This department welcomes short notes and problems believed to be new. Contributions should include solutions where known, or background material to the problem in case the problem is unsolved. Send all communications concerning this department to W. Moser, Department of Mathematics, University of Manitoba, Winnipeg, Manitoba.

## ON AN OBSERVATION OF M. RIESZ

P. Scherk

(received October 1, 1960)

On a recent visit to Toronto, Professor Riesz made an interesting remark on the invariance of a certain cross-ratio. I wish to present a simple proof of his result.

Let $x = (x_1, x_2, x_3, x_4)$, ... denote homogeneous coordinates in real projective three-space; $x(t)$, ... are differentiable curves in that space.

If $x = x(t)$ and $y = y(t)$ are two such curves, the straight lines $\alpha x(t) + \beta y(t)$ generate a developable surface if and only if the determinant $(x\ \dot{x}\ y\ \dot{y})$ vanishes.

Consider two such developable surfaces

$$(1) \qquad \alpha x(t) + \beta y(t), \qquad \alpha x(t) + \beta z(t)$$

through the same curve $x(t)$. Thus

$$(2) \qquad (x\ \dot{x}\ y\ \dot{y}) = (x\ \dot{x}\ z\ \dot{z}) = 0.$$

For each value of $t$ the two generators (1) are assumed to span a plane which does not contain the tangent of the curve $x(t)$. Thus

$$(3) \qquad (x\ \dot{x}\ y\ z) \neq 0.$$



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Consider now a third developable surface through the curve $x(t)$ whose generators always lie in the plane through the points $x, y, z$ and are distinct from the straight lines (1). We may write its equation in the form

$$(4) \qquad \alpha x(t) + \beta w(t)$$

where

$$(5) \qquad w(t) = \eta(t)y(t) + \xi(t)z(t); \qquad \eta(t) \neq 0, \qquad \xi(t) \neq 0.$$

Since $y(t)$, $z(t)$ and $w(t)$ are differentiable, we may assume $\eta(t)$ and $\xi(t)$ to be so too. On account of (2), (3), and (5), the equation $(x\,\dot{x}\,w\,\dot{w}) = 0$ readily implies

$$(6) \qquad \frac{\dot{\xi}}{\xi} - \frac{\dot{\eta}}{\eta} = \frac{(x\,\dot{x}\,\dot{y}\,z) - (x\,\dot{x}\,y\,\dot{z})}{(x\,\dot{x}\,y\,z)}.$$

If

$$w_i = \eta_i(t)y(t) + \xi_i(t)z(t) \qquad [i = 1, 2]$$

are two curves satisfying the above assumptions, (6) implies

$$\frac{\dot{\xi}_1}{\xi_1} - \frac{\dot{\eta}_1}{\eta_1} = \frac{\dot{\xi}_2}{\xi_2} - \frac{\dot{\eta}_2}{\eta_2}$$

or

$$\frac{d}{dt}\left(\frac{\xi_1}{\eta_1} \Big/ \frac{\xi_2}{\eta_2}\right) = 0.$$

Hence

$$\frac{\xi_1}{\eta_1} \Big/ \frac{\xi_2}{\eta_2} = \text{constant}$$

and (4)-(5) maps the planes through $x(t)$, $y(t)$, $z(t)$ projectively onto one another.

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