \in K_i$, $i \in \{1, \dots, d\} \setminus \{i_0\}$, the one-variable function, $K_{i_0} \ni z \mapsto f(z_1, \dots, z_d)$ with $z_{i_0} = z$ and $z_i = w_i$ for all $i \in \{1, \dots, d\} \setminus \{i_0\}$, belongs to $A_D(K_{i_0})$. See also [1].

A compact set $K \subseteq \mathbb{C}^d$ is called regular closed if every point in the boundary of K is the limit of a sequence of points in the interior of K . Thus, if K is regular closed and $K \neq \emptyset$, then necessarily $K^\circ \neq \emptyset$. In [2], it was proved that, for nonvoid regular closed planar compact sets K_1, \dots, K_d and $K = K_1 \times \dots \times K_d$, we have $A_D(K) = A(K)$. In the second section of this note, we shall improve this result and characterize the equality of the two algebras when K is a product of planar compact sets (see Theorem 2.1). The third section includes the proof of this result and some corollaries. Sections 4 and 5 are devoted to investigating the equality of the two algebras on compact sets that are not necessarily product domains. In particular, polynomial, rational, and holomorphic approximation are considered.

The sixth section of this note is devoted to the study of functions plausibly approximable by polynomials, rational functions, and holomorphic functions. More precisely, denoting by \widehat{K} the polynomially convex hull of K , we show that a natural algebra containing $P(K)$ is the algebra $A_D(\widehat{K})$; that is, the functions $f : K \rightarrow \mathbb{C}$ which admit a continuous extension \widehat{f} to \widehat{K} that belongs to $A_D(\widehat{K})$ are plausibly approximable, in the sense that every function that can be uniformly approximated on K by polynomials necessarily belongs to $A_D(\widehat{K})$. If we replace polynomials by rational functions which are holomorphic on K , then the functions $f : K \rightarrow \mathbb{C}$ which admit an extension to the rational convex hull \widehat{K}^r of K that belongs to $A_D(\widehat{K}^r)$ are plausibly approximable.

Polynomially convex hulls and rationally convex hulls have played an extremely important role in approximation by polynomials and rational functions respectively. In order to replace polynomials and rational functions by functions holomorphic on some neighborhood of K , it is natural to seek an appropriate hull associated with such holomorphic approximation. There is no consensus in the literature as to the best way of defining such a hull for a compact set K and we shall introduce a new hull for this purpose, which we call an O -hull of K . We show that the functions $h : K \rightarrow \mathbb{C}$ which admit an extension to an O -hull X and belong to $A_D(X)$ are plausibly approximable.

One can always choose as X the set K itself, but in general there are more choices of X , and, in some cases, there is no maximal O -hull for K .

In Section 7, we strengthen a density result from [2], for compact sets of the form $K = K_1 \times \cdots \times K_d$. Finally, Section 8 contains some concluding remarks and problems for future research.

2 Main results

One of our main goals in this note is to prove the following theorem, which establishes conditions for the equality of the algebras $A_D(K)$ and $A(K)$ when K is a product of nonvoid compact planar sets. We recall that a planar compact set F is called regular closed if $\overline{F^\circ} = F$. We also recall that if $K = K_1 \times \cdots \times K_d$, where each K_i is a compact set in \mathbb{C} , then a continuous function f belongs to $A_D(K)$ if and only if the slice functions belong to $A(K_i)$ (see [2, Proposition 5.3]).

Theorem 2.1 *Let K_1, \dots, K_d be nonvoid compact planar sets and $K = K_1 \times \cdots \times K_d$. Then, $A_D(K) = A(K)$ if and only if one of the two following assertions holds:*

- (a) $K_1^\circ = K_2^\circ = \cdots = K_d^\circ = \emptyset$,
- (b) $K_i^\circ \neq \emptyset$ for all $i \in \{1, \dots, d\}$ and all K_i are regular closed sets for $i \in \{1, \dots, d\}$.

In Section 3, we prove this result and provide some corollaries. Our other main results, that will be discussed in the subsequent sections, are Theorem 4.1, Example 5.1, Proposition 6.4, and Theorem 7.3.

3 Proof and corollaries of Theorem 2.1

Proof of Theorem 2.1 We distinguish four cases which cover the different scenarios of Theorem 2.1.

