PREDICTIVE STOP-LOSS PREMIUMS

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Abstract

Based on a representation of the aggregate claims random variable as linear combination of counting random variables, a linear multivariate Bayesian model of risk theory is defined. In case of the classical risk theoretical assumptions, that is conditional Poisson likelihood counting variates and Gamma structural density, the model is shown to identify with a Bayesian version of the collective model of risk theory. An interesting multivariate credibility formula for the predictive mean is derived. A new type of recursive algorithm, called three-stage nested recursive scheme, allows to evaluate the predictive density and associated predictive stop-loss premiums in an effective way.

1. Introduction

It is well-known that the fluctuation of basic probabilities in a portfolio of risks plays an important role. Early work has been done by Ammeter (1948/1949) and the subject is emphasized in Beard et al. (1984). The actuarial literature devoted to stochastic variation of mortality and other types of mortality variation is relatively scarce. A review of known studies is given by Wolthuis and Van Hoek (1986), Section 7. More recent work includes Norberg (1987) and Klugman (1989). Random variation in uncertain payments taking into account other sources of variation is discussed in De Jong (1983).

In life insurance the observed mortality experience of a group contract may deviate considerably from the expected mortality given by a life table. This means that the expected value of aggregate claims may also deviate considerably from the value obtained from a life table. Since priorities of stop-loss contracts are usually expressed as percentages of the mean aggregate claims and stop-loss premiums are very sensitive with respect to this quantity, it follows that the impact of the variation in basic probabilities on the aggregate

claims distribution and derived quantities such as mean, standard deviation and stop-loss premiums may be important. When tariffing stop-loss contracts and in order to avoid mismatches in priorities, the estimation of the mean aggregate claims needs special attention. In this paper an attempt is made to take into account claims experience when evaluating aggregate claims distributions and related stop-loss premiums. The considered Bayesian (analytical) model is derived from the well-known standard tools of Decision Theory. The mathematical steps involved in the construction of a feasible computational algorithm are however rather formidable and are based on previous results of the author. Analogous results for alternate Bayesian models chosen from among other natural conjugate families may be possible but are not considered in this paper. It is also most desirable to develop distribution-free formulas following eventually new paradigms as suggested by JEWELL (1990). In the following let us give a more detailed outline of the paper with its main results.

In Section 2 the random variable of aggregate claims associated to a portfolio of risks is viewed as a linear combination of counting random variables for which there exist computational algorithms to evaluate the corresponding distribution function (see HÜRLIMANN (1990a)). It is assumed that risk units produce claims of known amount and that the probability of occurrence of a given claim is an unknown following some structural density. Based on these assumptions a *linear multivariate Bayesian model* of risk theory is defined. If one adds further the natural model assumptions used in life and general insurance, one obtains what we call *linear multivariate Poisson Gamma Bayesian model* of risk theory. In the present work only this special model is studied. To illustrate the results of the paper a simple life insurance example is presented and used throughout.

In Section 3 a link to classical collective risk theory is given. It is shown that the linear multivariate Poisson Gamma Bayesian model coincides with a well-defined Bayesian version of the collective model of risk theory.

The needed Bayesian formulas to perform later on effective stop-loss premium calculations are dereived in Section 4. As main results we obtain an appealing multivariate credibility formula for the predictive mean of aggregate claims and an analytical representation of the predictive density defined earlier by JEWELL (1974).

Using two recursive algorithms to evaluate the probability distribution function of a linear combination of independent random variables first derived in HÜRLIMANN (1990a) and reviewed in the Appendix, we derive in Section 5 a three-stage nested recursive scheme to evaluate the predictive density of our Bayesian model. Previous numerical algorithms for aggregate claims probability models involved so far only one and two-stage nested recursive calculations as can be seen from the last advances in the insurance field by PANJER (1981), DE PRIL (1986/1989) and HÜRLIMANN (1990a/b).

Finally in Section 6 a numerical example illustrates the important impact claims experience may have on mean, cumulative probability and stop-loss premiums evaluated in a Bayesian framework.

2. A LINEAR MULTIVARIATE BAYESIAN MODEL OF RISK THEORY

For the purpose of rating stop-loss and other non-proportional reinsurance contracts, we consider the random variable of aggregate claims which is associated to a given portfolio of risks. Assume that the risks in question are subject to the following *risk classification system*:

— each risk unit (= individual policy exposed to risk) can produce a claim of a known amount m_k, $k = 1, \ldots, r$, where r is the number of possible amounts

— a claim is characterised by an unknown probability of occurrence θ_i, $i = 1, \ldots, s$, where s is the number of different probabilities of occurrence

Example 2.1

Consider a life insurance portfolio subject to the risk of death and/or disability. Each life consists of at most three insurable risk units, two for death and one for disability, producing claims whose amounts at risk can be evaluated using computer programs. In pension insurance they are routinely calculated using well-known formulas (e.g. BERTRAM and FEILMEIER (1987), Section 3.2.1, pp. 61-64). Given a life aged x, a claim for the risk of death may occur with the unknown probability q_x^a if the insured dies as active member, or it may occur with the unknown probability q_x^i if the insured dies as disabled member, both with different amounts at risk. A claim for the risk of disability may occur with the unknown probability i_x. It is straightforward to obtain the above risk classification system by renaming the variables appropriately.

