Glasgow Math. J. **46** (2004) 301–304. © 2004 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089504001776. Printed in the United Kingdom

WEYL'S THEOREM FOR TENSOR PRODUCTS

YEONG-MOO SONG

Department of Mathematics Education, Sunchon National University, Sunchon 540-742, Korea
e-mail: ymsong@sunchon.ac.kr

and AN-HYUN KIM

Department of Mathematics, Changwon National University, Changwon 641-773, Korea
e-mail: ahkim@changwon.ac.kr

(Received 21 March, 2003; accepted 29 October, 2003)

Abstract. Suppose that A and B are 'isoloid' operators acting on a complex Banach space, that is, every isolated point of their spectra is an eigenvalue. In this note it is shown that if Weyl's theorem holds for both A and B then it holds for $A \otimes B$.

2000 *Mathematics Subject Classification.* 47A10, 47A53.

Throughout this note let $\mathcal{X}$ denote an infinite dimensional complex Banach space. Let $\mathcal{L}(\mathcal{X})$ denote the algebra of bounded linear operators on $\mathcal{X}$. If $T \in \mathcal{L}(\mathcal{X})$ write $N(T)$ and $R(T)$ for the null space and range of T; $\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the set of eigenvalues of T. Recall ([**4**], [**5**]) that $T \in \mathcal{L}(\mathcal{X})$ is called *upper semi-Fredholm* if it has closed range with finite-dimensional null space and *lower semi-Fredholm* if it has closed range with its range of finite co-dimension. If T is either upper or lower semi-Fredholm, we call it *semi-Fredholm* and if T is both upper and lower semi-Fredholm, we call it *Fredholm*. The *index* of a semi-Fredholm operator $T \in \mathcal{L}(\mathcal{X})$ is given by

$$\mathrm{ind}(T) = \dim N(T) - \dim X/R(T).$$

An operator $T \in \mathcal{L}(\mathcal{X})$ is called *Weyl* if it is Fredholm of index zero. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\omega(T)$ of $T \in \mathcal{L}(\mathcal{X})$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\};$$
$$\omega(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}:$$

then (cf. [**5**])

$$\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_e(T) \cup \mathrm{acc}\,\sigma(T) \quad \text{and} \quad \omega(T) \subseteq \eta\,\sigma_e(T),$$

where we write $\mathrm{acc}\,K$ and $\eta\,K$ for the *accumulation points* and the *polynomially-convex hull*, respectively, of $K \subseteq \mathbb{C}$. We also write $\mathrm{iso}\,K = K \backslash \mathrm{acc}\,K$ and

$$\pi_{00}(T) := \{\lambda \in \mathrm{iso}\,\sigma(T) : 0 < \dim\,(T - \lambda I)^{-1}(0) < \infty\}$$

This paper was supported in part NON DIRECTED RESEARCH FUND from Suncheon National University.

for the isolated eigenvalues of finite multiplicity. We say that *Weyl's theorem holds for* $T \in \mathcal{L}(\mathcal{X})$ if there is equality

$$\sigma(T)\backslash\omega(T) = \pi_{00}(T). \tag{0.1}$$

H. Weyl [**10**] discovered that the equality (0.1) holds for every hermitian operator. Weyl's theorem has been extended from hermitian operators to hyponormal operators and to Toeplitz operators by L. Coburn [**3**], to several classes of operators including seminormal operators by S. Berberian [**1**], [**2**], and to a few classes of Banach space operators [**6**], [**7**], [**8**]. In this note we examine Weyl's theorem for $A \otimes B$ when Weyl's theorem holds for A and B.

Recall that an operator $T \in \mathcal{L}(\mathcal{X})$ is called an *isoloid operator* if $\text{iso}\,\sigma(T) \subseteq \pi_0(T)$, i.e., every isolated point of the spectrum is an eigenvalue (cf. [**2**], [**8**]).

Our main theorem now follows.

THEOREM 1. *Suppose* $A, B \in \mathcal{L}(\mathcal{X})$ *are isoloid. If Weyl's theorem holds for both* A *and* B *then it holds for* $A \otimes B$.

Proof. We first show that

$$\sigma(A \otimes B)\backslash\omega(A \otimes B) \subseteq \pi_{00}(A \otimes B). \tag{1}$$

