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Indicators, Chains, Antichains, Ramsey Property

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Abstract. We introduce two Ramsey classes of finite relational structures. The first class contains finite structures of the form $\left(A, (I_i)_{i=1}^n, \leq, (\preceq_i)_{i=1}^n\right)$, where $\leq$ is a total ordering on A and $\preceq_i$ is a linear ordering on the set $\{a \in A : I_i(a)\}$. The second class contains structures of the form $\left(a, \leq, (i_i)_{i=1}^n, \preceq\right)$, where $(A, \leq)$ is a weak ordering and $\preceq$ is a linear ordering on A such that A is partitioned by $\{a \in A : I_i(a)\}$ into maximal chains in the partial ordering $\leq$ and each $\{a \in A : I_i(a)\}$ is an interval with respect to $\preceq$.

1 Introduction

We consider a signature L and a class $\mathcal{K}$ of finite structures in the signature L. Let $\mathbb{A}$ and $\mathbb{B}$ be structures in $\mathcal{K}$. If $\mathbb{A}$ and $\mathbb{B}$ are isomorphic, then we write $\mathbb{A} \cong \mathbb{B}$. If there is an embedding from $\mathbb{A}$ into $\mathbb{B}$, we write $\mathbb{A} \hookrightarrow \mathbb{B}$, and if $\mathbb{A}$ is a substructure of $\mathbb{B}$, then we write $\mathbb{A} \leq \mathbb{B}$. The collection of all substructures of $\mathbb{B}$ isomorphic to $\mathbb{A}$ is denoted by $\binom{\mathbb{B}}{\mathbb{A}} = \{\mathbb{C} \leq \mathbb{B} : \mathbb{C} \cong \mathbb{A}\}$. If $\mathbb{C} \in \mathcal{K}$ and r is a natural number such that for any coloring

$$c\colon \binom{\mathbb{B}}{\mathbb{A}} \longrightarrow \{1, \dots, r\},$$

there is $\mathbb{B}' \in \binom{\mathbb{C}}{\mathbb{B}}$ such that the restriction $c\restriction\binom{\mathbb{B}'}{\mathbb{A}}$ is constant, then we write

$$\mathbb{C} \longrightarrow (\mathbb{B})_r^{\mathbb{A}}.$$

We say that the class $\mathcal{K}$ satisfies the *Ramsey property (RP)*or that $\mathcal{K}$ is a *Ramsey class* if for all $\mathbb{A}, \mathbb{B} \in \mathcal{K}$, and all natural numbers r there is $\mathbb{C} \in \mathcal{K}$ such that $\mathbb{C} \rightarrow (\mathbb{B})_r^{\mathbb{A}}$.

This paper is motivated by questions from the structural Ramsey theory and by the analysis in [10]. In the sequel we consider the following two problems:

- Most examples of Ramsey classes are classes of structures with linear orderings; see [3–6]. In all of these examples we have structures with only one linear ordering, for example, linearly ordered graphs or linearly ordered hypergraphs. So it is natural to ask for *Ramsey classes of structures with more than one linear ordering*. The Ramsey property for the class of finite sets with two linear orderings is given in [9], and it is generalized to the class of finite sets with finite number of linear orderings in [11]. In this paper we consider a Ramsey property for the class of finite sets with a finite number of linear orderings that are not necessary total. An example of such a class is given in Theorem 1.1.

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- Let $\mathcal{A}$ be a list of finite structures in a given signature L and let $\mathcal{F}$ be a class of finite structures in L. Let $\mathcal{F}(\mathcal{A})$ be the class of finite structures $\mathbb{B} \in \mathcal{F}$ with the property that there is no $\mathbb{A} \in \mathcal{A}$ satisfying $\mathbb{A} \hookrightarrow \mathbb{B}$. In this case we say that $\mathcal{F}(\mathcal{A})$ is given by forbidden configurations. The class of ordered triangle-free graphs and the class of finite ordered metric spaces are examples of such Ramsey classes; see [3, 6]. In this paper we extend the list of *Ramsey classes with forbidden configuration*; see Theorem 1.2 and an explanation after the statement of Theorem 1.2.

In order to simplify our exposition we fix notation. For a given set A, we denote the cardinality of the set A by $|A|$, and we denote the collection of all linear orderings on A by $\mathrm{lo}(A)$. If $\leq$, $\preceq$, and $\sqsubseteq$ denote linear orderings, then we denote by $<$, $\prec$, and $\sqsubset$ their strict parts respectively. For a given natural number $n \geq 1$ we denote the set $\{1, \ldots, n\}$ by $[n]$. We assume that all classes of finite structures are closed under isomorphic images.