Given the above risk classification system, let us consider the following mathematical model of aggregate claims. Let X_{ki} be the random variable counting the number of claims of amount at risk m_k with probability of occurrence θ_i. Then the random variable representing the aggregate claims is given by

$$(2.1) \qquad X = \sum_{k=1}^{r} m_k \sum_{i=1}^{s} X_{ki}.$$

The uncertainty about θ_i is modelled by a prior or structural density denoted $u_i(\theta_i)$, $i = 1, \ldots, s$. We assume independence between the θ_i's. Therefore the structural density of the parameter vector $\theta = (\theta_1, \ldots, \theta_s)$ is given by

$$(2.2) \qquad u(\theta) = \prod_{i=1}^{s} u_i(\theta_i).$$

Specifying different assumptions on the conditional probabilities $\Pr(X_{ki} = j|\theta_i)$ that j claims of amount m_k (given θ_i) occur, and on the structural densities $u_i(\theta_i)$, it is possible to obtain different Bayesian models to describe and analyze the aggregate claims random variable X. The following natural *model assumptions* are widely used in life as well as in general insurance:

(I) Conditionally on $\boldsymbol{\theta}$ the random variables X_{ki} are independent and depend upon θ_i through the Poisson likelihood

$$(2.3)\qquad \mathrm{Pr}\,(X_{ki} = j|\theta_i) = \mathrm{Po}\,(\,j;\,\theta_i n_{ki}) = e^{-\theta_i n_{ki}}\frac{(\theta_i n_{ki})^j}{j!}\,,\qquad j = 0,\,1,\,2,\,\ldots,$$

where n_{ki} is the number of risk units with amount at risk m_k and probability of occurrence θ_i.

(II) The structural densities $u_i(\theta_i)$ are Gamma densities given by

$$(2.4)\quad u_i(\theta_i) = \mathrm{Gamma}\,(\theta_i;\,\alpha_i,\,\beta_i) = \frac{\beta_i^{\alpha_i}\theta_i^{\alpha_i-1}e^{-\beta_i\theta_i}}{\Gamma(\alpha_i)}\,,\ i = 1,\,2,\,\ldots,\,s,\ \alpha_i,\,\beta_i > 0.$$

At this stage it is possible to get formulas for the moments of X. The expected value, which will be needed later on, is calculated as follows:

$$(2.5)\qquad E[X] = \sum_{k=1}^{r} m_k \sum_{i=1}^{s} E[X_{ki}] = \sum_{k=1}^{r} m_k \sum_{i=1}^{s} E_{\theta_i}[E[X_{ki}|\theta_i]]$$

$$= \sum_{i=1}^{s} \frac{\alpha_i}{\beta_i} \sum_{k=1}^{r} m_k n_{ki}$$

The results of the present paper will be illustrated numerically at the following simple life insurance portfolio.

Example 2.2

Given is a portfolio of 1500 active persons insured against the risk of death. It is divided into $s = 3$ *age classes* corresponding to the approximate ages $x = 30,\,40,\,50$ with probabilities of death $q_{30} = 0.00051$, $q_{40} = 0.00114$, $q_{50} = 0.00344$ borrowed from the EVK80 table, which is the life table of the "Eidgenössiche Versicherungskasse" used in Swiss pension insurance for rating risk of death and disability. Each age class is subdivided into $r = 5$ *risk sums subclasses* with lump sums 500'000, 1'000'000, 1'500'000, 2'000'000 and 2'500'000. Choosing a risk unit of $\Delta = 500'000$, this means that $m_k = k$, $k = 1,\,2,\,\ldots,\,5$. The number n_{ki} of persons in each of the 15 subclasses is as follows:

	n_{ki}				
	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$i = 1$	200	150	50	50	50
$i = 2$	100	100	100	100	100
$i = 3$	50	50	200	100	100

The structural parameters α_i, β_i defining the structural density in (2.4) are to be estimated. Usually this step depends heavily on the application as well on the knowledge of the situation. Let us illustrate a simple estimation procedure that applies to the typical case of the mortality risk in life insurance. One can set $\theta_i = q_x$ for a certain age x, which is interpreted as un unknown conditional probability of death, given alive at age x. As a possible method to estimate α_i, β_i, we propose to use the estimate q_x^* of q_x given by the life table as well as an estimate of the uncertainty in the estimation of q_x. In other words estimate α_i, β_i by solving the moment equations

$$(2.6) \qquad E[q_x] = \frac{\alpha_i}{\beta_i}, \qquad \text{Var}\,[q_x] = \frac{\alpha_i}{\beta_i^2},$$

and using estimators of the mean and variance of q_x. One can take $q_x^* \cong E[q_x]$ and a good approximation of the variance of q_x is given by

$$(2.7) \qquad \text{Var}\,[q_x] \cong \frac{q_x^*(1-q_x^*)}{E_x},$$

where E_x is the exposure, that is the number of risk years under observation for the estimation of q_x in a life table (e.g. LONDON (1988), chap. 6.2, p. 115). It follows that

$$(2.8) \qquad \beta_i = \frac{E_x}{1-q_x^*}, \qquad \alpha_i = \beta_i q_x^*$$

For the EVK80 table it is known that the total exposure $\Sigma\,E_x$ for the active ages between 20 and 65 is 470'937. A rough approximation is thus $E_x \approx 10'000$. In our example one has

$$(2.9) \qquad \beta_1 = \frac{10000}{1-q_{30}} = 10005.103, \qquad \alpha_1 = \beta_1 q_{30} = 5.103,$$

$$\beta_2 = \frac{10000}{1-q_{40}} = 10011.413, \qquad \alpha_2 = \beta_2 q_{40} = 11.413,$$

$$\beta_3 = \frac{10000}{1-q_{50}} = 10034.519, \qquad \alpha_3 = \beta_3 q_{50} = 34.519,$$

According to (2.5) the expected value of the aggregate claims is equal to $E[X] = 3'973'500$. Since $\alpha_i/\beta_i = q_x$ this is equal to the expected value of aggregate claims evaluated using the life table in the traditional way.

Remarks

(i)   In Section 5 we will assume that the amounts at risk m_k are non-negative integers. This is an assumption made in most papers of present-day

applied risk theory. However in pension insurance negative amounts at risk often occur. In this case special mathematical treatment is needed (see e.g. HÜRLIMANN (1991) and the relevant references mentioned there).

(ii) Since $\Pr(X_{ki} = j|\theta_i)$ and $u_i(\theta_i)$ are Poisson, resp. Gamma distributed, it follows that the unconditional probability $\Pr(X_{ki} = j)$ belongs to the negative binomial distribution.

3. LINK TO THE COLLECTIVE MODEL OF RISK THEORY

Before undertaking the Bayesian analysis of the linear multivariate Poisson Gamma Bayesian model, we show that it actually identifies with a Bayesian version of the classical collective model of risk theory.