To show this it suffices to show that $\sigma(A \otimes B)\backslash\omega(A \otimes B) \subseteq \text{iso}\,\sigma(A \otimes B)$. Assume to the contrary that $\lambda \in \sigma(A \otimes B)\backslash\omega(A \otimes B)$ and $\lambda \in \text{acc}\,\sigma(A \otimes B)$. Since $\lambda \in \text{acc}\,(\sigma(A) \cdot \sigma(B))$, it follows that $\lambda \in [\text{acc}\,\sigma(A) \cdot \sigma(B)] \cup [\sigma(A) \cdot \text{acc}\,\sigma(B)]$: indeed, more generally, if H and K are compact subsets of $\mathbb{C}$ then $\text{acc}\,(H \cdot K) \subseteq [(\text{acc}\,H) \cdot K] \cup [H \cdot (\text{acc}\,K)]$. But since Weyl's theorem holds for A and B, we have that $\text{acc}\,\sigma(A) \subseteq \omega(A)$ and $\text{acc}\,\sigma(B) \subseteq \omega(B)$. Therefore

$$\lambda \in \omega(A) \cdot \sigma(B) \cup \sigma(A) \cdot \omega(B) = \omega(A \otimes B),$$

giving a contradiction. This proves (1). For the reverse inclusion we first observe

$$[N(A) \otimes \mathcal{H}] \cup [\mathcal{H} \otimes N(B)] \subseteq N(A \otimes B); \tag{2}$$

$$N(A - \mu I) \otimes N(B - \nu I) \subseteq N(A \otimes B - \mu\nu(I \otimes I)) \quad \text{for each } \mu, \nu \in \mathbb{C}: \tag{3}$$

indeed the inclusion (2) is evident and the inclusion (3) comes from the observation

$$\begin{aligned}[A \otimes B - \mu\nu(I \otimes I)](x \otimes y) &= [(A - \mu I) \otimes B + \mu I \otimes (B - \nu I)](x \otimes y) \\ &= (A - \mu I)x \otimes By + \mu x \otimes (B - \nu I)y.\end{aligned}$$

Suppose $\lambda \in \pi_{00}(A \otimes B)$. We then proceed as follows.

Claim 1. $\lambda \neq 0$.

Claim 2. If $\lambda = \mu\nu$ with $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$, then $\mu \in \text{iso}\,\sigma(A)$ and $\nu \in \text{iso}\,\sigma(B)$.

Claim 3. If $\lambda = \mu\nu$ with $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$ then $A - \mu I$ and $B - \nu I$ are both Weyl.

For Claim 1, we assume to the contrary that $\lambda = 0$. Thus $0 \in \text{iso}\,\sigma(A \otimes B)$, and hence $0 \in \text{iso}\,\sigma(A)$ or $0 \in \text{iso}\,\sigma(B)$. But since A and B are isoloid it follows that $0 \in \pi_0(A)$ or $0 \in \pi_0(B)$. Therefore by (2) $N(A \otimes B)$ is infinite dimensional, which contradicts our assumption $0 \in \pi_{00}(A \otimes B)$.

To prove Claim 2 we write $\lambda = \mu\nu$ with $\mu \in \sigma(A)$, $\nu \in \sigma(B)$ and $\lambda \neq 0$. Assume to the contrary that $\mu \in \operatorname{acc}\sigma(A)$. Then we can find a sequence $\{\mu_n\}$ of distinct numbers in $\sigma(A)$ such that $\lim \mu_n = \mu$, so that $\lim \mu_n \nu = \lambda$, which shows that $\lambda \in \operatorname{acc}\sigma(A \otimes B)$, a contradiction; therefore $\mu \in \operatorname{iso}\sigma(A)$ and similarly $\nu \in \operatorname{iso}\sigma(B)$.

Towards Claim 3, note that $\mu \in \pi_0(A)$ and $\nu \in \pi_0(B)$ by Claim 2 because A and B are isoloid. We assume to the contrary that $A - \mu I$ is not Weyl. Thus $\mu \notin \pi_{00}(A)$ because A obeys Weyl's theorem. So we have that $N(A - \mu I)$ is infinite dimensional. Also since $N(B - \nu I) \neq \{0\}$, it follows from (3) that $N(A \otimes B - \mu\nu(I \otimes I))$ is infinite dimensional, which contradicts our assumption $\lambda \in \pi_{00}(A \otimes B)$. This compltes the proof of Claim 3.

From Claims 1,2, and 3 we can conclude that if $\lambda \in \pi_{00}(A \otimes B)$ then $\lambda \notin \omega(A) \cdot \sigma(B) \cup \sigma(A) \cdot \omega(B)$, and hence $\lambda \in \sigma(A \otimes B) \setminus \omega(A \otimes B)$. Therefore

$$\pi_{00}(A \otimes B) \subseteq \sigma(A \otimes B) \setminus \omega(A \otimes B). \tag{4}$$

By (1) and (4) we can conclude that Weyl's theorem holds for $A \otimes B$. □

Example 2. (a) The "isoloid" condition is essential in Theorem 1. To see this let T be an injective quasinilpotent operator on ℓ_2 and define

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & T+2 \end{pmatrix} : \mathbb{C} \oplus \mathbb{C} \oplus \ell_2 \longrightarrow \mathbb{C} \oplus \mathbb{C} \oplus \ell_2.$$

Then

$$\sigma(A) = \{1, 2, 4\}, \quad \omega(A) = \{2\}, \quad \text{and} \quad \pi_{00}(A) = \{1, 4\};$$

so Weyl's theorem holds for A, while

$$\begin{aligned} \sigma(A \otimes A) &= \sigma(A) \cdot \sigma(A) = \{1, 2, 4, 8, 16\}; \\ \omega(A \otimes A) &= \sigma(A) \cdot \omega(A) = \{2, 4, 8\}; \\ \pi_{00}(A \otimes A) &= \{1, 4, 16\}; \end{aligned}$$

so Weyl's theorem fails for $A \otimes A$. Note that A is not isoloid.

