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BOUNDARY-VALUE PROBLEMS OF A DEGENERATE SOBOLEV-TYPE DIFFERENTIAL EQUATION

BY

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ABSTRACT. The purpose of this paper is to study a degenerate Sobolev type partial differential equation in the form of $Mu_t + Lu = f$, where M and L are second order partial differential operators defined in a domain $(0, T] \times \Omega$ in R^{n+1}. The degenerate property of the equation is in the sense that both M and L are not necessarily strongly elliptic and their coefficients may vanish or be negative in some part of the domain $(0, T] \times \Omega$. Two types of boundary conditions are investigated.

1. **Introduction.** Let Ω be a bounded domain in R^n and let L, M be differential operators defined by:

$$Lu = \sum_{i,j=1}^{n} (a_{ij}(t, x)u_{x_j})_{x_i} - a(t, x)u, \qquad Mu = \sum_{i,j=1}^{n} (b_{ij}(t, x)u_{x_j})_{x_i} - b(t, x)u.$$

We consider the following Sobolev type differential equation

$$Mu_t + Lu = f(t, x) \qquad (t \in (0, T], x \in \Omega). \tag{1.1}$$

This equation is of regular Sobolev type when the operators L and M are uniformly strongly elliptic and the function b is positive on the closure $\bar{D}$ of $D \equiv (0, T] \times \Omega$. In this paper, we treat a degenerate equation in the sense that the operators L and M are not necessarily strongly elliptic and the function b may not be strictly positive in D. Specifically, we allow the function b taking zero or negative values in D and the matrices $A \equiv (a_{ij})$, $B = (b_{ij})$ being positive semi-definite in D. (In fact, A and B may even be indefinite.) In particular, if $b_{ij} \equiv 0$ for all i, j Eqn. (1.1) becomes a degenerate parabolic equation and if, in addition, $b \equiv 0$, it is reduced to a degenerate elliptic equation. When b_{ij} and b are not all zero we consider the following boundary and initial conditions

$$u(t, x) = 0 \qquad (t \in (0, T], x \in \Gamma) \tag{1.2}$$

$$u(0, x) = u_0(x) \qquad (x \in \Omega), \tag{1.3}$$

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where Γ is the boundary of Ω. However, if L, M are in the form

(1.4) $$Lu=\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_j})_{x_i}-a(t,x)u, \qquad Mu=\sum_{i,j=1}^{n}(\alpha(x)a_{ij}(x)u_{x_j})_{x_i}-b(t,x)u$$

we treat the following more general boundary condition

(1.2)′ $$\partial u/\partial\nu+\beta(x)u=0\ (x\in\Gamma_1), \qquad u(t,x)=0\ (x\in\Gamma_2), \qquad (t\in(0,T])$$

where $\alpha\geq 0$, $\beta\geq 0$, $\Gamma=\Gamma_1\cup\Gamma_2$ and $\partial/\partial\nu$ is the conormal derivative on Γ_1 with respect to the matrix $B=(\alpha a_{ij})$, that is, $\partial u/\partial\nu=\nu\cdot(B\nabla u)$. In the boundary condition (1.2)′, either Γ_1 or Γ_2 is allowed to be empty. The purpose of this paper is to study the existence and uniqueness of a weak solution for the above boundary value problems.

Sobolev type equations arise from various physical phenomena such as in the non-steady flow of fluids, heat conduction, resonant radiation in a gas and seepage of liquids in fissured rocks (cf. [1, 2, 11, 12]). These equations and their generalizations have recently been discussed in [5, 9, 10]. In most of these papers, it is assumed that L and M are uniformly strongly elliptic and b is positive on $\bar{D}$. These requirements insure that M is invertible and the composite operator $M^{-1}L$ generates a semi-group (in fact, a group) of bounded operators in some function space. However, for a degenerate operator M the invertibility of M no longer holds and even if M is not degenerate it is not clear whether $M^{-1}L$ is the generator of a semi-group when L is degenerate. In this paper, we use a variational approach to the problem and seek a weak solution in a suitable function space. Our essential idea is the construction of a suitable norm for this function space.