First of all we derive a simple formula for the likelihood $f(x|\theta)$ of an aggregate claims observable

$$(3.1) \qquad x = \sum_{k=1}^{r} m_k \sum_{i=1}^{s} x_{ki}$$

in the Bayesian set-up of Section 2. Consider the matrix $x = (x_{ki})^T = (x_1, \ldots, x_s)^T$ of dimension sxr, where $x_i = (x_{1i}, \ldots, x_{ri})$, $i = 1, \ldots, s$, are row vectors of dimension r, and the scalar

$$(3.2) \qquad x_i = (x \circ 1_r)_i = \sum_{k=1}^{r} x_{ki},$$

where $\circ$ denotes matrix multiplication and $1_r = (1, \ldots, 1)^T$ is a unit vector of dimension r.

Let $m = (m_1, \ldots, m_r)^T$ be the vector of possible claim amounts. The scalar product of vectors is denoted by the bracket $\langle \cdot, \cdot \rangle$. Then the aggregate claims observable x may be indifferently identified as scalar product, sum of scalar products, and sum of scalars:

$$(3.3) \qquad x = \langle x \circ m, 1_s \rangle = \sum_{i=1}^{s} \langle x_i^T, m \rangle = \sum_{i=1}^{s} \sum_{k=1}^{r} x_{ki} m_k,$$

where $1_s = (1, \ldots, 1)^T$ is a unit vector of dimension s. The above notations are also defined for the random variable X instead of x and are used throughout. By assumption one has

$$(3.4) \quad f(x|\theta) = \Pr(\langle X \circ m, 1_s \rangle = x|\theta)$$

$$= \sum_{\langle x \circ m, 1_s \rangle = x} \Pr(X_{ki} = x_{ki}, k = 1, \ldots, r, i = 1, \ldots, s|\theta)$$

$$= \sum_{\langle x \circ m, 1_s \rangle = x} \prod_{i=1}^{s} \prod_{k=1}^{r} p(x_{ki}|\theta_i),$$

where the sum goes over all non-negative integer solutions x_{ki}, $k = 1, \ldots, r$, $i = 1, \ldots, s$, to the linear inhomogeneous Diophantine equation $\langle x \circ m, 1_s \rangle = x$, and $p(x_{ki}|\theta_i) = \Pr(X_{ki} = x_{ki}|\theta_i)$. For later use we set for short

$$(3.5) \qquad p(x|\theta) = \prod_{i=1}^{s} \prod_{k=1}^{r} p(x_{ki}|\theta_i).$$

Therefore the likelihood density is given by

$$(3.6) \qquad f(x|\theta) = \sum_{\langle x \circ m, 1_s \rangle = x} p(x|\theta).$$

To obtain the desired link with classical collective risk theory, let us show the following mathematical result.

Proposition 3.1

Assume that the stochastic system of aggregate claims (X, θ) satisfies the model assumptions (I) and (II) of Section 2. Then the likelihood density of aggregate claims is conditional compound Poisson of the form

$$(3.7) \qquad f(x|\theta) = \sum_{n=0}^{\infty} q(n|\theta) h^{*n}(x|\theta), \text{ with}$$

$$q(n|\theta) = e^{-\lambda(\theta)} \frac{\lambda(\theta)^n}{n!}, \qquad \lambda(\theta) = \sum_{i=1}^{s} \theta_i n_i,$$

where $n_i = \sum_{k=1}^{r} n_{ki}$ is the number of risk units producing claims with probability θ_i,

$$h(x|\theta) = \begin{cases} \dfrac{\lambda_k(\theta)}{\lambda(\theta)}, & \text{if } x = m_k, \ k = 1, \ldots, r, \\ 0, & \text{else}, \end{cases}$$

$$\lambda_k(\theta) = \sum_{i=1}^{s} \theta_i n_{ki}.$$

Proof:

(GERBER (1979), pp. 13-14). Since X_{ki} given θ_i is Poisson distributed with parameter $\theta_i n_{ki}$, the conditional moment generating function of the random variable $m_k X_{ki}$ is equal to

$$(3.8) \qquad M_{ki}(t, \theta_i) = E[e^{tm_k X_{ki}}|\theta_i] = e^{\theta_i n_{ki}(e^{tm_k} - 1)}.$$

With the conditional independence assumption of (I) one obtains

$$(3.9) \qquad M_x(t, \theta) = E[e^{tX}|\theta] = \prod_{i=1}^{s} \prod_{k=1}^{r} M_{ki}(t, \theta_i)$$

$$= \prod_{k=1}^{r} e^{\lambda_k(\theta)(e^{tm_k} - 1)} = e^{\lambda(\theta)\left(\sum_{k=1}^{r} \frac{\lambda_k(\theta)}{\lambda(\theta)} e^{tm_k} - 1\right)}.$$

But this is the conditional moment generating function of a conditional compound Poisson distribution with likelihood (3.7). The result is shown.

The above proof actually identifies the linear multivariate Poisson Gamma Bayesian model with the following *compound Poisson Gamma Bayesian collective model* of risk theory. One has

$$(3.10) \qquad\qquad\qquad X = \sum_{i=1}^{N} Y_i,$$

and the following model assumptions are fulfilled:

(I') Conditionally on θ the random variables $Y_1, \ldots, Y_N$, N are independent and the Y_k's are identically distributed, that is $Y_k = Y$ for all k. The random variable N depends upon θ through the likelihood density $q(n|\theta)$ and Y depends upon θ through $h(x|\theta)$ both defined in (3.7).

(II') The structural densities $u_i(\theta_i)$ are Gamma densities given by (2.4).

Research problem

It has been shown that the Bayesian model (I), (II) identifies with the Bayesian model (I'), (II'). In general, that is when $p(x_{ki}|\theta_i)$ and $q(n|\theta)$ are not conditionally Poisson distributed, the models (I), (II) and (I'), (II') will not coincide. It seems true that they will coincide only in the Poisson case.

4. BAYESIAN ANALYSIS OF THE LINEAR MULTIVARIATE POISSON GAMMA MODEL

Given statistically independent observations of the claims for the different amounts at risk, Bayesian analysis allows to up-date the aggregate claims model. Taking into account available past information, it is thus possible to make predictions about the "true" aggregate claims model associated to a portfolio of risks.