Let L and L' be two signatures such that $L \subset L'$. Let $\mathbb{A}$ and $\mathbb{A}'$ be structures in L and L', respectively, defined on the same set. If the interpretation of the symbols from L is the same in both structures, then we say that $\mathbb{A}$ is a *reduct* of $\mathbb{A}'$ or that $\mathbb{A}'$ is an *expansion* of $\mathbb{A}$, and write $\mathbb{A} = \mathbb{A}'|L$. We denote the class of finite substructures embeddable into a given structure $\mathbb{A}$ by $\mathrm{Age}(\mathbb{A})$.

For a given natural number n we consider unary relational symbols $(I_i)_{i=1}^n$ and $n+1$ binary relational symbols $\leq, (\preceq_i)_{i=1}^n$. We consider the class $\mathcal{OM}_n$ that contains structures $\mathbb{A} = (A, (I_i^A)_{i=1}^n, \leq^A, (\preceq_i^A)_{i=1}^n)$ with the property that for every $i \in [n]$,

(a) A is a non empty finite set,
(b) I_i^A is a unary relation on A,
(c) $\leq^A \in \mathrm{lo}(A)$,
(d) $\preceq_i^A \in \mathrm{lo}(\{a \in A : I_i^A(a)\})$.

We prove the following result.

Theorem 1.1 *For natural number $n \geq 1$, the class $\mathcal{OM}_n$ is a Ramsey class.*

We recall the definition of the poset $(C_n, \leq^{C_n})$ from the Schmerl list in [7]. We point out that this poset is denoted by $(C_n, <)$ in [7]. Let $\mathbb{Q}$ be the set of rational numbers, let $n \geq 1$ be a natural number, and let $C_n = [n] \times \mathbb{Q}$. We use $\leq$ to denote the natural orderings on $\mathbb{Q}$ and $\mathbb{N}$. We define partial ordering $\leq^{C_n}$ on the set C_n such that for all $(i, x), (j, y) \in C_n$ we have

$$(i, x) \leq^{C_n} (j, y) \iff (x < y \text{ or } (i = j \text{ and } x = y)).$$

Therefore, we have poset $\mathbb{C}_n' = (C_n, \leq^{C_n})$. In the structure $\mathbb{C}_n'$ each point belongs to a maximal antichain of size n. For a fixed $x \in \mathbb{Q}$, the set $\{(i, x) : i \in [n]\}$ is a maximal antichain in $\mathbb{C}_n'$. There are automorphisms of $\mathbb{C}_n'$ that permute each maximal antichain. In order to avoid such automorphisms we consider the structure $\mathbb{C}_n = (C_n, \leq^{C_n}, (I_i^{C_n})_{i=1}^n)$ such that for all $i \in [n]$, $I_i^{C_n}$ is a unary relation on C_n given by

$$I_i^{C_n}((j, y)) \iff i = j$$

for $(j, y) \in C_n$. Note that we have a partition $C_n = \bigcup_{i=1}^n \{x : I_i^{C_n}(x)\}$ with the property that $\{x : I_i^{C_n}(x)\}$ is a maximal chain with respect to $\leq^{C_n}$ for all $i \in [n]$.

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We consider the class $\mathcal{C}_n = \mathrm{Age}(\mathbb{C}_n)$, and we point out that the structures from the class $\mathrm{Age}(\mathbb{C}'_n)$ are called *weak orderings*.

Let $\mathbb{A} = (A, \leq^A, (I_i^A)_{i=1}^n)$ be a structure from $\mathcal{C}_n$. We say that $\preceq \in \mathrm{lo}(A)$ is *convex on* $\mathbb{A}$ if for all $i \in [n]$ and all x, y, z from A we have

$$I_i^A(x), I_i^A(z), x \preceq y \preceq z \Longrightarrow I_i^A(y).$$

The set of convex linear orderings on $\mathbb{A}$ we denote by $co(\mathbb{A})$. Adding arbitrary linear orderings that are convex, we have the class

$$\mathcal{COC}_n = \left\{ \left(A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A\right) : \left(A, \leq^A, (I_i^A)_{i=1}^n\right) \in \mathcal{C}_n, \preceq^A \in co(A, \leq^A, \left(I_i^A\right)_{i=1}^n)\right) \right\}.$$

Note that for $n = 1$, the class $\mathcal{COC}_1$ can be seen as the class of finite sets with two linear orderings (see [9]), so $\mathcal{COC}_1$ is a Ramsey class.