2. **The main results.** Throughout the paper we assume that $a_{ij}=a_{ji}$, $b_{ij}=b_{ji}$ and the coefficients of L and M together with their first partial derivatives and the mixed partial derivatives of b_{ij} in t and x are all bounded measurable in D. The functions f, u_0, α, β are assumed bounded measurable in their respective domains. We also assume that $b(0,x)\geq 0$ in Ω and the matrix $B_0(x)\equiv(b_{ij}(0,x))$ is positive semi-definite in Ω.

Let $C^2(D)$ be the set of functions $\phi(t,x)$ such that ϕ is continuous on $\bar{D}$ and ϕ_t and its second partial derivatives in x are continuous in D. Set

$$C_0^2(D)=\{\phi\in C^2(D);\quad \phi(t,x)=0 \text{ in } [0,T]\times\Gamma \text{ and } \phi(T,x)=\phi_{x_i}(T,x)=0 \text{ in } \Omega\}.$$

For any $\phi,\psi\in C_0^2(D)$ and any $n\times n$ matrix $P\equiv(p_{ij}(t,x))$ we set

(2.1) $$\begin{cases}\langle\phi,\psi\rangle=\int_D\phi(z)\psi(z)\,dz, \qquad \|\phi\|=\langle\phi,\phi\rangle^{1/2}\\ \langle\phi,\psi\rangle_P=\int_D\sum_{i,j=1}^{n}p_{ij}(z)\phi_{x_i}(z)\psi_{x_j}(z)\,dz\end{cases}$$

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where $dz = dx\,dt$. When P is positive semi-definite in D we write $\|\phi\|_P = \langle\phi, \phi\rangle_P^{1/2}$. Similarly, we set for $\phi, \psi \in C_0^2(D)$,

$$(2.2)\quad \begin{cases} (\phi, \psi)_{b_0} = \int_\Omega b'(0, x)\phi(0, x)\psi(0, x)\,dx, & \|\phi\|_{b_0} = (\phi, \phi)_{b_0}^{1/2} \\ (\phi, \psi)_{B_0} = \int_\Omega \sum_{i,j=1}^n b_{ij}(0, x)\phi_{x_i}(0, x)\psi_{x_j}(0, x)\,dx, & \|\phi\|_{B_0} = (\phi, \phi)_{B_0}^{1/2} \end{cases}$$

Our main idea for insuring the existence problem of (1.1)–(1.3) is the introduction of the functional

$$\langle\psi, \phi\rangle_H = \langle\psi, \phi\rangle_A - \tfrac{1}{2}\langle\psi, \phi\rangle_{B_t} + \langle\psi, (a - b_t/2)\phi\rangle + \tfrac{1}{2}(\psi, \phi)_{B_0} + \tfrac{1}{2}(\psi, \phi)_{b_0}$$
$$(2.3)\qquad (\phi, \psi \in C_0^2(D))$$

where $\langle\psi, \phi\rangle_A$ and $\langle\psi, \phi\rangle_{B_t}$ are defined in (2.1) with $P = A$ and $P = B_t \equiv ((b_{ij})_t)$, respectively. Since the matrices A and B are symmetric it is clear that $\langle\circ, \circ\rangle_H$ is a symmetric bilinear functional on $C_0^2(D)$. Assume that for some constant $\delta > 0$,

$$(2.4)\qquad \langle\phi, \phi\rangle_H \geq \delta\langle\phi, \phi\rangle \qquad (\phi \in C_0^2(D)).$$

Then $\langle\circ, \circ\rangle_H$ defines an inner product on $C_0^2(D)$. We denote the completion of $C_0^2(D)$ with respect to the norm $\|\phi\|_H = \langle\phi, \phi\rangle_H^{1/2}$ by H. In view of (2.4), the space H is contained in $L^2(D)$ both algebraically and topologically.