- (i) $K_1^\circ = K_2^\circ = \cdots = K_d^\circ = \emptyset$.
- (ii) There exists $i_0, i_1 \in \{1, 2, \dots, d\}$ such that $K_{i_0}^\circ = \emptyset$ and $K_{i_1}^\circ \neq \emptyset$.
- (iii) $K_i^\circ \neq \emptyset$, for all $i \in \{1, 2, \dots, d\}$, and there is an $i_0 \in \{1, 2, \dots, d\}$ such that K_{i_0} is not regular closed.
- (iv) $K_i^\circ \neq \emptyset$, for all $i \in \{1, 2, \dots, d\}$, and K_i is regular closed for all $i \in \{1, 2, \dots, d\}$.

We examine each case separately. For simplicity, we assume that if the interiors of K_{i_1}, \dots, K_{i_m} are not empty, then zero is in $(K_{i_1} \times \cdots \times K_{i_m})^\circ$.

Case (i). In this case, $A(K_i) = A_D(K_i) = C(K_i)$. It follows that for every $f \in C(K)$, the slice functions belong to $A_D(K_i)$. Hence, $f \in A_D(K)$. Therefore, $C(K) \subseteq A_D(K)$. On the other hand, $K^\circ = \emptyset$, which implies that $A(K) = C(K)$. Thus, in Case (i), we have that $A_D(K) \subseteq A(K) = C(K) \subseteq A_D(K)$.

Case (ii). In this case, $K^\circ = \emptyset$, so $A(K) = C(K)$. We shall show that $A_D(K) \neq C(K)$. Indeed, let $f(z_1, \dots, z_d) = \overline{z_{i_1}}$. Then, $f \in C(K) = A(K)$, but the slice function corresponding to the coordinate i_1 is not holomorphic in $K_{i_1}^\circ$ and does not belong to $A(K_{i_1}) = A_D(K_{i_1})$. It follows that $f \notin A_D(K)$. Thus, $A_D(K) \neq C(K) = A(K)$ in this case.

Case (iii). Since K_{i_0} is not regular closed, there exists $w \in K_{i_0} \setminus \overline{K_{i_0}^\circ}$. By the well-known Urysohn extension lemma, there exists a function $\phi \in C(K_{i_0})$ such that $\phi \equiv 0$ on $K_{i_0}^\circ$ and $\phi(w) = 1$. Since $d > 1$, there exists $i_1 \in \{1, 2, \dots, d\} \setminus \{i_0\}$. We consider

the function $f(z_1, \dots, z_d) = \phi(z_{i_0}) \cdot \overline{z_{i_1}}$. Then, $f \in C(K)$. Moreover, if $(z_1, \dots, z_d) \in K^\circ$, then $z_{i_0} \in K_{i_0}^\circ$, and thus $f(z_1, \dots, z_d) = \phi(z_{i_0}) \cdot \overline{z_{i_1}} = 0$ on K° . It follows that $f \in \mathcal{O}(K^\circ)$. Hence, $f \in C(K) \cap \mathcal{O}(K^\circ) = A(K)$. Now, if $z_{i_0} = w$, the slice function corresponding to the coordinate i_1 is $\phi(z_{i_0})\overline{z_{i_1}} = \phi(w)\overline{z_{i_1}} = \overline{z_{i_1}}$ and is not holomorphic in $K_{i_1}^\circ$, and hence it does not belong to $A_D(K_{i_1}) = A(K_{i_1})$. It follows that $f \notin A_D(K)$. Therefore, $A_D(K) \neq A(K)$ in this case.

Case (iv). In this case, we know from [2] that $A_D(K) = A(K)$.

Since Cases (i)–(iv) cover the general case, the proof of Theorem 2.1 is complete. ■

In [2], it is also proved that, if for the planar compact sets $K_i, i = 1, \dots, d$, we have $A_D(K_i) = \overline{\mathcal{O}}(K_i)$ for all $i = 1, \dots, d$, then for the Cartesian product $K = K_1 \times \dots \times K_d$, we also have that $A_D(K) = \overline{\mathcal{O}}(K)$. One can easily see that the converse also holds. Combining results from [2] with Theorem 2.1, we obtain the following corollaries.