Theorem 1.2 *For a natural number n, the class $\mathcal{COC}_n$ is a Ramsey class.*

Now we explain how $\mathcal{COC}_n$ is a class of structures with forbidden configurations; *i.e.,* it is of the form $\mathcal{F}(\mathcal{A})$. We take $\mathcal{F}$ to be the class of finite structures $(A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A)$ such that $(A, \leq^A) \in \mathcal{C}_n$, $\preceq^A \in \mathrm{lo}(A)$, and I_i^A is an unary relation on A for each $i \in [n]$. The list $\mathcal{A}$ contains the following structures:

(a) For every $I \subseteq [n]$ there is an $\mathbb{A}_I = (A_I, \leq^{A_I}, (I_i^{A_I})_{i=1}^n, \preceq^{A_I}) \in \mathcal{A}$, where $A_I = \{a_I\}$ and $I_i^{A_I}(a_I) \Leftrightarrow i \in I$. Note that in this case we allow $I = \varnothing$.

(b) For all distinct $k, l \in [n]$ and all $t \in [5]$ we have

$$\mathbb{A}_{k,l,t} = (A_{k,l,t}, \leq^{A_{k,l,t}}, (I_i^{A_{k,l,t}})_{i=1}^n, \preceq^{A_{k,l,t}}) \in \mathcal{A},$$

where the following hold:

$$A_{k,l,t} = \{a_{k,1,t}, a_{k,2,t}, a_{l,0,t}\},$$
$$I_i^{A_{k,l}}(a_{k,1,t}) \Longleftrightarrow i = k, \quad I_i^{A_{k,l}}(a_{k,2,t}) \Longleftrightarrow i = k, \quad I_i^{A_{k,l}}(a_{l,0,t}) \Longleftrightarrow i = l,$$
$$t \neq t' \Longrightarrow \mathbb{A}_{k,l,t} \restriction \{\leq, (I_i)_{i=1}^n\} \ncong \mathbb{A}_{k,l,t} \restriction \{\leq, (I_i)_{i=1}^n\},$$
$$a_{k,1,t} \preceq^{A_{k,l,t}} a_{l,0,t} \preceq^{A_{k,l,t}} a_{k,2,t}.$$

By forbidding embeddability of the structures $\mathbb{A}_I$ we ensure that an underlying set of a structure from $\mathcal{F}(\mathcal{A})$ is partitioned by indicators. That structures in $\mathcal{F}(\mathcal{A})$ have convex linear orderings is provided by forbidding embeddability of the structures $\mathbb{A}_{k,l,t}$. Note that the list $\mathcal{A}$ is an irreducible system according to the definition in [6, p. 184]. In contrast with [6, Theorem A], where the Ramsey class is obtained starting with the $\mathrm{Soc}(\Delta)$, in Theorem 1.2 we start with a subset of $\mathrm{Soc}(\Delta)$; see [6, p. 184] for the definition of $\mathrm{Soc}(\Delta)$.

Our proofs of Ramsey statements are based on the cross-construction developed in [8]. The idea is to construct structures on a product where each coordinate gives some information about the structures. Note that some of these proofs can be conducted by using the partite construction developed in [5, 6].

2 Background

Let X be a non empty set, and let k, l, m, r be natural numbers. Then $[X]^k = \binom{X}{k} = \{S \subseteq X : |S| = k\}$. If for every set C with $|C| = m$ and every coloring $c\colon \binom{C}{k} \to \{1, \dots, r\}$ there is $B \subseteq C$ with $|B| = l$ such that $c\restriction\binom{B}{k} = \text{const}$, then we write

$$m \longrightarrow (l)^k_r.$$

The following is the well-known classical Ramsey theorem.

Theorem 2.1 ([2]) *For all natural numbers r, k, l there is a natural number m_0 such that for all $m \geq m_0$ we have $m \to (l)^k_r$.*

Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a sequence of nonempty finite sets. A triple $\mathbb{X} = (X, f^X, \preceq^X)$ is called an *α-colored set* if $\preceq^X \in \operatorname{lo}(X)$ and f^X is a function from $\cup_{i=1}^k [X]^i$ to $\cup_{i=1}^k \alpha_i$ such that for all $i \in [k]$ and $x \in [X]^i$ we have $f^X(x) \in \alpha_i$. If $\mathbb{Y} = (Y, f^Y, \preceq^Y)$ is also an α-colored set, then the map $F\colon X \to Y$ is an *embedding* if it is $1-1$. For all x, $x' \in X$ we have $x \preceq^X x' \Leftrightarrow F(x) \preceq^Y F(x')$, and for all $i \in [k]$, all $z \in [X]^i$ we have $f^X(z) = f^Y(F(z))$. If there is an embedding from $\mathbb{X}$ into $\mathbb{Y}$, then we write $\mathbb{X} \hookrightarrow \mathbb{Y}$, and if the embedding is realized by the identity map, then we say that $\mathbb{X}$ is a substructure of $\mathbb{Y}$, or $\mathbb{X} \leq \mathbb{Y}$. An embedding that is a bijection is called an isomorphism; we write $\mathbb{X} \cong \mathbb{Y}$. The class of finite α-colored sets with the notion of embedding as defined above we denote by $\mathcal{K}(\alpha)$. Our proofs will use the following result.