A function $u \in H$ is said to be a weak solution of (1.1)–(1.3) if

$$\langle u, \phi\rangle_A - \langle u, \phi\rangle_{B_t} - \langle u, \phi_t\rangle_B + \langle u, a\phi - (b\phi)_t\rangle = (u_0, \phi)_{B_0} + (u_0, \phi)_{b_0} - \langle f, \phi\rangle$$
$$(2.5)\qquad (\phi \in C_0^2(D)).$$

Equation (2.5) is obtained from (1.1) by a formal integration by parts and using the conditions (1.2), (1.3). In obtaining the equation we have used the relations

$$\langle Mu_t, \phi\rangle = -\langle u_t, \phi\rangle_B - \langle u_t, b\phi\rangle$$
$$(2.6)\qquad = (u_0, \phi)_{B_0} + \langle u, \phi\rangle_{B_t} + \langle u, \phi_t\rangle_B + (u_0, \phi)_{b_0} + \langle u, (b\phi)_t\rangle, \qquad (\phi \in C_0^2(D))$$

$$(2.7)\qquad \langle Lu, \phi\rangle = -\langle u, \phi\rangle_A - \langle u, a\phi\rangle, \qquad (\phi \in C_0^2(D)).$$

Let $\phi \in C_0^2(D)$ be fixed. Define a linear functional $B[\circ, \phi]$ on H by:

$$(2.8)\quad B[\psi, \phi] = \langle\psi, \phi\rangle_A - \langle\psi, \phi\rangle_{B_t} - \langle\psi, \phi_t\rangle_B + \langle\psi, a\phi - (b\phi)_t\rangle \qquad (\psi \in H).$$

It will be shown in the following section that for each $\phi \in C_0^2(D)$, $B[\circ, \phi]$ is a bounded linear functional on H and there exists a closable operator $S: C_0^2(D) \to H$ such that $B[u, \phi] = \langle u, S\phi\rangle_H$ for $u \in H$, $\phi \in C_0^2(D)$. Denote the closure of S by $\bar{S}$. Then we have the following result.

THEOREM 1. *Assume that* (2.4) *holds for some* $\delta > 0$. *Then the problem* (1.1)–(1.3) *has a weak solution* $u \in H$. *Furthermore, for any two solutions* $u_1, u_2 \in H$ *there exists* $w \in R^\perp(\bar{S})$ *such that* $u_1 = u_2 + w$, *where*

$$R^\perp(\bar{S}) = \{\psi \in H; \langle \psi, \phi\rangle = 0 \quad \text{for all} \quad \phi \in R(\bar{S})\}.$$

For the mixed boundary-value problem (1.1), (1.2)′, (1.3), where L and M are in the form of (1.4) we seek a solution in the Hilbert space $\tilde{H}$ which is defined as follows: Let

$$\zeta_0^2(D) = \{\phi \in C_0^2(D); \phi(t, x) = 0 \quad \text{on} \quad [0, T] \times \Gamma_2, \phi(T, x) = \phi_{x_i}(T, x) = 0 \text{ in } \Omega\}.$$

Define a symmetric bilinear functional on $\zeta_0^2(D)$ by

$$\langle \psi, \phi\rangle_{\tilde{H}} = \langle \psi, \phi\rangle_A + \langle \psi, \phi\rangle_\beta + \langle \psi, (a - b_t/2)\phi\rangle + \tfrac{1}{2}[(\psi, \phi)_{B_0} + (\psi, \phi)_{\beta_0} + (\psi, \phi)_{b_0}] \tag{2.9}$$
$$(\psi, \phi \in \zeta_0^2(D)),$$

where $\langle \psi, \phi\rangle_A$, $\langle \psi, \phi\rangle$, $(\psi, \phi)_{B_0}$, $(\psi, \phi)_{b_0}$ are given in (2.1), (2.2) and

$$\text{(2.10)} \quad \begin{aligned} \langle \psi, \phi\rangle_\beta &= \int_0^T \int_{\Gamma_1} \beta(x)\psi(t, x)\phi(t, x)\, dS\, dt, \qquad \|\phi\|_\beta = \langle \phi, \phi\rangle_\beta^{1/2} \\ (\psi, \phi)_{\beta_0} &= \int_{\Gamma_1} \alpha(x)\beta(x)\psi(0, x)\phi(0, x)\, dS, \qquad \|\phi\|_{\beta_0} = (\phi, \phi)_{\beta_0}^{1/2}. \end{aligned}$$