Corollary 3.1 *Let K_1, \dots, K_d be nonvoid compact planar sets and $K = K_1 \times \dots \times K_d$. Then, polynomials are uniformly dense in $A_D(K)$ if and only if $\mathbb{C} \setminus K_i$ are connected for all $i \in \{1, 2, \dots, d\}$.*

Proof For $d = 1$, $A(K_1) = A_D(K_1)$, so this is just Mergelyan’s theorem (see Theorem 1.1).

Assume now that $d \geq 1$. If polynomials are uniformly dense in $A_D(K)$, it follows easily that polynomials are uniformly dense in $A_D(K_i)$ for $i = 1, \dots, d$. Therefore, according to the previous case, $\mathbb{C} \setminus K_i$ is connected. Suppose now that all $\mathbb{C} \setminus K_i$ are connected for $i = 1, \dots, d$. Mergelyan’s theorem (Theorem 1.1) implies that polynomials are dense in each $A(K_i) = A_D(K_i)$. Hence, $\overline{\mathcal{O}}(K_i) = A_D(K_i)$ for all $i = 1, \dots, d$. According to [2, Theorem 4.1], it follows that $\overline{\mathcal{O}}(K) = A_D(K)$.

Since $\mathbb{C} \setminus K_i$ are connected, according to [2, Theorem 4.6], it follows that polynomials are uniformly dense on $\overline{\mathcal{O}}(K)$. Thus, polynomials are uniformly dense in $A_D(K)$. See also [1, 6, 7]. ■

Corollary 3.2 *Let K be a countable compact set in \mathbb{C}^d . Then, $C(K) = P(K)$.*

Proof The projections of K are denumerable compact planar sets. Let F be their product. Then, $A(F) = P(F)$, $A(F) = C(F)$, and $A(K) = C(K)$. Now, if f is continuous on K , then, by Tietze’s theorem, it has a continuous extension h on F and h can be uniformly approximated on F (and on K) by polynomials. ■

Corollary 3.3 *Let K_1, \dots, K_d be nonvoid compact planar sets and $K = K_1 \times \dots \times K_d$. Then, polynomials are uniformly dense in $A(K)$ if and only if $\mathbb{C} \setminus K_i$ are connected for all $i \in \{1, 2, \dots, d\}$ and one of the conditions (a) or (b) of Theorem 2.1 holds.*

Proof Denote by $P(K)$ the set of uniform limits of polynomials on K . Then, we have

$$P(K) \subseteq \overline{\mathcal{O}}(K) \subseteq A_D(K) \subseteq A(K).$$

Thus, $P(K) = A(K)$ is equivalent to $P(K) = A_D(K)$ and $A_D(K) = A(K)$. A combination of Theorem 2.1 and Corollary 3.1 completes the proof. ■

Corollary 3.4 *Let K_1, \dots, K_d be nonvoid compact planar sets and $K = K_1 \times \dots \times K_d$. Then, $A(K) = \overline{O}(K)$ if and only if $A(K_i) = \overline{O}(K_i)$, for all $i = 1, 2, \dots, d$, and one of the conditions (a) or (b) of Theorem 2.1 holds.*

Proof Since $\overline{O}(K) \subseteq A_D(K) \subseteq A(K)$, it follows that $\overline{O}(K) = A(K)$ is equivalent to $\overline{O}(K) = A_D(K)$ and $A_D(K) = A(K)$. According to [2], $\overline{O}(K) = A_D(K)$ is equivalent to $\overline{O}(K_i) = A_D(K_i)$ for all $i = 1, \dots, d$. This combined with Theorem 2.1 completes the proof. ■

Remark 3.5 Corollary 3.4 is in contradiction with [4, Corollary 9.2], which implies the following: if K_1, \dots, K_d are nonvoid planar compact sets, $K = K_1 \times \dots \times K_d$ and $A(K_i) = \overline{O}(K_i)$ for all $i = 1, \dots, d$, then $A(K) = \overline{O}(K)$. According to Corollary 3.4, it suffices to take $d = 2$, K_1 a nonempty set with empty interior, and K_2 a set with nonempty interior with $A(K_1) = \overline{O}(K_1)$ and $A(K_2) = \overline{O}(K_2)$ to obtain a contradiction. For instance, take $K_1 = \{0\}$ and $K_2 = \{w \in \mathbb{C} : |w| \leq 1\}$. This was the counterexample given in [2], which led us to introduce the new natural function algebra $A_D(K)$.

