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FUNCTION SPACES CONTINUOUSLY PAIRED BY OPERATORS OF CONVOLUTION-TYPE

BY
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ABSTRACT. Certain operators essentially defined by convolution are considered. Their possible domain and range spaces are determined; then conditions are given under which the construction of the optimal continuous partner may be carried out for a suitable domain or range. Special cases of operators of convolution-type are useful in studying the boundedness properties of conjugate function operators and, more generally, classes of operators satisfying restricted weak-type conditions.

1. **Introduction.** In this paper we fix on a positive operator T of convolution-type and give conditions under which one can construct with respect to it an optimal continuous partner for a proposed domain or range. Such a T has the form

$$(1.1) \qquad (Tf)(t)=\int_0^\infty a(s)f(st)\,ds, \qquad t>0;$$

the domain consists of all functions f in the class of Lebesgue–measurable functions on $(0,\infty)$, denoted by $M(0,\infty)$, for which the integral exists a.e.; the kernel $a(t)$ is a nonnegative function in $M(0,\infty)$, $a(t)\not\equiv 0$. Motivation for the term "convolution-type" may be found in [5] and references cited there.

It has been shown that the boundedness of certain conjugate function operators between a pair of rearrangement invariant function spaces is equivalent to that of a T with kernel

$$(1.2) \qquad \min[t^{1/p-1}, t^{1/q-1}] \qquad 1\le p<q\le\infty.$$

The first theorem of this kind was proved in Boyd [3] for the Hilbert transformation. Further results and references are given in [7]. Such operators also play a special role in the theory of operators of restricted weak-type. See Calderón [6]—in particular the discussion of optimal pairs in section 3—and Boyd [4]; also [8].

Theorems 2.2 and 2.2′ give the conditions for the construction of optimal continuous partners. These apply, in particular, to T having kernels (1.2). The continuous pairs thus determined are the same as those for the conjugate function operators mentioned above.

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As shown in Theorem 2.1 there is a maximum domain and a minimum range for the T of the above two results; Theorems 2.3 and 2.3′ describe their optimal partners when T has a kernel (1.2).

Background material on rearrangement invariant spaces and convolution-type operators may be found in [4] and [5]. We will use the notation $[X, Y]$ for the space of linear operators bounded from X to Y, abbreviating $[X, X]$ by $[X]$. Finally, if T is a positive operator of convolution-type with kernel $a(t)$, the operator T' with kernel $(1/t)a(1/t)$ will be called its associate operator if

$$\int_0^\infty f(t)(Tg)(t)\,dt = \int_0^\infty g(t)(T'f)(t)\,dt \tag{1.3}$$

for all nonnegative f, $g \in M(0, \infty)$.

2. **Continuous Pairs**. If a postitive T of convolution-type is in $[X, Y]$, X and Y being rearrangement invariant with respect to Lebesgue's measure m on $(0, \infty)$, then $T\chi_{(0,1)}$ must be locally integrable. It is then a consequence of Theorem 2.1 below that should T be bounded between a pair of rearrangement invariant spaces there will exist two Lorentz spaces, one of which is the largest possible domain space for T; the other, the smallest possible range space. As is well-known, given a nonegative, nonincreasing function ϕ on $(0, \infty)$ with

$$\Phi(t) = \int_0^t \phi(u)\,du < \infty, \qquad t > 0, \tag{2.1}$$

the (rearrangement invariant) Lorentz spaces $\Lambda(\phi)$ and $M(\phi)$ have their norms given at nonnegative $f \in M(0, \infty)$ by

$$\sigma(f) = \int_0^\infty f^*(t)\phi(t)\,dt, \tag{2.2}$$

and

$$\sigma'(f) = \sup_{t>0} \int_0^t f^*(u)\,du/\Phi(t),$$

respectively. Here f^* is the nonincreasing rearrangement of f. Further, as the notation in (2.2) suggests, $M(\phi)$ is the space associate to $\Lambda(\phi)$.