Assume that for some constant $\delta > 0$,

$$\langle \phi, \phi\rangle_{\tilde{H}} \geq \delta\langle \phi, \phi\rangle \qquad (\phi \in \zeta_0^2(D)). \tag{2.11}$$

Then $\langle \circ, \circ\rangle_{\tilde{H}}$ defines an inner product in $\zeta_0^2(D)$. We denote by $\tilde{H}$ the completion of $\zeta_0^2(D)$ with respect to the norm $\|\phi\|_{\tilde{H}} = \langle \phi, \phi\rangle_{\tilde{H}}^{1/2}$. A function $u \in \tilde{H}$ is called a weak solution of (1.1), (1.2)′, (1.3) if

$$\langle u, \phi\rangle_A - \langle u, \phi_t\rangle_B + \langle u, \phi - \alpha\phi_t\rangle_\beta + \langle u, a\phi - (b\phi)_t\rangle$$
$$= (u_0, \phi)_{B_0} + (u_0, \phi)_{\beta_0} + (u_0, \phi)_{b_0} - \langle f, \phi\rangle \qquad (\phi \in \zeta_0^2(D)). \tag{2.12}$$

As in the previous case the definition of a weak solution is obtained from (1.1) by a formal integration by parts and using the conditions (1.2)′, (1.3). In the present situation the formal integration yields the relations

$$\begin{cases} \langle Mu_t, \phi\rangle = \langle u, \phi_t\rangle_B + \langle u, \alpha\phi_t\rangle_\beta + \langle u, (b\phi)_t\rangle + (u_0, \phi)_{B_0} + (u_0, \phi)_{\beta_0} + (u_0, \phi)_{b_0} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\phi \in \zeta_0^2(D)) \\ \langle Lu, \phi\rangle = -\langle u, \phi\rangle_A - \langle u, \phi\rangle_\beta - \langle u, a\phi\rangle \end{cases}$$

(2.13)

For each $\phi \in \zeta_0^2(D)$ we define a linear functional on $\tilde{H}$ by

$$\tilde{B}[\psi, \phi] = \langle \psi, \phi \rangle_A - \langle \psi, \phi_t \rangle_B + \langle \psi, \phi - \alpha\phi_t \rangle_\beta + \langle \psi, a\phi - (b\phi)_t \rangle \qquad (\phi \in \zeta_0^2(D)). \tag{2.14}$$

It is easily shown that for each $\phi \in \zeta_0^2(D)$, $\tilde{B}[\circ, \phi]$ is a bounded linear functional on $\tilde{H}$ and there exists a closable operator $S_1 : \zeta_0^2(D) \to \tilde{H}$ such that $\tilde{B}[\psi, \phi] = \langle \psi, S_1\phi \rangle$ for $\psi \in \tilde{H}$, $\phi \in \zeta_0^2(D)$. Denoting by $\bar{S}_1$ the closure of S_1, we have the following conclusion:

THEOREM 2. *Assume that* (2.11) *holds for some* $\delta > 0$. *Then the problem* (1.1), (1.2)′, (1.3) *with M and L given by* (1.4) *has a weak solution* $u \in \tilde{H}$. *Furthermore, for any two solutions* $u_1, u_2 \in \tilde{H}$ *there exists* $w \in R^\perp(\bar{S}_1)$ *such that* $u_1 = u_2 + w$.

REMARKS. (a) By a transformation $u \to e^{-\lambda t}u$ in the problem (1.1)–(1.3) for some real constant λ, the condition (2.4) is satisfied if either one of the following conditions holds:

(i) $(A + \lambda B - B_t/2)$ is positive semi-definite and $a + \lambda b - b_t/2 \ge \delta$ in $\bar{D}$.
(ii) $(A + \lambda B - B_t/2)$ is positive definite and $a + \lambda b - b_t/2 \ge 0$ in $\bar{D}$.