4 The algebra $A_D(K)$ on general compact sets

We continue by studying compact sets which are not necessarily product domains. We start with a result which provides a simple necessary condition to ensure that the algebras $A_D(K)$ and $A(K)$ coincide. This result was stated as a remark in the last section of [2].

Theorem 4.1 *Let $K \subseteq \mathbb{C}^d$ be a compact set. Then, if $A_D(K) = A(K)$, we have that $K \setminus \overline{K}^\circ$ contains no analytic disk.*

Proof We shall prove the result by showing that if $K \setminus \overline{K}^\circ$ contains an analytic disk, then $A_D(K) \neq A(K)$. Since there is a holomorphic embedding of the open disk into $K \setminus \overline{K}^\circ$, it is easy to see that there is also a holomorphic embedding φ of the closed disk \overline{D} into $K \setminus \overline{K}^\circ$. Consider g any function in $A(K^\circ)$. Then, since $K \setminus \overline{K}^\circ$ has empty interior, any continuous extension of g belongs to the algebra $A(K)$. Consider the mapping $h : \varphi(\overline{D}) \rightarrow \mathbb{C}$ given by $h(z) = \overline{\varphi^{-1}(z)}$. Since $\varphi(\overline{D})$ and \overline{K}° are disjoint compact sets, we can find a continuous mapping f on K which coincides with g on the set \overline{K}° and coincides with h on \overline{D} . Then, the function f belongs to $A(K)$, but does not belong to $A_D(K)$. Thus, the algebras $A(K)$ and $A_D(K)$ must be different. ■

Corollary 4.2 *Let $K \subseteq \mathbb{C}^d$ be a compact set. If $K \setminus \overline{K}^\circ$ contains an analytic disk, then $P(K) \subseteq R(K) \subseteq \overline{O}(K) \subseteq A_D(K) \subsetneq A(K)$. Thus, $P(K)$, $R(K)$, $\overline{O}(K)$, and $A_D(K)$ are different from $A(K)$.*

5 A counterexample to the converse statement of Theorem 4.1

It is natural to ask whether the converse statement of Theorem 4.1 holds; that is, if $A_D(K) \neq A(K)$, then does $K \setminus \overline{K^\circ}$ always contain an analytic disk? As the following example demonstrates, the answer is negative.

Before showing the example, we recall that a compact set K is said to be regular closed if $K = \overline{K^\circ}$.

Example 5.1 There is a regular closed compact set $K \subseteq \mathbb{C}^2$ such that K° is connected and nonempty, and

$$A_D(K) \not\subseteq A(K).$$

We remark that if we do not require that K° be connected and nonempty, it is easier to give such an example.

Details of the example:

Let \overline{D} be the closed unit disk in \mathbb{C} and $K_0 = \{(z, w) \in \mathbb{C}^2 : z \in \overline{D}, w = 0\}$. We will consider a sequence P_n , $n = 1, 2, \dots$, of polydomains, whose closures \overline{P}_n are disjoint from K_0 , such that $P_n \cap P_m \neq \emptyset$ if and only if $|n - m| \leq 1$. We will also arrange that, for every $w \in K_0$, there is a sequence $w_n \in P_n$ converging to w , and conversely, if $w_n \in P_n$ for all n and a subsequence of w_n converges to a point s , then $s \in K_0$. Set

$$(5.1) \quad K = K_0 \cup \bigcup_{n=1}^{\infty} \overline{P}_n.$$

Then, $K^\circ = \cup_n P_n$ is connected, $K = \overline{K^\circ}$, and the disk K_0 is contained in the boundary ∂K . We shall choose the polydomains P_n more carefully momentarily. For their construction, we shall make use of the following result (for its proof, see [5]).

Lemma 5.2 Let φ be continuously differentiable, and let ε be positive and continuous on $[0, 1)$. Then, there is a function ϕ holomorphic in the unit disk, such that

$$\max\{|\varphi(t) - \phi(t)|, |\varphi'(t) - \phi'(t)|\} < \varepsilon(t), \quad \text{for } 0 \leq t < 1.$$

We note that, from [5, Theorem 1.3], we also have this result for $I = (-1, +1)$.