THEOREM 2.1. *Suppose T is a positive operator of convolution-type with associate T'. Let $\phi = T\chi_{(0,1)}$ and $\psi = T'\chi_{(0,1)}$. It follows that*

(i) *If σ_1 is a rearrangement invariant norm on $M(0, \infty)$ for which another such norm σ_2 exists with $T \in [L^{\sigma_1}, L^{\sigma_2}]$, then $L^{\sigma_1} \subset \Lambda(\psi)$.*

(ii) *If σ_2 is a rearrangement invariant norm on $M(0, \infty)$ for which another such norm σ_1 exists with $T \in [L^{\sigma_1}, L^{\sigma_2}]$, then $L^{\sigma_2} \supset M(\phi)$.*

Proof. Suppose $f \in L^{\sigma_1}$. Then $Tf^* \in L^{\sigma_2}$. As a result, since $\chi_{(0,1)}$ belongs to every rearrangement invariant space,

$$\int_0^\infty (Tf^*)(t)\chi_{(0,1)}(t)\,dt < \infty, \tag{2.3}$$

or, equivalently,

$$\int_0^\infty f^*(t)\psi(t)\,dt < \infty. \tag{2.4}$$

$T \in [L^{\sigma_1}, L^{\sigma_2}]$ implies $T' \in [L^{\sigma_2'}, L^{\sigma_1'}]$. By (i), $L^{\sigma_2'} \subset \Lambda(\phi)$, or equivalently, $L^{\sigma_2} \supset M(\phi)$.

Theorem 2.2. *Let T be a positive operator of convolution-type having kernel $a(t)$ for which*

$$\int_0^\infty \min[1, 1/u]a(u)\,du < \infty. \tag{2.5}$$

Then the function $\phi = T_{\chi_{(0,1)}}$ is nonnegative and nonincreasing on $(0, \infty)$ with $\int_0^1 \phi(t)\,dt < \infty$. Moreover, to each rearrangement invariant norm σ on $M(0,\infty)$ with $L^\sigma \supset M(\phi)$ there corresponds a rearrangement invariant norm $\boldsymbol{\sigma}$ such that $T \in [L^{\boldsymbol{\sigma}}, L^\sigma]$.

Proof. Observe that (2.5) is simply the condition that $\Phi(1) = \int_0^1 \phi(t)\,dt$ be finite. This means that $\Phi(t)$ and hence $\|\chi_{(0,t)}\|_{M(\phi)} = t/\Phi(t)$ will be finite for all $t > 0$.

Given nonnegative $f \in M(0, \infty)$, define $\boldsymbol{\sigma}(f)$ by

$$\boldsymbol{\sigma}(f) = \sigma(Tf^*). \tag{2.6}$$

We show $\boldsymbol{\sigma}$ satisfies the definitive properties of a rearrangement invariant norm given in [4]. In what follows, f, f_n, and g are nonnegative functions in $M(0, \infty)$.

Now, $\sigma(Tf^*) \geq 0$ with equality if and only if

$$\int_0^\infty a(s)f^*(st)\,ds = 0, \quad \text{a.e.} \tag{2.7}$$

or, equivalently,

$$\int_0^\infty a(s/t)f^*(s)\,ds = 0, \quad \text{a.e.} \tag{2.8}$$

The assumption that $f = 0$ a.e. is false ensures the existence of $s_0 > 0$ such that $f^*(s) > 0$ when $0 < s < s_0$. But, for all sufficiently small t, the function $a(s/t)$ is greater than zero on a subset of $(0, s_0)$ of positive Lebesgue measure. Hence $f = 0$ a.e.

The subadditivity of $\boldsymbol{\sigma}$ will follow by duality given

$$\int_0^\infty h(t)[T(f+g)^*](t)\,dt \le \sigma(Tf^*)+\delta(Tg^*), \tag{2.9}$$

for all nonnegative, nonincreasing $h \in L^{\sigma'}$ with $\sigma'(h) \le 1$. But, the first term in (2.9) is equal to

$$\int_0^\infty a(u)\,du \int_0^\infty h(t)(f+g)^*(ut)\,dt. \tag{2.10}$$

Further,

$$\int_0^t (f+g)^*(us)\,ds \le \int_0^t f^*(us)\,ds + \int_0^t g^*(us)\,ds \tag{2.11}$$

together with a well-known result of Hardy and Littlewood, ensures (2.10) is dominated by

$$\int_0^\infty a(u)\,du \int_0^\infty h(t)f^*(ut)\,dt + \int_0^\infty a(u)\,du \int_0^\infty h(t)g^*(ut)\,dt. \tag{2.12}$$

After inverting the order of the integrals in (2.12), an appeal to the generalized Hölder inequality will yield (2.9).

To verify $\boldsymbol{\sigma}$ satisfies the Fatou property, observe that $0 \le f_n \uparrow f$ implies $Tf_n^* \uparrow Tf^*$ and hence, by the corresponding property of σ, $\boldsymbol{\sigma}(f_n) \uparrow \boldsymbol{\sigma}(f)$.

At this point we obtain from [9, p. 42] that $\boldsymbol{\sigma}$ gives rise to a Banach space in the usual way.