In particular, if Eqn. (1.1) is of the form

$$\sum_{i=1}^{n} (b^*(t, x)u_{tx_i})_{x_i} - b(t, x)u_t + \sum_{i=1}^{n} (a^*(t, x)u_{x_i})_{x_i} - a(t, x)u = f(t, x)$$

which was considered in [1, 2, 10, 11, 12] then the above conditions become, respectively,

(i) $a^* + \lambda b^* - b_t^*/2 \ge 0$ and $a + \lambda b - b_t/2 \ge \delta$ in $\bar{D}$,
(ii) $a^* + \lambda b^* - b_t^*/2 \ge \delta$ and $a + \lambda b - b_t/2 \ge 0$ in $\bar{D}$.

(b) If $B \equiv 0$, the problem (1.1)–(1.3) becomes the degenerate parabolic equation considered in [8] (see also [3, 4, 7]) and if, in addition, $b \equiv 0$ it reduces to a degenerate elliptic equation (cf. [8]). In the latter situation, the initial condition (1.3) should be disregarded.

3. Proof of the theorems

Proof of Theorem 1. For any $\phi, \psi \in C_0^2(D)$, the relation

$$\begin{aligned} |\langle \psi, \phi \rangle_P| &= \left| \int_D \sum_{i,j=1}^{n} p_{ij}(z)\psi_{x_i}(z)\phi_{x_j}(z)\, dz \right| \\ &= \left| \int_D \psi(z) \sum_{i,j=1}^{n} (p_{ij}(z)\phi_{x_j}(z))_{x_i}\, dz \right| \end{aligned} \tag{3.1}$$

with P representing A, B, and B_t, respectively, implies that

$$|B[\psi, \phi]| \le K_\phi \, \|\psi\| \le \delta^{-1/2} K_\phi \, \|\psi\|_H \qquad (\psi \in C_0^2(\bar{D})), \tag{3.2}$$

where K_ϕ is a constant depending only on ϕ and the matrices A, B. Thus

$B[\circ, \phi]$ is a bounded linear functional on $C_0^2(D)$ and so it can be extended to H. In view of (2.5) and (2.8), it suffices to find a $u \in H$ such that

$$B[u, \phi] = (u_0, \phi)_{B_0} + (u_0, \phi)_{b_0} - \langle f, \phi\rangle \qquad (\phi \in C_0^2(D)).$$

Now for each $\phi \in C_0^2(D)$ the Riesz Theorem insures the existence of $S\phi \in H$ such that

$$B[u, \phi] = \langle u, S\phi\rangle_H \qquad (u \in H, \phi \in C_0^2(D)). \tag{3.4}$$

Clearly, S is a linear operator on $C_0^2(D)$ to H. We show that S is closable. For each fixed $\psi \in C_0^2(D)$, the second equality in (2.6) and the relation (3.1) imply that

$$\begin{aligned} |\langle \psi, \phi\rangle_{B_t} + \langle \psi, \phi_t\rangle_B + \langle \psi, (b\phi)_t\rangle| &= |\langle \psi_t, \phi\rangle_B + \langle \psi_t, b\phi\rangle + (\psi, \phi)_{B_0} + (\psi, \phi)_{b_0}| \\ &\le K'_\psi \|\phi\| + \|\psi\|_{B_0} \|\phi\|_{B_0} + \|\psi\|_{b_0} \|\phi\|_{b_0} \\ &\le K''_\psi \|\phi\|_H \qquad (\phi \in C_0^2(D)), \end{aligned} \tag{3.5}$$

where K'_ψ, K''_ψ are some constants independent of ϕ. In view of (3.4), (2.8), (3.5), and (3.1) we have

$$|\langle \psi, S\phi\rangle_H| \le |\langle \psi, \phi\rangle_A + \langle \psi, a\phi\rangle| + K''_\psi \|\phi\|_H \le K_\psi \|\phi\|_H \qquad (\phi \in C_0^2(D))$$