In the first subsection we derive the ubiquitous parameter posterior density $f(\theta|D)$ given a data set D. It allows to calculate the posterior-to-data expected value $E[g(\theta)|D]$ for any function $g(\theta)$ of the parameter vector θ. In particular one obtains $E[\theta|D]$.

In the second subsection we are interested in predictions about the random variable Y of future aggregate claims in the Bayesian model (I), (II). Given a

data set D, the predictive mean $E[Y|D]$ is directly obtained from the formula for $E[\theta|D]$, while for the calculation of the predictive density $f(y|D)$, defined by JEWELL (1974), one needs the formula for $f(\theta|D)$ obtained in the first subsection.

4.1 Posterior-to-data parameter estimation

The *observation likelihood density* of the Bayesian model (I), (II) has been derived in Section 3, formula (3.5). Let us write

$$(4.1) \qquad p(x|\theta) = \prod_{i=1}^{s} e^{-\theta_i n_i} \theta_i^{x_i} C(x_i), \quad \text{with}$$

$$(4.2) \qquad C(x_i) = \prod_{k=1}^{r} \frac{n_{ki}^{x_{ki}}}{x_{ki}!}$$

Consider now the *data likelihood density* $f(D|\theta)$ for an observation data matrix $D = (x^{(1)}, \ldots, x^{(n)})$ of dimension $sx(n \cdot r)$ containing n statistically indepedent observation matrices $x^{(j)} = (x_{ki}^{(j)})^T$ of dimension sxr, which represent the claims (same notation as in Section 2 with the additional superscript (j) numbering the observation).

Related quantities of interest are the row vectors $x_i^{(j)} = (x_{1i}^{(j)}, \ldots, x_{ri}^{(j)})$ of dimension r and the scalars

$$(4.3) \qquad x_i^{(j)} = \sum_{k=1}^{r} x_{ki}^{(j)}.$$

Consider the vector $D_i = (x_i^{(1)}, \ldots, x_i^{(n)})$ of dimension $n \cdot r$ such that $D = (D_1, \ldots, D_s)^T$. The scalar

$$(4.4) \qquad T_i = \sum_{j=1}^{n} x_i^{(j)}, \qquad i = 1, \ldots, s,$$

represents the total number of claims with unknown probability of occurrence θ_i, and turns out to be a sufficient statistic for the considered model. One obtains

$$(4.5) \qquad f(D|\theta) = \prod_{j=1}^{n} p(x^{(j)}|\theta) = \prod_{i=1}^{s} e^{-n\theta_i n_i} \theta_i^{T_i} \prod_{j=1}^{n} C(x_i^{(j)}).$$

To simplify set

$$(4.6) \qquad C(D) = \prod_{i=1}^{s} C(D_i) = \prod_{i=1}^{s} \prod_{j=1}^{n} C(x_i^{(j)}).$$

Then one has

(4.7)
$$f(\boldsymbol{D}|\theta) = C(\boldsymbol{D}) \prod_{i=1}^{s} e^{-n\theta_i n_i} \theta_i^{T_i}.$$

From this representation one gets the *data probability density* by evaluation of a multiple integral, which separates as follows:

(4.8) $f(\boldsymbol{D}) = \displaystyle\int\int f(\boldsymbol{D}|\theta)\, u(\theta)\, d\theta = C(\boldsymbol{D}) \prod_{i=1}^{s} \int \frac{\beta_i^{\alpha_i} \theta_i^{\alpha_i + T_i - 1}}{\Gamma(\alpha_i)} e^{-(\beta_i + nn_i)\theta_i} d\theta_i$

$$= C(\boldsymbol{D}) \prod_{i=1}^{s} \frac{\Gamma(\alpha_i + T_i)}{\Gamma(\alpha_i)} \frac{\beta_i^{\alpha_i}}{(\beta_i + nn_i)^{\alpha_i + T_i}}.$$

It follows that the *parameter posterior density* is given by

(4.9) $$f(\theta|\boldsymbol{D}) = \frac{f(\boldsymbol{D})\, u(\theta)}{f(\boldsymbol{D})} = \prod_{i=1}^{s} \text{Gamma}\,(\theta_i; a_i, b_i),\quad \text{with}$$

(4.10) $a_i = \alpha_i + T_i, \qquad b_i = \beta_i + nn_i, \qquad i = 1, \dots, s.$

One sees that it is of the same form as the structural density with up-dated hyperparameters.

At this stage the posterior-to-data expected value $E[g(\theta)|\boldsymbol{D}]$ of any function $g(\theta)$ is obtained by evaluation of the multiple integral

(4.11) $$E[g(\theta)|\boldsymbol{D}] = \int\int g(\theta) f(\theta|\boldsymbol{D})\, d\theta.$$

In particular one gets

(4.12) $$E[\theta|\boldsymbol{D}] = (\dots, E_\theta[\theta_i|\boldsymbol{D}], \dots) = \left(\dots, \int\int \theta_i f(\theta|\boldsymbol{D})\, d\theta, \dots \right)$$

$$= \left(\dots, \frac{a_i}{b_i}, \dots \right).$$

4.2 Posterior-to-data predictions

Future observations of the aggregate claims and related quantities are written with the letter y instead of x. Our object of study is now the random variable

(4.13) $$Y = \langle Y \circ \boldsymbol{m}, 1_s \rangle = \sum_{k=1}^{r} m_k \sum_{i=1}^{s} Y_{ki}$$

representing future aggregate claims in the Bayesian model (I), (II). Immediate

and of primary importance is the *predictive mean of the aggregate claims*. Using (4.12) it is obtained as follows:

$$(4.14) \qquad E[Y|D] = E_\theta[E[Y|\theta, D]] = E_\theta\left[\sum_{i=1}^{s} \theta_i \sum_{k=1}^{r} m_k n_{ki} | D\right]$$

$$= \sum_{i=1}^{s} E_\theta[\theta_i|D] \sum_{k=1}^{r} m_k n_{ki} = \sum_{i=1}^{s} \frac{a_i}{b_i} \sum_{k=1}^{r} m_k n_{ki}.$$

If D is the empty set, corresponding to the situation in which no observation is available about the portfolio of risk, one has

$$(4.15) \qquad E[Y] = E_\theta[E[Y|\theta]] = \sum_{i=1}^{s} \frac{\alpha_i}{\beta_i} \sum_{k=1}^{r} m_k n_{ki},$$

clearly the same expression as given in (2.5). In the case of life insurance this means that the expected value of aggregate claims is evaluated according to the life table when no data experience is available.