Let $z : [0, 1) \rightarrow D$ be an analytic curve whose “end” (cluster set) is \overline{D} and $z'(t) \neq 0$ for each $t \in [0, 1)$. Moreover, let $\varepsilon : [0, 1) \rightarrow (0, 1]$ be a positive continuous function such that $\varepsilon(t) \rightarrow 0$ as $t \rightarrow 1$. By Lemma 5.2, there is a nonconstant holomorphic function $\phi : D \rightarrow \mathbb{C}$, in the unit disk D , such that

$$\max\{|z(t) - \phi(t)|, |z'(t) - \phi'(t)|\} < \min\{\varepsilon(t), |z'(t)|\} \quad \text{for } 0 \leq t < 1.$$

Since $z'(t) \neq 0$ for all $0 \leq t < 1$, it follows that $\phi'(t) \neq 0$ for all $0 \leq t < 1$. Invoking Lemma 5.2 again, there is a holomorphic function h in the unit disk D such that

$$(5.2) \quad |h(t) - \overline{\phi(t)}| < \varepsilon(t), \quad 0 \leq t < 1.$$

We may choose a sequence $0 < t_1 < t_2 < \dots < t_n < \dots$, $t_n \rightarrow 1$, and positive numbers r_n , such that the open disks $D_n = D(t_n, r_n)$ with centers t_n and radii r_n form a chain in

D (that is, $D_n \cap D_m \neq \emptyset$ if and only if $|n - m| \leq 1$) and the function $\phi(\zeta)$ is invertible in a neighborhood of each $\overline{D_n} \cup \overline{D_{n+1}}$. Moreover, since $h - \bar{\phi}$ is continuous on D and (5.2) holds, we may choose the t_n and r_n such that

$$(5.3) \quad |h(\zeta) - \overline{\phi(\zeta)}| < \varepsilon(t_n) \quad \text{for all } \zeta \in D(t_n, r_n).$$

For $n = 1, 2, \dots$, consider the open disks $W_n = D(2^{-n}, 2^{-n-1})$ and the polydomains $P_n = V_n \times W_n$, where V_n is the Jordan domain $V_n = \phi(D_n) := \{z \in \mathbb{C} : z = \phi(\zeta), \zeta \in D_n\}$. For these P_n , the set K defined by (5.1) has the required properties.

We now define a function $f \in A(K)$, by setting $f(z, w) = \bar{z}$, for $(z, w) \in K_0$ and $f(z, w) = h(\zeta_n(z))$, for $(z, w) \in \overline{P_n}$, where $\zeta_n(z)$ is the inverse function of the restriction of the function $\phi(\zeta)$ to $\overline{D_n}$. The function f is well defined and continuous on $K \setminus K^\circ$ because if $z \in \overline{D_n} \cap \overline{D_{n+1}}$, then $\zeta_n(z) = \zeta_{n+1}(z)$. It is easy to see that f is holomorphic on K° .

There remains to show that f is continuous at points of K° . Since the restriction of f to K° is continuous, it is sufficient to show that if (z_k, w_k) is a sequence in $K \setminus K^\circ$ converging to a point $(z, 0) \in K^\circ$, then $f(z_k, w_k) \rightarrow f(z, 0) = \bar{z}$. We may assume that $(z_k, w_k) \in P_{n(k)}$, $n(k) \rightarrow \infty$. Moreover, if $(z_k, w_k) \in \overline{P_{n(k)}}$, then $z_k \in \phi(\overline{D_{n(k)}})$, and so there is $y_k \in \overline{D_{n(k)}}$ such that $z_k = \phi(y_k)$. Thus, using (5.3), we conclude that

$$|f(z_k, w_k) - \bar{z}| = |h(\zeta_n(z_k)) - \overline{\phi(y_k)}| = |h(y_k) - \overline{\phi(y_k)}| \leq \varepsilon(t_{n(k)}),$$

which implies that $f \in C(K)$ since $\varepsilon(t_n) \rightarrow 0$ as $n \rightarrow \infty$. Consequently, $f \in A(K)$. Finally, it is easy to see that $f \notin A_D(K)$, since $f(z, w) = \bar{z}$ on the analytic disk K_0 .