for some constant K_ψ. The above relation shows that $C_0^2(D)$ is contained in the domain $D(S^*)$ of S^*, where S^* is the adjoint operator of S. Now if $\{\phi_k\}$ is a sequence in $C_0^2(D)$ such that $\phi_k \to 0$ and $S\phi_k \to g$ as $k \to \infty$ then for each $\psi \in C_0^2(D)$,

$$\langle \psi, g\rangle_H = \lim_{k\to\infty} \langle \psi, S\phi_k\rangle_H = \lim_{k\to\infty} \langle S^*\psi, \phi_k\rangle_H = 0.$$

Since $C_0^2(D)$ is dense in H we conclude that $g = 0$ and thus S is closable. It follows from the closed property of $\bar{S}$ and (3.4) that

$$B[u, \phi] = \langle u, \bar{S}\phi\rangle_H \qquad (u \in H, \quad \phi \in D(\bar{S})). \tag{3.6}$$

where $D(\bar{S})$ is the domain of $\bar{S}$. We next show that

$$\langle \phi, \bar{S}\phi\rangle_H = \|\phi\|_H^2 \qquad (\phi \in D(\bar{S})) \tag{3.7}$$

Since for $\phi \in C_0^2(D)$,

$$\begin{aligned} \langle \phi, (b\phi)_t\rangle &= \langle \phi, b_t\phi\rangle + \langle \phi, b\phi_t\rangle = \langle \phi, b_t\phi\rangle - \tfrac{1}{2}(\phi, \phi)_{b_0} - \tfrac{1}{2}\langle \phi, b_t\phi\rangle \\ &= \tfrac{1}{2}(\langle \phi, b_t\phi\rangle - (\phi, \phi)_{b_0}) \end{aligned} \tag{3.8}$$

$$\langle \phi, \phi\rangle_{B_t} + \langle \phi, \phi_t\rangle_B = \int_D \sum_{i,j=1}^n \phi_{x_i}(b_{ij}\phi_{x_j})_t \, dz = -(\phi, \phi)_{B_0} - \langle \phi, \phi_t\rangle_B,$$

and since the latter relation implies that

$$\langle \phi, \phi\rangle_{B_t} + \langle \phi, \phi_t\rangle_B = \tfrac{1}{2}(\langle \phi, \phi\rangle_{B_t} - (\phi, \phi)_{B_0}) \qquad (\phi \in C_0^2(\bar{D})), \tag{3.9}$$

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we see from (2.8), (3.8), (3.9), and (2.3) that

$$B[\phi, \phi] = \langle \phi, \phi \rangle_A - \tfrac{1}{2}\langle \phi, \phi \rangle_{B_t} + \tfrac{1}{2}(\phi, \phi)_{B_0} + \langle \phi, (a - \tfrac{1}{2}b_t)\phi \rangle + \tfrac{1}{2}(\phi, \phi)_{b_0} = \langle \phi, \phi \rangle_H.$$

It follows from (3.4) that the relation (3.7) holds for $\phi \in C_0^2(D)$. The closed property of $\bar{S}$ implies that (3.7) also holds for $\phi \in D(\bar{S})$. At this point, the proof of the existence of a solution follows from a theorem of Lion's (cf. [6]). However, in order to show the second part of the theorem we use a different argument. In view of (3.7), the inverse $\bar{S}^{-1}$ exists and $\|\bar{S}^{-1}\psi\|_H \le \|\psi\|_H$ for $\psi \in R(\bar{S})$. By the closed range theorem we have $R(\bar{S}^*) = H$, where $\bar{S}^*$ is the adjoint of $\bar{S}$. But the functional

$$F(\phi) \equiv (u_0, \phi)_{B_0} + (u_0, \phi)_{b_0} - \langle f, \phi \rangle \qquad (\phi \in C_0^2(D))$$

is bounded on $C_0^2(D)$. By extending F to H we can find $v \in H$ such that $F(\phi) = \langle v, \phi \rangle_H$ $(\phi \in C_0^2(D))$. Let $u \in D(\bar{S}^*)$ such that $\bar{S}^* u = v$. Then by (3.6),

$$B[u, \phi] = \langle u, \bar{S}\phi \rangle_H = \langle \bar{S}^* u, \phi \rangle_H = \langle v, \phi \rangle_H = F(\phi) \qquad (\phi \in C_0^2(D)).$$