Theorem 2.2 ([1, 6]) *For any finite sequence $\alpha = (\alpha_1, \dots, \alpha_k)$ of finite non empty sets, the class $\mathcal{K}(\alpha)$ satisfies RP.*

Let $\mathcal{L}_2$ be the class of finite structures of the form $(A, \leq^A, \preceq^A)$, where $\leq^A$ and $\preceq^A$ are linear orderings on the set A. Let $\mathbb{A} = (A, \leq^A, \preceq^A)$ and $\mathbb{B} = (B, \leq^B, \preceq^B)$ be structures from $\mathcal{L}_2$. An embedding from $\mathbb{A}$ into $\mathbb{B}$ is a map $f\colon A \to B$ such that for all $a_1, a_2 \in A$ we have

$$a_1 \leq^A a_2 \iff f(a_1) \leq^B f(a_2) \quad \text{and} \quad a_1 \preceq^A a_2 \iff f(a_1) \preceq^B f(a_2).$$

Theorem 2.3 ([9]) *$\mathcal{L}_2$ is a Ramsey class*

We need the following result about a product of Ramsey classes.

Theorem 2.4 ([10]) *Let $(\mathcal{A}_i)_{i=1}^l$ be a sequence of Ramsey classes of finite structures and let r be a natural number. Let $(\mathbb{A}_i)_{i=1}^l$ and $(\mathbb{B}_i)_{i=1}^l$ be sequences of finite structures such that $\mathbb{A}_i \in \mathcal{A}_i, \mathbb{B}_i \in \mathcal{A}_i$, and $\binom{\mathbb{B}_i}{\mathbb{A}_i} \neq \varnothing$ for $i \in [l]$. Then there is a sequence $(\mathbb{C}_i)_{i=1}^l$ such that $\mathbb{C}_i \in \mathcal{A}_i$ for all $i \in [l]$ and such that for every coloring*

$$p\colon \binom{\mathbb{C}_1}{\mathbb{A}_1} \times \cdots \times \binom{\mathbb{C}_l}{\mathbb{A}_l} \longrightarrow \{1, \dots, r\},$$

there is a sequence of structures $(\mathbb{E}_i)_{i=1}^l$, where $\mathbb{E}_i \in \binom{\mathbb{C}_i}{\mathbb{B}_i}$ for $i \in [l]$ and such that

$$p\restriction\binom{\mathbb{E}_1}{\mathbb{A}_1} \times \cdots \times \binom{\mathbb{E}_l}{\mathbb{A}_l} = \text{const}\,.$$

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For structures that satisfy the statement of the previous theorem we use arrow notation

$$(\mathbb{C}_1, \dots, \mathbb{C}_l) \longrightarrow (\mathbb{B}_1, \dots, \mathbb{B}_l)_r^{(\mathbb{A}_1,\dots,\mathbb{A}_l)}, \quad \text{or} \quad \overrightarrow{\mathbb{C}} \to (\overrightarrow{\mathbb{B}})_r^{\overrightarrow{\mathbb{A}}},$$

where $\overrightarrow{\mathbb{C}} = (\mathbb{C}_1, \dots, \mathbb{C}_l)$, $\overrightarrow{\mathbb{B}} = (\mathbb{B}_1, \dots, \mathbb{B}_l)$, and $\overrightarrow{\mathbb{A}} = (\mathbb{A}_1, \dots, \mathbb{A}_l)$.

Suppose that, in the previous theorem, for some nonempty $I \subseteq [n]$ we have $\mathbb{A}_i = \varnothing \Leftrightarrow i \in I$. Then we also write $\overrightarrow{\mathbb{C}} \to (\overrightarrow{\mathbb{B}})_r^{\overrightarrow{\mathbb{A}}}$, where $\mathbb{C}_i = \mathbb{B}_i$ for $i \in I$, and if $[n]\backslash I \neq \varnothing$, then $\mathbb{C}_i = \mathbb{D}_i$ for $i \in [n]\backslash I = \{i_1 < i_2 < \cdots < i_l\}$, where

$$(\mathbb{D}_{i_1}, \dots, \mathbb{C}_{i_l}) \leftrightarrow (\mathbb{B}_{i_1}, \dots, \mathbb{B}_{i_l})_r^{(\mathbb{A}_{i_1},\dots,\mathbb{A}_{i_l})}.$$

In particular, if in the previous theorem we take $\mathcal{A}_i = \cdots = \mathcal{A}_i$ to be the class of finite sets, then we get the product Ramsey theorem as stated in [2].