6 Plausibly approximable functions

6.1 Polynomial approximation

Let K be a compact planar set, and let P_n be a sequence of polynomials uniformly Cauchy on K . Then, by the maximum principle, it is uniformly Cauchy on every bounded component of the complement of K . Thus, if $h \in C(K)$ is the uniform limit of P_n on K , then h has an extension f on the polynomially convex hull \widehat{K} of K , which is the union of K with all bounded components of the complement of K . Furthermore, f belongs to $A(K)$. Conversely, if f belongs to $A(\widehat{K})$, since the complement of \widehat{K} is connected, according to Mergelyan's theorem in one variable, f belongs to $P(\widehat{K}) = P(K)$. Thus, in one variable, the problem is solved: a function h on K belongs to $P(K)$ if and only if it has an extension f belonging to $A(\widehat{K})$.

If K is a compact subset of \mathbb{C}^d , $d > 1$, then the polynomially convex hull \widehat{K} of K is the set of $z \in \mathbb{C}^d$, such that for every polynomial P , the number $|P(z)|$ is less than or equal to the supremum on K of the modulus of P . It follows easily that if h belongs to $P(K)$, then it has an extension f in $A_D(\widehat{K})$, so such functions are plausibly approximable by polynomials on K . Consider the following example.

Example 6.1 Let K be the unit sphere in \mathbb{C}^d . Then, \widehat{K} is the closed unit ball \overline{B} and $A_D(K) = C(K)$, $A_D(\widehat{K}) = A(\overline{B}) = P(\overline{B})$. If $z = (z_1, \dots, z_d)$, then the continuous function $h(z) = \bar{z}_1$, the complex conjugate of z_1 , does not have an extension belonging to $A_D(\widehat{K})$, so it is not uniformly approximable by polynomials on K .

It is easy to see that, for $d = 1$, if every h in $A_D(K)$ has an extension belonging to $A_D(\widehat{K})$, then K is polynomially convex. However, the result does not hold in general if $d > 1$ as the following example shows. That is, there exists a set K that is not polynomially convex such that every function h in $A_D(K)$ has an extension belonging to $A_D(\widehat{K})$.

Example 6.2 Let $d > 1$ and $K_1 := \{z \in \mathbb{C}^d : 1/2 \leq |z| \leq 1\}$, which is not polynomially convex. Then, \widehat{K}_1 is the closed unit ball B and every function h in $A_D(K_1)$ has an extension to $A_D(\widehat{K}_1)$ and can be approximated by polynomials.

We shall show now that for every compact, convex set K with nonempty interior, $P(K) = A(K)$, and for this, we shall invoke the following lemma. Although the lemma is well known, for the sake of completeness, we include the proof here.

Lemma 6.3 If $K \subseteq \mathbb{C}^d$ is a compact and convex set with $K^\circ \neq \emptyset$, then for every $z_0 \in K^\circ$ and every $z_1 \in K$, the segment $[z_0, z_1)$ is in K° . Consequently, K is regular closed.

Proof Without loss of generality, we can assume that $z_0 = 0$. Fix $\varepsilon > 0$ such that the open ball of center zero and radius ε is in K° , $B(0, \varepsilon) \subseteq K^\circ$. Consider z any fixed point in K . We claim that $z \in \overline{K^\circ}$. Indeed, for every $r \in (0, 1)$ and every $y \in B(rz, (1-r)\varepsilon)$, we have that $\frac{y-rz}{1-r} \in B(0, \varepsilon) \subseteq K$. By the convexity of K , we obtain that $y = rz + (1-r)\frac{y-rz}{1-r} \in K$. Therefore, $B(rz, (1-r)\varepsilon) \subseteq K$, and hence $rz \in K^\circ$. We have shown that $[0, z) \subseteq K^\circ$, and since $z = \lim_{r \rightarrow 1} rz$, we obtain that $K \subseteq \overline{K^\circ}$. ■

Proposition 6.4 If $K \subseteq \mathbb{C}^d$ is a compact, convex set with nonempty interior, then $P(K) = A(K)$. In particular, $P(K) = A_D(K)$.