(b) On the other hand, the condition "Weyl's theorem holds for both A and B" is essential in Theorem 1. If Weyl's theorem does not hold for either A or B, then Theorem 1 may fail. To see this, consider the operators on $\ell_2 \oplus \ell_2$ defined by

$$A = U \oplus U^* \quad \text{and} \quad B = (I - UU^*) \oplus 0_\infty,$$

where U is the unilateral shift on ℓ_2. Let $\mathbb{D}$ and $\mathbb{T}$ denote the closed unit disk and the unit circle, respectively. Then we have that (i) A and B are both isoloid; (ii) $\sigma(A) = \mathbb{D}$ and $\omega(A) = \mathbb{T}$, and hence Weyl's theorem fails for A; (iii) $\sigma(B) = \{0, 1\}$, $\omega(B) = \{0\}$ and $\pi_{00}(B) = \{1\}$, and hence Weyl's theorem holds for B; (iv) $\sigma(A \otimes B) = \mathbb{D}$ and $\omega(A \otimes B) = \mathbb{T} \cup \{0\}$, and hence Weyl's theorem fails for $A \otimes B$.

(c) The converse of Theorem 1 may not be true in general. Indeed if $A = U \oplus U^*$ as in (b) then $\sigma(A \otimes 1) = \mathbb{D}$, $\omega(A \otimes 1) = \mathbb{D}$ and $\pi_{00}(A \otimes 1) = \emptyset$, which implies that Weyl's theorem holds for $A \otimes 1$ although A does not satisfy Weyl's theorem.

If $\mathcal{H}$ is a complex Hilbert space and $T \in \mathcal{L}(\mathcal{H})$, write $W(T)$ for the numerical range of T. It is also familiar that $W(T)$ is convex and $\operatorname{conv}\sigma(T) \subseteq \operatorname{cl} W(T)$. An operator T is called *convexoid* if $\operatorname{conv}\sigma(T) = \operatorname{cl} W(T)$. Also T is called *restriction-convexoid* if the restriction of T to every invariant subspace is convexoid and is called *reduction-convexoid* if every direct summand of T is convexoid. It is known [**2**] that hyponormal $\Rightarrow$ restriction-convexoid $\Rightarrow$ isoloid.

COROLLARY 3. *If $\mathcal{H}$ is a complex Hilbert space and $A, B \in \mathcal{L}(\mathcal{H})$ are restriction-convexoid then Weyl's theorem holds for $A \otimes B$.*

Proof. By an argument of Prasanna [**9**, Theorem 2.1], Weyl's theorem holds for restriction-convexoid operators. Thus the result immediately follows from Theorem 1. □

Weyl's theorem may fail for reduction-convexoid operators. For example if $A = U \oplus U^*$, where U is the unilateral shift on ℓ_2, then A is reduction-convexoid because U and U^* are both convexoid and have no nontrivial reducing subspaces, while Weyl's theorem fails for A. Note that A is not restriction-convexoid.

ACKNOWLEDGEMENT. The authors are grateful to the referee for helpful comments concerning this paper.

REFERENCES

1. S. K. Berberian, An extension of Weyl's theorem to a class of not necessarily normal operators, *Michigan Math. J.* **16** (1969), 273–279.

2. S. K. Berberian, The Weyl spectrum of an operator, *Indiana Univ. Math. J.* **20** (1970), 529–544.

3. L. A. Coburn, Weyl's theorem for nonnormal operators, *Michigan Math. J.* **13** (1966), 285–288.

4. R. E. Harte, Fredholm, Weyl and Browder theory, *Proc. Royal Irish Acad.* Sect. A **85** (1985), 151–176.

5. R. E. Harte, *Invertibility and singularity for bounded linear operators* (Dekker, New York, 1988).

6. V. I. Istrătescu, On Weyl's spectrum of an operator. I, *Rev. Roum. Math. Pures Appl.* **17** (1972), 1049–1059.

7. W. Y. Lee and H. Y. Lee, On Weyl's theorem, *Math. Japan* **39** (1994), 545–548.

8. K. K. Oberai, On the Weyl spectrum, *Illinois J. Math.* **18** (1974), 208–212.

9. S. Prasanna, Weyl's theorem and thin spectra, *Indian Acad. Sci. (Math. Sci.)* **91** (1982), 59–63.

10. H. Weyl, Über beschränkte quadratische Formen, deren Differenz , vollsteig ist, *Rend. Circ. Mat. Palermo* **27** (1909), 373–392.

https://doi.org/10.1017/S0017089504001776 Published online by Cambridge University Press