3 Main Proof

Proof of Theorem 1.1 Let r be a natural number. Let $\mathbb{A} = (A, (I_i^A)_{i=1}^n, \leq^A, (\preceq_i^A)_{i=1}^n)$ and $\mathbb{B} = (B, (I_i^B)_{i=1}^n, \leq^B, (\preceq_i^B)_{i=1}^n)$ be structures from $\mathcal{OM}_n$ such that $\binom{\mathbb{B}}{\mathbb{A}} \neq \varnothing$.

First, we consider the class $\mathcal{K}(\alpha)$ of α-colored sets, where

$$\alpha = (\alpha_1), \alpha_1 = \{0, 1\}.$$

To each $\mathbb{F} = (F, (I_i^F)_{i=1}^n, \leq^F, (\preceq_i^F)_{i=1}^n)$ from $\mathcal{OM}_n$ we assign sequences $(\Delta_i(\mathbb{F}))_{i=1}^n$, $(\sigma_i(\mathbb{F}))_{i=1}^n$, $(\Phi_i(\mathbb{F}))_{i=1}^n$ with the property that for $i \in [n]$ we have the following:

- $\Delta_i(\mathbb{F}) = (F, f_i^F, \leq^F) \in \mathcal{K}(\alpha)$ where f_i^F is defined by using the unary relation I_i^F, *i.e.*,

$$f_i^F(x) = 1 \Longleftrightarrow I_i^F(x), \text{ for } x \in F.$$

- $\sigma_i(\mathbb{F}) = \{x \in F : I_i^F(x)\} \subseteq F.$
- $\Phi_i(\mathbb{F}) = (\sigma_i(\mathbb{F}), \leq^F \restriction \sigma_i(\mathbb{F}), \preceq_i^F) \in \mathcal{L}_2.$

In particular, for every $i \in [n]$ we have:

- $\Delta_i(\mathbb{A}) = (A, f_i^A, \leq^A), \Delta_i(\mathbb{B}) = (B, f_i^B, \leq^B) \in \mathcal{K}(\alpha).$
- $\sigma_i(\mathbb{A}) \subseteq A, \sigma_i(\mathbb{B}) \subseteq B.$
- $\Phi_i(\mathbb{A}) = (\sigma_i(\mathbb{A}), \leq^A \restriction \sigma_i(\mathbb{A}), \preceq_i^A), \Phi_i(\mathbb{B}) = (\sigma_i(\mathbb{B}), \leq^B \restriction \sigma_i(\mathbb{B}), \preceq_i^B) \in \mathcal{L}_2.$

At this point we have sequences

$$\overrightarrow{\mathbb{B}} = \big(\Delta_1(\mathbb{B}), \dots, \Delta_n(\mathbb{B}), \Phi_1(\mathbb{B}), \dots, \Phi_n(\mathbb{B})\big),$$

$$\overrightarrow{\mathbb{A}} = \big(\Delta_1(\mathbb{A}), \dots, \Delta_n(\mathbb{A}), \Phi_1(\mathbb{A}), \dots, \Phi_n(\mathbb{A})\big).$$

By Theorem 2.2, $\mathcal{K}(\alpha)$ is a Ramsey class, and by Theorem 2.3, $\mathcal{L}_2$ is a Ramsey class. Then by Theorem 2.4, there is a sequence of structures $\overrightarrow{\mathbb{C}} = (\mathbb{C}_i)_{i=1}^{2n}$ such that $\mathbb{C}_i = (C_i, f^{C_i}, \leq^{C_i}) \in \mathcal{K}(\alpha)$ for $i \in [n]$, $\mathbb{C}_i = (C_i, \leq^{C_i}, \preceq_i^{C_i}) \in \mathcal{L}_2$ for $n < i \leq 2n$, and

$$\overrightarrow{\mathbb{C}} \longrightarrow (\overrightarrow{\mathbb{B}})_r^{\overrightarrow{\mathbb{A}}}.$$

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We point out that this is well defined even in the case where there are i such that $\sigma_i(\mathbb{A}) = \varnothing$; see the second paragraph after Theorem 2.4.

We use sequence $(\mathbb{C}_i)_{i=1}^{2n}$ to define a structure $\mathbb{C} = (C, (I_i^C)_{i=1}^n, \leq^C, (\preceq_i^C)_{i=1}^n)$ in $\mathcal{OM}_n$. Let $\star$ be such that $\star \notin \bigcup_{i=1}^{2n} C_i$. The underlying set of the structure $\mathbb{C}$ is given as a subset

$$C \subseteq \Omega = \Big(\prod_{i=1}^n C_i\Big) \times \Big(\prod_{i=1}^n \big(C_{n+i} \cup \{\star\}\big)\Big)$$

such that for $c = (c_i)_{i=1}^{2n} \in \Omega$ we have

$$c \in C \iff (\forall i \in [n])(f^{C_i}(c_i) = 1 \iff (c_{n+i} \neq \star)).$$

In order to define linear orderings and unary relations in $\mathbb{C}$ we consider points $c = (c_i)_{i=1}^{2n}$ and $c' = (c'_i)_{i=1}^{2n}$ in C. For $i \in [n]$ we first define the following.