If D is a non empty set, one has $n \geq 1$, and using (4.10) the formula (4.14) may be rearranged to yield the following *multivariate credibility formula*:

$$(4.16) \quad E[Y|D] = \sum_{i=1}^{s} \left\{(1-Z_i) \cdot E[\theta_i n_i] + Z_i \frac{T_i}{n}\right\} \cdot \left\{\frac{1}{n_i} \sum_{k=1}^{r} m_k n_{ki}\right\}, \text{ where}$$

$Z_i = \dfrac{nn_i}{\beta_i + nn_i}$ is a credibility factor,

$E[\theta_i n_i] = n_i \dfrac{\alpha_i}{\beta_i}$ is the expected number of claims with unknown probability of occurrence θ_i,

$\dfrac{T_i}{n}$ is the observed mean number of claims with unknown probability of occurrence θ_i

$\dfrac{1}{n_i} \sum_{k=1}^{r} m_k n_{ki}$ is the mean amount at risk for risk units subject to claims with probability θ_i.

To evaluate predictive or exact Bayesian stop-loss premiums, one needs besides the predictive mean also the *predictive density* $f(y|D)$ of the future aggregate claims. Due to the simple structure of our model, it is not difficult to obtain an explicit analytical formula for the predictive density. With the future observation of aggregate claims $y = \langle y \circ m, 1_s \rangle$ one has from (3.6) and (4.1)

$$(4.17) \qquad f(y|\theta) = \sum_{\langle y \circ m, 1_s \rangle = y} p(y|\theta) = \sum_{\langle y \circ m, 1_s \rangle = y} \prod_{i=1}^{s} C(y_i) e^{-\theta_i n_i} \theta_i^{y_i}.$$

The predictive density, which by the way is known to be the unbiased least-squares estimate of $f(y|\theta_T)$ for the true value θ_T of θ (e.g. JEWELL (1974)), is obtained by evaluating the multiple integral

$$(4.18) \qquad f(y|D) = \int \int f(y|\theta) f(\theta|D)\, d\theta.$$

Using (4.9) and (4.17) the multiple integral separates as follows:

$$(4.19) \qquad f(y|D) = \sum_{\langle y \circ m, 1_s \rangle = y} \prod_{i=1}^{s} C(y_i) \int \frac{b_i^{a_i} \theta_i^{a_i + y_i - 1}}{\Gamma(a_i)}\, e^{-(b_i + n_i)\theta_i}\, d\theta_i$$

$$= \sum_{\langle y \circ m, 1_s \rangle = y} \prod_{i=1}^{s} C(y_i) \frac{\Gamma(a_i + y_i)}{\Gamma(a_i)} \frac{b_i^{a_i}}{(b_i + n_i)^{a_i + y_i}}$$

Using (4.2) and rearranging one obtains the analytical representation

$$(4.20) \quad f(y|D) = \sum_{\langle y \circ m, 1_s \rangle = y} \prod_{i=1}^{s} C(y_i) \frac{\Gamma(a_i + y_i)}{\Gamma(a_i)} \left(\frac{b_i}{b_i + n_i} \right)^{a_i} \prod_{k=1}^{r} \frac{\left(\dfrac{n_{ki}}{b_i + n_i} \right)^{y_{ki}}}{y_{ki}!}$$

5. A RECURSIVE ALGORITHM FOR PREDICTIVE STOP-LOSS PREMIUMS

In this section we assume for simplicity that $m_k = k$, $k = 1, \ldots, r$, that is $m = (1, \ldots, r)$. Other notations remain the same.

Consider the new random variables

$$Y^i = \sum_{k=1}^{r} k Y_{ki}, \quad \text{with realizations}$$

$$y^i = \sum_{k=1}^{r} k y_{ki}, \quad i = 1, \ldots, s.$$

The set of all non-negative integer solutions to the Diophantine equation

$$(5.1) \qquad \langle y \circ m, 1_s \rangle = \sum_{i=1}^{s} \sum_{k=1}^{r} k y_{ki} = y$$

is in one-to-one correspondence with the set of all non-negative integer solutions to the simultaneous Diophantine equations

$$(5.2) \qquad \sum_{i=1}^{s} y^i = y, \quad \langle y_i^T, m \rangle = \sum_{k=1}^{r} k y_{ki} = y^i, \quad i = 1, \ldots, s.$$

Moreover each Diophantine equation

$$\langle y_i{}^T, m \rangle = y^i, \qquad i = 1, \ldots, s,$$

is in one-to-one correspondence with the infinite number of simultaneous Diophantine equations

(5.3) $\qquad \langle y_i{}^T, m \rangle = y^i, \qquad y_i = \sum_{k=1}^{r} y_{ki} = k_i, \qquad k_i = 0, 1, 2, \ldots$

Using these correspondences the expression (4.20) for the predictive density can be rearranged as follows

(5.4) $\quad f(y|D) = \sum_{\Sigma y^i = y} \sum_{k_1, \ldots, k_s = 0}^{\infty} \prod_{i=1}^{s} \frac{\Gamma(a_i + k_i)}{\Gamma(a_i)} \left(\frac{b_i}{b_i + n_i} \right)^{a_i} \cdot S_{k_1, \ldots, k_s}(y^1, \ldots, y^s),$

$$S_{k_1, \ldots, k_s}(y^1, \ldots, y^s) = \sum_{\substack{\langle y_i{}^T, m \rangle = y^i \\ y_i = k_i \\ i = 1, \ldots, s}} \prod_{i=1}^{s} \prod_{k=1}^{r} \frac{\left(\dfrac{n_{ki}}{b_i + n_i} \right)^{y_{ki}}}{y_{ki}!}$$