Proof We may assume that $0 \in K^\circ$, and by Lemma 6.3, we obtain that for each $r < 1$ and every $z \in K$, the point rz belongs to K° . Fix a function $f \in A(K)$. Then, the function $f_r(z) = f(rz)$ is holomorphic on a neighborhood of K . Furthermore, for fixed $\varepsilon > 0$, there exists $r_\varepsilon < 1$ such that $|f_{r_\varepsilon}(z) - f(z)| < \varepsilon/2$ for every $z \in K$. Since K is compact and convex, K is polynomially convex. By the Oka–Weil theorem, there exists a polynomial p such that $|p(z) - f_{r_\varepsilon}(z)| < \varepsilon/2$ for every $z \in K$. Thus, $|p(z) - f(z)| < \varepsilon$ for every $z \in K$. Hence, polynomials are dense in $A_D(K)$. ■

6.2 Approximation by rational functions

Let K be a compact set in \mathbb{C}^d , $d > 1$. The rationally convex hull \widehat{K}^r of K is the set of z in \mathbb{C}^d such that for every rational function f holomorphic on an open set containing K , the number $|f(z)|$ is less than or equal to the supremum on K of the modulus of f . It follows easily that if a function h in $C(K)$ is in $R(K)$, then h has an extension H in $A_D(\widehat{K}^r)$. Thus, the functions h in $C(K)$ admitting an extension H in $A_D(\widehat{K}^r)$ are plausibly approximable by rational functions. In Examples 6.1 and 6.2, the rational convex hull and the polynomial convex hull coincide. Thus, Example 6.2 shows that there exists a set K that is not rationally convex such that every function h in $A_D(K)$ has an extension belonging to $A_D(\widehat{K}^r)$.

If L is a rationally convex compact subset of \mathbb{C}^d , then it is not always true that every function f in $A_D(L)$ can be approximated by rational functions. For example, let K_1 be the Swiss cheese and put $L = K_1 \times \{0\}$. Then, L is rationally convex, but $R(L) \neq A_D(L)$.

6.3 Approximation by functions in $\mathcal{O}(K)$

Let us start with an example. Let K be the unit sphere in \mathbb{C}^d , $d > 1$, and let f be a function holomorphic on a connected open set containing K . Then, f has an extension F holomorphic in an open set containing the closed unit ball \bar{B} . Furthermore, for every z in \bar{B} , $|F(z)|$ is less than or equal to the supremum of $|f|$ on K . It follows easily that if h is a function in $\bar{\mathcal{O}}(K)$, then h has an extension in $A_D(\bar{B})$. As before, these functions are plausibly approximable by elements of $\mathcal{O}(K)$. More generally, let K be a compact set in \mathbb{C}^d , $d > 1$, and let X be a compact set in \mathbb{C}^d containing K and satisfying the following: for every open set V containing K so that every component of V meets K , and for every function f holomorphic on V , the function f has an extension F , to an open set containing X , that is holomorphic, and satisfies that for every $z \in X$ the number $|F(z)|$ is less than or equal to the supremum of $|f|$ on K .

We call such a compact set X an O -hull of K . Obviously, $X = K$ is always such a hull. We note that in general there is no maximal such hull. It follows that if K is rationally convex, then the only O -hull of K is K itself.

Example 6.5 Let $d > 1$ and $K := \{z \in \mathbb{C}^d : |z| = 1/2 \text{ or } |z| = 1\}$. Let r be a number in $(1/2, 1)$ and let X_r be the set of $z \in \mathbb{C}^d$ such that $|z| \in [0, 1/2] \cup [r, 1]$. Then X_r is an O -hull of K and their union \bar{B} too.

Example 6.6 Set $K = S \times \{0\}$, where S is the Swiss cheese. Then, the only O -hull of K is K itself (since K is rationally convex), but $A_D(K)$ and $\bar{\mathcal{O}}(K)$ are different from each other.

7 A density result

We start by stating two of the main results which were proved in [2] (see Theorem 3.8 and Corollary 3.10, respectively).

Theorem 7.1 Let X be a compact subset of \mathbb{C} such that $\bar{\mathcal{O}}(X) = A_D(X)$, and let Y be a compact subset of \mathbb{C}^m such that $\bar{\mathcal{O}}(Y) = A_D(Y)$. Then, $\bar{\mathcal{O}}(X \times Y) = A_D(X \times Y)$.