This shows that u is a solution of (3.3). Now if u_1, u_2 are two solutions of (3.3) then $w \equiv u_1 - u_2$ satisfies the relation

$$\langle w, \bar{S}\phi \rangle_H = B[w, \phi] = 0 \qquad (\phi \in D(\bar{S})).$$

Hence $w \in R^{\perp}(\bar{S})$ which completes the proof of the theorem.

Proof of Theorem 2. It is readily seen from the positive semi-definite property of the matrix $\{a_{ij}(x)\}$ that for each $\phi \in \zeta_0^2(D)$

$$\begin{aligned} |\tilde{B}[\psi, \phi]| &\le \|\psi\|_A \, \|\phi\|_A + \|\psi\|_B \, \|\phi_t\|_B + \|\psi\|_\beta \, \|\phi - \alpha\phi_t\|_\beta + \|\psi\| \, \|a\phi - (b\phi)_t\| \\ &\le \tilde{K}_\phi \, \|\psi\|_{\tilde{H}}, \qquad (\psi \in \zeta_0^2(D)) \end{aligned}$$

where $\tilde{K}_\phi$ is a constant independent of ψ. Thus we may extend $\tilde{B}[\circ, \phi]$ to $\tilde{H}$. In view of (2.12), (2.14) it suffices to find a $u \in \tilde{H}$ such that

$$\tilde{B}[u, \phi] = (u_0, \phi)_{B_0} + (u_0, \phi)_{\beta_0} + (u_0, \phi)_{b_0} - \langle f, \phi \rangle \tag{3.10}$$

By the Riesz Theorem, there exists a closable operator $S_1 : \zeta_0^2(D) \to \tilde{H}$ such that

$$\tilde{B}[\psi, \phi] = \langle \psi, \bar{S}_1 \phi \rangle_{\tilde{H}} \qquad (\phi \in D(\bar{S}_1), \psi \in \tilde{H}) \tag{3.11}$$

where $\bar{S}_1$ is the closure of S_1. Since by direct integration,

$$2\langle \phi, \phi_t \rangle_B = -(\phi, \phi)_{B_0}, \; 2\langle \phi, \alpha\phi_t \rangle_\beta = -(\phi, \phi)_{\beta_0}, \qquad (\phi \in \zeta_0^2(D)). \tag{3.12}$$

We obtain from (2.14), (3.12), (3.8), and (2.9) that

$$\begin{aligned} \tilde{B}[\phi, \phi] = \langle \phi, \phi \rangle_A + \tfrac{1}{2}(\phi, \phi)_{B_0} + (\phi, \phi)_\beta + \tfrac{1}{2}(\phi, \phi)_{\beta_0} + \langle \phi, (a - b_t/2)\phi \rangle + \tfrac{1}{2}(\phi, \phi)_{b_0} \\ = \langle \phi, \phi \rangle_{\tilde{H}} \qquad (\phi \in \zeta_0^2(D)). \end{aligned}$$

It follows from the closed property of $\bar{S}_1$ and (3.11) that

$$\langle \phi, \bar{S}_1\phi\rangle = \|\phi\|^2_{\tilde{H}} \qquad (\phi \in D(\bar{S})).$$

Using the above relation and the closed range theorem, a similar argument as in the proof of Theorem 1 leads to the existence of $u \in D(\bar{S}_1^*)$ satisfying the relation

$$\tilde{B}[u, \phi] = \langle \bar{S}_1^* u, \phi\rangle_{\tilde{H}} = (u_0, \phi)_{B_0} + (u_0, \phi)_{\beta_0} + (u_0, \phi)_{b_0} - \langle f, \phi\rangle.$$

This proves the existence problem. The second part of the theorem follows directly from the above relation.

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