Using Poisson likelihoods this last expression can be written as

(5.5) $\quad S_{k_1, \ldots, k_s}(y^1, \ldots, y^s) = \sum_{\substack{\langle y_i{}^T, m \rangle = y^i \\ y_i = k_i \\ i = 1, \ldots, s}} \prod_{i=1}^{s} e^{\frac{n_i}{b_i + n_i}} \prod_{k=1}^{r} \mathrm{Po}\left(y_{ki}; \frac{n_{ki}}{b_i + n_i} \right)$

$$= \prod_{i=1}^{s} e^{\frac{n_i}{b_i + n_i}} \cdot \left\{ \sum_{\substack{\langle y_i{}^T, m \rangle = y^i \\ y_i = k_i \\ i = 1, \ldots, s}} \prod_{k=1}^{r} \mathrm{Po}\left(y_{ki}; \frac{n_{ki}}{b_i + n_i} \right) \right\}.$$

Let us show how an expression in curly brackets can be evaluated. Each of these sums defines a function of the form

(5.6) $\qquad g_k(y) = \sum_{x \in S_{y,k}} \prod_{j=1}^{r} \mathrm{Po}(x_j; \lambda_j), \qquad \lambda_j > 0$ the Poisson parameter,

$$S_{y,k} = \left\{ x \in S_y : \sum_{j=1}^{r} x_j = k \right\}, \qquad k = 0, 1, \ldots, y,$$

$$S_y = \left\{ x = (x_1, \ldots, x_r) : \sum_{j=1}^{r} j x_j = y \right\}.$$

Applying algorithm 2 in the Appendix, one finds that $g_k(y)$ can be computed recursively using the formulas

$$(5.7) \quad g_0(y) = e^{-\lambda} \cdot \delta(y), \quad \lambda = \sum_{j=1}^{r} \lambda_j, \quad kg_k(y) = \sum_{i=1}^{k} i(c_i^* g_{k-i})(y),$$

where the functions $c_i(z)$ are defined by

$$(5.8) \qquad c_i(z) = \begin{cases} c(i,j), & \text{if} \quad z = ij, \quad j = 1, \dots, r, \quad i = 1, 2, \dots, \\ 0, & \text{else}, \end{cases}$$

with $c(i,j)$ determined by the recursive relation

$$(5.9) \qquad ic(i,j) \text{ Po }(0; \lambda_j) = i \text{ Po }(i; \lambda_j) - \sum_{k=1}^{i-1} kc(k,j) \text{ Po }(i-k; \lambda_j).$$

One checks immediately that

$$c(i,j) = \begin{cases} \lambda_j, & i = 1, \quad j = 1, \dots r \\ 0, & \text{else}. \end{cases}$$

Hence one has $c_i(z) = 0$, $i > 1$, and

$$c_1(j) = \begin{cases} \lambda_j, & j = 1, \dots, r \\ 0, & \text{else}. \end{cases}$$

Therefore one obtains the following recursive formula

$$(5.10) \qquad g_0(y) = e^{-\lambda} \delta(y), \quad g_k(0) = 0,$$

$$kg_k(y) = \sum_{j=1}^{\min(r,y)} \lambda_j g_{k-1}(y-j), \quad k = 1, 2, \dots$$

Apply this result to the recursive evaluation of the sum in (5.5) above. Define functions $g_k^i(y)$, $i = 1, \dots, s$, recursively as follows:

$$(5.11) \qquad g_0^i(y) = e^{-\frac{n_i}{b_i+n_i}} \delta(y), \quad g_k^i(0) = 0,$$

$$kg_k^i(y) = \sum_{j=1}^{\min(r,y)} \frac{n_{ji}}{b_i+n_i} g_{k-1}^i(y-j), \quad k = 1, 2, \dots$$

For use in numerical evaluation note that

$$(5.12) \qquad g_k^i(y) = 0, \quad \text{if} \quad y = 0, 1, \dots, k-1 \quad \text{or} \quad y > k \cdot r,$$
$$g_k^i(y) \neq 0, \quad \text{if} \quad y = k, k+1, \dots, k \cdot r.$$

With (5.11) one obtains the following formula

$$(5.13) \qquad S_{k_1, \dots, k_s}(y^1, \dots, y^s) = \prod_{i=1}^{s} e^{\frac{n_i}{b_i+n_i}} g_{k_i}^i(y^i).$$

Since $g^i_{k_i}(y^i) = 0$ for $k_i > y^i$, it follows that

$$(5.14) \quad f(y|D) = \sum_{\Sigma\, y^i = y} \prod_{i=1}^{s} \left\{ \sum_{k=0}^{y^i} \frac{\Gamma(a_i+k)}{\Gamma(a_i)} \left(\frac{b_i}{b_i+n_i} \right)^{a_i} e^{\frac{n_i}{b_i+n_i}} g^i_k(y^i) \right\}.$$

Remark

In the special case $s = r = 1$, one recovers the well-known negative binomial density

$$f(y|D) = \frac{\Gamma(a+y)}{\Gamma(a)\, y!} \left(\frac{b}{b+n} \right)^{a} \left(\frac{n}{b+n} \right)^{y}, \quad y = 0, 1, 2, \dots$$

Using algorithm 1 in the Appendix, it is possible now to evaluate formula (5.14) in a recursive manner as follows. Define

$$(5.15) \quad f_i(x) = \sum_{k=0}^{x} \frac{\Gamma(a_i+k)}{\Gamma(a_i)} \left(\frac{b_i}{b_i+n_i} \right)^{a_i} e^{\frac{n_i}{b_i+n_i}} g^i_k(x).$$