(a) $I_i^C(c) \Leftrightarrow (f^{C_i}(c_i) = 1)$.
(b) $c \leq^C c' \Leftrightarrow ((c = c')$ or $(c \neq c'$ and $c_{i_0} \leq^{C_{i_0}} c'_{i_0}))$, where $i_0 = \min\{i : c_i \neq c'_i\}$ for $c \neq c'$.
(c) The linear ordering $\preceq_i^C$ is defined only on the set $\{e \in C : I_i^C(e)\}$ as follows: $c \preceq_i^C c'$ if and only if $c = c'$, or if $c \neq c'$ and either
- $c_{n+i} = c'_{n+i}$ and $c_{i'_0} \leq^{C_{i'_0}} c'_{i'_0}$, where $i'_0 = \min\{j \neq i : c_j \neq c'_j\}$,
- or $c_{n+i} \neq c'_{n+i}$ and $c_{n+i} \preceq_{n+i}^{C_{n+i}} c'_{n+i}$.

Note that $\leq^C$ and $\preceq_i^C$ are well-defined linear orderings, because if $i_0 > n$ or $i'_0 > n$, then we have

$$(c_i)_{i=1}^n = (c'_i)_{i=1}^n \implies (\forall i \in [n])(c_{n+i} = \star \iff c'_{c+i} = \star).$$

We claim that $\mathbb{C} \to (\mathbb{B})_r^{\mathbb{A}}$. So, let

$$p \colon \binom{\mathbb{C}}{\mathbb{A}} \longrightarrow \{1, \dots, r\}$$

be a given coloring. Our goal is to pay attention only to specific substructures inside $\mathbb{C}$. Therefore we consider a sequence of structures $\overrightarrow{\mathbb{K}} = (\mathbb{K}_i)_{i=1}^{2n}$ given by the following:

(a) $\mathbb{K}_i = (K_i, f^{K_i}, \leq^{K_i}) \leq \mathbb{C}_i$ for $i \in [n]$.
(b) $\mathbb{K}_i = (K_i, \leq^{K_i}, \preceq^{K_i}) \leq \mathbb{C}_i$ for $n < i \leq 2n$.
(c) $|K_i| = |K_j| = a$ for all $i, j \in [n]$, for some natural number a.
(d) $|K_{n+i}| = a_i$, where $a_i = |\{x \in K_i : f^{K_i}(x) = 1\}|$ for $i \in [n]$.

Note that we can have $a_i = 0$, and in that case we will obtain a structure without linear ordering $\preceq_i^C$. For each $i \in [n]$, we take $K_i = \{k_{i,j}\}_{j=1}^a$ and assume that $k_{i,j} <^{K_i} k_{i,j'}$ for all $j < j' \in [a]$, where $<^{K_i}$ is the strict part of the linear ordering $\leq^{K_i}$. Also, for each $i \in [n]$ we take $K_{n+i} = \{k_{n+i,j}\}_{j=1}^{a_i}$ and assume that $k_{n+i,j} <^{K_{n+i}} k_{n+i,j'}$ for all $j < j' \in [a_i]$, where $<^{K_{n+1}}$ is the strict part of the linear ordering $\leq^{K_{n+1}}$. Now we assign to the sequence $\overrightarrow{\mathbb{K}}$ a unique substructure $\varphi(\overrightarrow{\mathbb{K}})$ of $\mathbb{C}$ with the underlying set $\{u^i\}_{i=1}^a$, where for $j \in [a]$ we take $u^j = (u_i^j)_{i=1}^{2n}$ such that for all $i \in [n]$ we have:

(a) $(u_i^j)_{i=1}^n = (k_{i,j})_{i=1}^n$,
(b) $f^{K_i}(k_{i,j}) = 0 \Rightarrow u_{n+i}^j = \star$,
(c) if $f^{K_i}(k_{i,j}) = 1$ and $k_{i,j}$ is the s-th element of the set $\{x \in K_i : f^{K_i}(x) = 1\}$ with respect to $\leq^{K_i}$, then u_{n+1}^j is the s-th element in the sequence

$$\{\, k_{n+i,1} <^{K_{n+i}} k_{n+i,2} <^{K_{n+i}} \cdots <^{K_{n+i}} k_{n+i,a_i} \,\}.$$