Corollary 7.2 Let $K_i, i = 1, \dots, d$, be compact subsets of \mathbb{C} , and let L_i a subset of $\mathbb{C} \cup \{\infty\}$ meeting every complementary component of K_i . We also assume that $\bar{\mathcal{O}}(K_i) = A_D(K_i)$. Then, every f in $A_D(K_1 \times \dots \times K_d)$ can be uniformly approximated on $K_1 \times \dots \times K_d$ by finite sums of finite products of rational functions of one variable z_i with poles only in L_i .

For $d = 2$, we have the following.

Theorem 7.3 Let X be a compact subset of \mathbb{C} such that $\overline{\mathcal{O}}(X) = A_D(X)$, and let Y be a compact subset of \mathbb{C}^m such that $\overline{\mathcal{O}}(Y) = A_D(Y)$. Let L_X be a subset of $\mathbb{C} \cup \{\infty\}$ meeting every complementary component of X , and let S be a dense subset of $A_D(Y)$. Then, every f in $A_D(X \times Y)$ can be uniformly approximated on $X \times Y$ by finite sums of products of the form $g(z)h(w)$, where g is a rational function of one variable z with poles only in L_X and $h \in S$.

Remark 7.4 This strengthens Corollary 7.2 in case $d = 2$, for which we can take S to be the rational functions in z_2 with poles in L_2 .

Proof According to Theorem 7.1, we can assume that f is holomorphic on a set $V \times W$, where X is a subset of $V \subseteq \mathbb{C}$ and Y is subset of $W \subseteq \mathbb{C}^m$ and V and W are open sets. Let l be a cycle in $V \setminus X$ such that $\text{Ind}(l, z) = 1$ for all $z \in X$ and $\text{Ind}(l, z) = 0$ for all $z \in \mathbb{C} \setminus V$. Cauchy's formula gives that, for $z \in X$ and $w \in Y$, the number $f(z, w)$ is equal to an integral on l . The quantity to be integrated is uniformly continuous on $X \times Y \times l$; therefore, $f(z, w)$ can be uniformly approximated by Riemann sums of the integral on $X \times Y$; that is, linear combinations of products of the form $(\zeta_j - z)^{-1}f(\zeta_j, w)$. Each function $f(\zeta_j, w)$ as a function of w can also be approximated by a function $h_j(w) \in S$, because ζ_j is fixed and varies in a finite set. Similarly, the functions $(\zeta_j - z)^{-1}$ can be approximated by rational functions $g_j(z)$ with poles in L_X . ■

Remark 7.5 In Theorem 7.3, in some scenarios, we can choose the functions g and h to belong to some specific families:

- If X is a compact subset of \mathbb{C} with connected complement, then the functions g can be chosen to be polynomials.
- If S is the set of polynomials of m variables, then of course the functions h are polynomials.
- If S is the set of rational functions of m variables holomorphic on Y , then of course the functions h are rational functions of m variables holomorphic on Y .
- If $S = \mathcal{O}(Y)$, then of course the functions h are in $\mathcal{O}(Y)$.

8 Concluding remarks

We conclude with some remarks and lines of future research.

- (1) It is natural to ask what happens if we replace the compact sets by closed sets, and require uniform approximation there, since our new algebra $A_D(K)$ can be defined for closed sets as well. Moreover, one can consider tangential (Carleman) approximation on closed (unbounded) sets. This problem was originally studied in [1] in the more general context of Riemann surfaces, where the authors avoided the error in [4] by not using the standard definition for A . Instead, they gave a definition for A on products [1, p. 98], which is equivalent to A_D . In [2], a new original proof was written completely independently of [1] and new Mergelyan-type results were given for the algebra A_D , for example, for certain graphs as well as for Cartesian products of an arbitrary (possibly infinite)

- indexed family of planar compact sets. Thus, it would be interesting to investigate whether we can obtain analogous results for Carleman approximation on closed sets.
- (2) There remains the open question of providing sufficient and necessary conditions for the existence of a maximal O -hull for a compact set K .
 - (3) Wu Jujie and the second author are presently investigating $A(K)$, $A_D(K)$ and approximation, in the context of holomorphic motions.

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