Then one has from (5.14)

$$(5.16) \quad f(y|D) = \sum_{\Sigma\, y^i = y} \prod_{i=1}^{s} f_i(y^i).$$

Using algorithm 1 one obtains the following recursive algorithm for the exact evaluation of the predictive density:

$$(5.17) \quad f(0|D) = \prod_{i=1}^{s} \left(\frac{b_i}{b_i+n_i} \right)^{a_i},$$

$$yf(y|D) = \sum_{j=1}^{y} j \cdot \left\{ \sum_{k=1}^{s} c(j,k) \right\} \cdot f(y-j|D), \quad y = 1, 2, \dots,$$

$$(5.18) \quad jc(j,k) f_k(0) = jf_k(j) - \sum_{i=1}^{j-1} ic(i,k) f_k(j-i),$$

the $f_k(j)$'s being themselves recursively computed using formulas (5.11) and (5.15). Note that according to (5.12) only summands for which $g^i_k(y) \neq 0$ are calculated in (5.15). It is important to remark that the numerical process to evaluate $f(y|D)$ involves a *three-stage nested recursive scheme*. Indeed the functions $g^i_k(y)$, $c(j,k)$ and $f(y|D)$ are successively recursively computed. However the computation process needs only finitely many operations. The numerical illustration of the next section is based on a concrete computer implementation of the present algorithm.

Formulas for the recursive evaluation of *predictive stop-loss premiums* are now easily obtained. For each non-negative integer T let

$$(5.19) \qquad SL(T|\boldsymbol{D}) = E[(Y-T)_+|\boldsymbol{D}]$$

be the predictive stop-loss premium to the priority T. Then one has the recursive relations

$$(5.20) \qquad SL(0|\boldsymbol{D}) = E[Y|\boldsymbol{D}],$$

$$SL(T+1|\boldsymbol{D}) = SL(T|\boldsymbol{D}) - 1 + F(T|\boldsymbol{D}), \qquad T = 0, 1, 2, \ldots,$$

$$F(0|\boldsymbol{D}) = f(0|\boldsymbol{D}),$$

$$F(T|\boldsymbol{D}) = F(T-1|\boldsymbol{D}) + f(T|\boldsymbol{D}), \qquad T = 1, 2, \ldots$$

Note that the predictive mean $E[Y|\boldsymbol{D}]$ is computed according to the credibility formula (4.16).

6. A NUMERICAL EXAMPLE

The following tables are based on the simple example 2.2. They illustrate several extreme situations of interest.

The needed structural parameters α_i, β_i, $i = 1, 2, 3$, have been estimated in Section 2. Given an n-year observation period, the up-dated hyperparameters are

$$b_i = \beta_i + nn_i, \quad a_i = \alpha_i + T_i, \quad i = 1, \ldots, s,$$

where T_i is the number of observed deaths in age class i over n years.

If in the linear multivariate model of Section 2 the θ_i's are assumed to be known with certainty (e.g. the traditional q_x of the life table) and the X_{ki} are independent and Poisson (θ_i) distributed, one gets the traditional collective model of risk theory, that is the usual *compound Poisson approximation* of the exact individual model of aggregate claims. In Table 1 we compare this classical model with the *no data predictive density* obtained by setting $n = 0$, $T_i = 0$, $i = 1, 2, 3$. Table 2 shows the dependency of the predictive distribution and stop-loss premiums upon claims experience. The time of observation is fixed to $n = 5$ years and T_i varies. In Table 3 the dependency upon time is illustrated assuming an extreme no claims experience over several periods of observation.

Concerning the displayed figures, note that sometimes, due to rounding effects, the cumulative probability may be one, while the corresponding stop-loss premium may not be zero.

TABLE 1

COMPOUND POISSON (CPM) VS. NO DATA PREDICTIVE MODEL (DPM)

	CPM	DPM
Expected value	3'973'500	3'973'500
Standard deviation	2'697'638	2'755'165

Aggregate claims	Cumulative probability		Stop-loss premiums	
	CPM	DPM	CPM	DPM
10 Δ	0.7131	0.7120	680'833	703'125
20 Δ	0.9769	0.9743	41'324	48'057
30 Δ	0.9993	0.9990	1'120	1'618
40 Δ	1.0000	1.0000	16	32

TABLE 2

DEPENDENCY ON EXPERIENCE BY FIXED TIME OF OBSERVATION ($n = 5$)

	Claims experience					
T_1	0	0	1	1	2	2
T_2	0	1	2	3	4	4
T_3	0	3	5	8	10	14

Predictive mean					
3'180'542	3'437'942	3'673'506	3'930'906	4'166'469	4'429'742

Standard deviation					
2'454'680	2'553'414	2'637'293	2'729'429	2'808'054	2'897'092

Aggregate claims	Predictive cumulative probability					
0	0.13568	0.11602	0.09920	0.08483	0.07253	0.06202
10 Δ	0.81224	0.78071	0.75113	0.71777	0.68700	0.65213
20 Δ	0.98971	0.98582	0.98155	0.97584	0.96979	0.96179
30 Δ	0.99976	0.99961	0.99941	0.99911	0.99875	0.99819
40 Δ	1.00000	0.99999	0.99999	0.99998	0.99997	0.99995

Aggregate claims	Predictive stop-loss premiums					
0	3'180'512	3'437'942	3'673'506	3'930'906	4'166'469	4'429'742
10 Δ	394'778	483'804	572'673	680'274	785'832	914'391
20 Δ	17'059	24'405	32'805	44'538	57'477	75'378
30 Δ	352	590	904	1'409	2'041	3'037
40 Δ	4	8	14	26	42	71

TABLE 3

DEPENDENCY ON TIME OF OBSERVATION FOR NO CLAIMS EXPERIENCE

	Observation period					
n	1	2	3	4	5	10

	Credibility factors					
Z_1	0.04760	0.09087	0.13038	0.16660	0.19992	0.33322
Z_2	0.04757	0.09081	0.13031	0.16651	0.19982	0.33308
Z_3	0.04746	0.09062	0.13004	0.16619	0.19945	0.33257

			Predictive mean			
	3'784'779	3'613'172	3'456'452	3'312'762	3'180'542	2'651'420

			Standard deviation			
	2'686'154	2'622'212	2'562'622	2'506'912	2'454'680	2'235'012