Suppose that $I = \{i \in [n] : K_{n+i} = \varnothing\}$. Note that the definition of $\varphi(\overrightarrow{\mathbb{K}})$ is well defined even for $I \neq \varnothing$, and in that case does not depend on $\mathbb{K}_{n+i}$ for $i \in I$. So for $I \neq \varnothing$, we consider φ as a map from $\prod_{i=1}^n \binom{\mathbb{C}_i}{\Delta_i(\mathbb{A})} \times \prod_{i>n: i-n\notin I} \binom{\mathbb{C}_{i+n}}{\Phi_i(\mathbb{A})}$ into $\binom{\mathbb{C}}{\mathbb{A}}$.

If $K_{n+i} \neq \varnothing$ for all $i \in [n]$, then we have an induced coloring:

$$\bar{p} : \prod_{i=1}^n \binom{\mathbb{C}_i}{\Delta_i(\mathbb{A})} \times \prod_{i=1}^n \binom{\mathbb{C}_{i+n}}{\Phi_i(\mathbb{A})} \to \{1, \dots, r\},$$
$$\bar{p}(\overrightarrow{\mathbb{K}}) = p(\varphi(\overrightarrow{\mathbb{K}})).$$

From the definition of the map φ and definition of the structure $\mathbb{C}$ we conclude that $\bar{p}$ is well defined. Moreover there is a sequence

$$\overrightarrow{\mathbb{E}} = (\mathbb{E}_i)_{i=1}^{2n} \in \prod_{i=1}^n \binom{\mathbb{C}_i}{\Delta_i(\mathbb{B})} \times \prod_{i=1}^n \binom{\mathbb{C}_{i+n}}{\Phi_i(\mathbb{B})}$$

such that

$$\bar{p} \restriction \prod_{i=1}^n \binom{\mathbb{E}_i}{\Delta_i(\mathbb{A})} \times \prod_{i=1}^n \binom{\mathbb{E}_{i+n}}{\Phi_i(\mathbb{A})} = \text{const}\,.$$

Now we have that $\varphi(\overrightarrow{\mathbb{E}}) \cong \mathbb{B}$ and that every $\mathbb{M} \in \binom{\mathbb{B}}{\mathbb{A}}$ is of the form $\varphi(\overrightarrow{\mathbb{U}})$ for some

$$\overrightarrow{\mathbb{U}} \in \prod_{i=1}^n \binom{\mathbb{E}_i}{\Delta_i(\mathbb{A})} \times \prod_{i=1}^n \binom{\mathbb{E}_{i+n}}{\Phi_i(\mathbb{A})}.$$

Consequently, we have

$$p \restriction \binom{\mathbb{B}}{\mathbb{A}} = \text{const},$$

so RP is verified for $\mathcal{OM}_n$. ■

4 Chains and Antichains

Since we can take $\mathcal{COC}_1$ as $\mathcal{L}_2$, Theorem 2.3 implies the Ramsey property for the class $\mathcal{COC}_1$. Therefore in the proof of Theorem 1.2 we discuss only the case $n \geq 2$. We emphasize that Theorem 1.2 is not a restatement of Theorem 1.1, because structures from $\mathcal{COC}_n$ are equipped with partial orderings that must be preserved under embeddings.

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Proof of Theorem 1.2 Let $\mathbb{A} = (A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A)$ be a structure from $\mathcal{COC}_n$. In this case $\leq^A$ denotes a partial ordering on A, while $\preceq^A$ denotes a linear ordering on A. Then the set A is decomposed into maximal antichains with respect to the partial ordering $\leq^A$ such that $A = A_1 \cup \cdots \cup A_a$, and without loss of generality we may assume that $\leq^A$ induces a linear ordering on the sets $\{A_1, \ldots, A_a\}$ by $A_1 \leq^A \cdots \leq^A A_a$. To the structure $\mathbb{A}$ we assign a structure $\Delta(\mathbb{A}) = ([a], (I_i^{[a]})_{i=1}^n, \leq^{[a]}, (\preceq_i^{[a]})_{i=1}^n)$ in $\mathcal{OM}_n$. For $i \in [n]$ and $x, x' \in [a]$ we take:

(a) $\leq^{[a]}$ to be given by $1 <^{[a]} 2 <^{[a]} \cdots <^{[a]} a$.
(b) $I_i^{[a]}$ to be given by

$$I_i^{[a]}(x) \iff (\exists y \in A)[I_i^A(y) \text{ and } y \in A_x].$$

(c) $\preceq_i^{[a]}$ to be a linear ordering defined on the set $\{y \in [a] : I_i^{[a]}(y)\}$ such that if $I_i^{[a]}(x)$ and $I_i^{[a]}(x')$, then

$$x \prec^{[a]} x' \iff (\exists y, y' \in A)[\, y \in A_x \text{ and } y' \in A_{x'} \text{ and } y \prec^A y'].$$

Note that $\preceq_i^{[a]}$ is a well-defined linear ordering, because for all $x \in [a]$ and all $i \in [n]$ we have $|\{p \in A_x : I_i^A(p)\}| \leq 1$.