Aggregate claims			Predictive cumulative probability			
0	0.09364	0.10401	0.11451	0.12507	0.13568	0.18815
10 Δ	0.73655	0.75856	0.77836	0.79618	0.81224	0.87230
20 Δ	0.97885	0.98249	0.98541	0.98778	0.98971	0.99532
30 Δ	0.99926	0.99945	0.99959	0.99969	0.99976	0.99993
40 Δ	0.99999	0.99999	0.99999	1.00000	1.00000	1.00000

Aggregate claims			Predictive stop-loss premiums			
0	3'784'779	3'613'172	3'456'452	3'312'762	3'180'542	2'651'420
10 Δ	621'345	551'480	491'478	439'688	394'778	241'494
20 Δ	38'469	31'048	25'251	20'685	17'059	7'106
30 Δ	1'164	849	626	467	352	98
40 Δ	21	14	9	6	4	1

APPENDIX

In order to be self-contained, as well as for the convenience of the reader, the main results of HÜRLIMANN (1990a) are reproduced without proof.

Let $X_j, j = 1, 2, \ldots,$ be mutually independent random variables taking values in the non-negative integers. Let $f_i(i) = \Pr(X_j = i)$ and assume that $f_j(0) > 0$. Consider a linear combination of random variables

$$(A.1) \qquad Y = \sum_{k=1}^{\infty} a_k X_k, \qquad a_k \in N_0.$$

Assume that $a_1 \geq a_2 \geq \ldots \geq a_k > 0,\ k = 1, 2, \ldots$. Given the a_k's and the $f_j(i)$'s, the following two-stage nested recursive algorithm for the exact computation of $f(y) = \Pr(Y = y)$ is available (Theorem 1 in HÜRLIMANN (1990a)).

Algorithm 1

Under the above conditions one has

$$(A.2) \qquad f(0) = \prod_{j \in I_\infty} f_j(0)$$

If $y > 0$ then one has

(A.3) $$yf(y) = \sum_{k \in I_y} a_k \sum_{s=1}^{[y/a_k]} sc(s, k) f(y - sa_k)$$

(A.4) $$= \sum_{s=1}^{y} s \sum_{k \in I_{[y/s]}} a_k c(s, k) f(y - sa_k).$$

In these formulas

(A.5) $$I_y = \{k : 0 < a_k \le y\},$$

(A.6) $$I_\infty = \{k : a_k > 0\},$$

(A.7) $[y]$ is the greatest integer contained in y,

and $c(s, k)$ is determined by the recursive relation

(A.8) $$sc(s, k) f_k(0) = sf_k(s) - \sum_{j=1}^{s-1} jc(j, k) f_k(s - j).$$

The apparent complexity of Algorithm 1 is reduced by abstraction as follows. For each $s = 1, 2, \ldots$, define

(A.9) $$c_s(z) = \begin{cases} c(s, k), & \text{if } z = sa_k \text{ for some } k, \\ 0, & \text{otherwise.} \end{cases}$$

Note the index notation error in HÜRLIMANN (1990a), where the indices s, k must be exchanged in order to yield correct practical results. With the change of variable $z = sa_k$ the relation (A.4) can be rewritten as

(A.10) $$yf(y) = \sum_{z=1}^{y} z \sum_{s=1}^{\infty} c_s(z) f(y - z).$$

Then set $\lambda = -\ln\{f(0)\}$ and define the function

(A.11) $$h(z) = \frac{1}{\lambda} \sum_{s=1}^{\infty} c_s(z).$$

Then Algorithm 1 is equivalent to the simple *Panjer-like recursion*

(A.12) $$f(0) = e^{-\lambda},$$

$$yf(y) = \lambda \sum_{z=1}^{y} zh(z) f(y - z).$$

The second important computational result contained in HÜRLIMANN (1990a), theorem 2, concerns an alternative recursive procedure to evaluate

$f(y)$. For this one considers for each $y \in N_0$ the Diophantine set

$$(A.13) \quad S_y = \left\{ x = (0, \ldots, 0, x_{k_y+1}, x_{k_y+2}, \ldots, x_{k_0}, x_{k_0+1}, \ldots) : x_k \in N_0, \sum_{k \in I_y} a_k x_k = y \right\}$$

consisting of all non-negative integer solutions of the linear equation

$$\sum_{k=1}^{\infty} a_k x_k = y.$$

In this definition

$$k_x = \begin{cases} \max \{k : a_k > x\}, & \text{if } x < a_1, \\ 0, & \text{if } x \geq a_1. \end{cases}$$

and it follows that

$$I_y = \{k_y + 1, k_y + 2, \ldots, k_0\}.$$

From the independence of the X_j's, one deduces that

$$(A.14) \qquad f(y) = c_y \sum_{x \in S_y} \prod_{j \in I_y} f_j(x_j), \qquad c_y = \prod_{j=1}^{k_y} f_j(0).$$

Consider the following subsets of S_y:

$$(A.15) \qquad S_{y,k} = \left\{ x \in S_y : \sum_{j \in I_y} x_j = k \right\}, \qquad k = 0, 1, \ldots, y,$$

and define functions

$$(A.16) \qquad g_k(y) = c_y \sum_{x \in S_{y,k}} \prod_{j \in I_y} f_j(x_j), \qquad k \in N_0,$$

with the convention that $g_k(y) = 0$ whenever the sum is empty. From (A.14) one sees that

$$(A.17) \qquad f(y) = \sum_{k=0}^{\infty} g_k(y).$$

Algorithm 2 shows how to compute $f(y)$ by the successive recursive evaluation of $g_k(y)$ using the $c_k(y)$'s defined in (A.9).

Algorithm 2
One has the following recursive formula
$$(A.18) \qquad\qquad g_0(y) = f(0) \, \delta(y),$$

$$k g_k(y) = \sum_{j=1}^{k} j (c_j^* g_{k-j})(y).$$

It is important to observe that Algorithm 2 is a Panjer-like recursion in the space of discrete distributions with addition and convolution as operations. It is worthwhile to mention that Algorithm 2 generalizes the results obtained through the shovelboard approach of VAN KLINKEN (1960) recently revisited by ALTING VON GEUSAU (1990).

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