Let $\mathbb{A} = (A, (I_i^A)_{i=1}^n, \leq^A, (\preceq_i^A)_{i=1}^n)$ be a structure in $\mathcal{OM}_n$. We consider a structure $\mathbb{B} = (B, \leq^B, (I_i^B)_{i=1}^n, \preceq^B)$ on the set $B = \bigcup_{i=1}^n \{i\} \times A$, where $\leq^B$ is a partial ordering on a subset of B, $(I_i^B)_{i=1}^n$ is a sequence of unary relations on B and $\preceq^B$ is a linear ordering on a subset of B. Let $i \in [n]$ and let $a' = (k, a)$ and $b' = (l, b)$ be distinct points in B.

(a) Define I_i^B by

$$I_i^B(a') \iff i = k.$$

Let $B_0 = \{a' \in B : (\exists i)[I_i^B(a')]\}$.
(b) Define $\leq^B$ on the set B_0 such that for $a', b' \in B_0$ we have

$$a' <^B b' \iff a <^A b.$$

(c) Define $\preceq^B$ on the set B_0 such that for $a', b' \in B_0$ we have

$$a' \prec^B b' \iff ((k < l) \text{ or } (k = l \text{ and } a \prec_k^A b)).$$

We denote by $\Phi(\mathbb{A})$ the substructure of $\mathbb{B}$ with the underlying set B_0. Moreover, we have $\Phi(\mathbb{A}) \in \mathcal{COC}_n$.

Note that for structures $\mathbb{A}_1$ and $\mathbb{A}_2$ in $\mathcal{OM}_n$ we have

$$\mathbb{A}_1 \leq \mathbb{A}_2 \Longrightarrow \Phi(\mathbb{A}_1) \leq \Phi(\mathbb{A}_2).$$

We point out that for $\mathbb{A} = (A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A) \in \mathcal{COC}_n$ we do not always have $\Phi(\Delta(\mathbb{A}))) \cong \mathbb{A}$, but under the assumption that $I_i^A(a), I_j^A(a') \Rightarrow a \prec^A a'$ for all $i < j$, we have $\Phi(\Delta(\mathbb{A}))) \cong \mathbb{A}$.

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Let $\mathbb{A} = (A, \leq^A, (I_i^A)_{i=1}^n, \preceq^A)$ and $\mathbb{B} = (B, \leq^B, (I_i^B)_{i=1}^n, \preceq^B)$ be structures from $\mathcal{COC}_n$ such that $\binom{\mathbb{B}}{\mathbb{A}} \neq \varnothing$.

Without loss of generality we may assume that we have $I_i^B(b), I_j^B(b') \Rightarrow b \prec^B b'$ for all $i < j$. Let r be a natural number. By Theorem 2.2 there is a structure $\mathbb{C} \in \mathcal{COC}_n$ such that

$$\mathbb{C} \longrightarrow \big(\Delta(\mathbb{B})\big)_r^{\Delta(\mathbb{A})},$$

and we claim that $\Phi(\mathbb{C}) \to (\mathbb{B})_r^{\mathbb{A}}$. Suppose that we have a coloring $p\colon \binom{\Phi(\mathbb{C})}{\mathbb{A}} \to \{1, \dots, r\}$. Then there is an induced coloring

$$\bar{p}\colon \binom{\mathbb{C}}{\Delta(\mathbb{A})} \longrightarrow \{1, \dots, r\},$$
$$\bar{p}(\mathbb{P}) = p(\Phi(\mathbb{P})).$$

By the choice of the structure $\mathbb{C}$ there is $\mathbb{R} \in \binom{\mathbb{C}}{\Delta(B)}$ such that $\bar{p} \restriction \binom{\mathbb{R}}{\Delta(\mathbb{A})} = \text{const}$. Since $\Phi(\mathbb{R}) \cong \mathbb{B}$ and for every $\mathbb{G} \in \binom{\Phi(\mathbb{C})}{\mathbb{A}}$ there is a $\mathbb{K} \in \binom{\mathbb{C}}{\Delta(\mathbb{A})}$ such that $\Phi(\mathbb{K}) = \mathbb{G}$, we obtain that $p \restriction \binom{\Phi(R)}{\mathbb{A}} = \text{const}$. This completes the verification of RP for the class $\mathcal{COC}_n$. ■

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