Suppose now $E \in \mathfrak{M}$, the class of Lebesgue-measurable subsets of $(0, \infty)$, and that $m(E) < \infty$. Then $T\chi_E^*$ will belong to L^σ if there exists $c > 0$ so that

$$\int_0^s [T\chi_{(0,\, m(E))}](u)\,du \le c \int_0^s [T\chi_{(0,\,1)}](u)\,du \tag{2.13}$$

for all $s > 0$. But, (2.13) just asks that

$$m(E)\Phi(s/m(E)) \le c\Phi(s) \qquad s > 0, \tag{2.14}$$

which is true with $c = \max(1, m(E))$, since Φ increases concavely from $\Phi(0) = 0$.

Finally, we show that to each $E \in \mathfrak{M}$, $m(E) < \infty$, there is associated a constant $k_E > 0$ so that

$$\int_E f(t)\,dt \le k_E \boldsymbol{\sigma}(f) \tag{2.15}$$

for all nonnegative $f \in M(0, \infty)$. It will be enough to show that for such f there is a $k > 0$ for which $\sigma(f) \le k\boldsymbol{\sigma}(f)$, because σ satisfies (2.15). To this end, fix f, g and suppose $\sigma'(g) \le 1$. Also, let $u > 0$ be such that $A(u) \equiv \int_0^u a(s)\,ds > 0$. We have

$$\int_0^\infty f^*(t)g^*(t)\,dt \le \max(1,u)\int_0^\infty f^*(ut)g^*(t)\,dt, \tag{2.16}$$

since $f^*(t) \le f^*(ut)$ for $0 < u \le 1$, while $g^*(t) \le g^*(t/u)$ for $u > 1$. Now, from Lemma 3.3 of [3],

$$A(u)\int_0^\infty f^*(ut)g^*(t)\,dt \le \int_0^u a(s)\,ds \int_0^\infty f^*(st)g^*(t)\,dt, \tag{2.17}$$

the latter being no bigger than

$$\begin{aligned}\int_0^\infty a(s)\,ds\int_0^\infty f^*(st)g^*(t)\,dt &= \int_0^\infty (Tf^*)(t)g^*(t)\,dt \\ &\le \sigma(Tf^*)\sigma'(g^*) = \boldsymbol{\sigma}(f)\end{aligned} \tag{2.18}$$

Thus,

$$\int_0^\infty f^*(t)g^*(t)\,dt \le k\boldsymbol{\sigma}(f) \tag{2.19}$$

where $k = \max(1,u)[A(u)]^{-1}$. The argument is completed on taking the supremum over g.

Clearly, $T \in [L^{\boldsymbol{\sigma}}, L^{\sigma}]$ by the very definition of $\boldsymbol{\sigma}$.

The result dual to Theorem 2.2 is

THEOREM 2.2′. *Let T be a positive operator of convolution-type having kernel $a(t)$ and associate T'. Suppose*

$$\int_0^\infty \min(1, 1/u)a(1/u)\frac{du}{u} < \infty. \tag{2.20}$$

Then the function $\psi = T'\chi_{(0,1)}$ is nonnegative and nonincreasing on $(0,\infty)$ with $\int_0^1 \psi(t)\,dt < \infty$. Moreover, to each rearrangement invariant norm σ on $M(0,\infty)$ with $L^\sigma \subset \Lambda(\psi)$ there corresponds a rearrangement invariant norm $\tilde{\sigma}$ such that $T \in [L^\sigma, L^{\tilde{\sigma}}]$.

Proof. Condition (2.20) is just condition (2.5) of Theorem 2.2 for T' and its kernel $(1/t)a(1/t)$. Further, $L^\sigma \subset \Lambda(\psi)$ implies $L^{\sigma'} \supset M(\psi)$. Let $\boldsymbol{\sigma}'$ be the norm guaranteed by Theorem 2.2 for T' and σ'. Take $\tilde{\sigma} = (\boldsymbol{\sigma}')'$.

REMARKS. 1. One may give $\tilde{\sigma}$ a somewhat more explicit form in Theorem 2.2′ using a construction analogue to that in Bennett [2]. Thus, firstly, $\tilde{\sigma}^0$ is defined at nonnegative $g \in M(0,\infty)$ by

$$\tilde{\sigma}^0(g) = \inf\{\sigma(|f|) : g^{**} \le (Tf^*)^{**}, f \in L^\sigma\}, \tag{2.21}$$

with the convention that $\tilde{\sigma}^0(g) = \infty$ if no such f exists. Then, $\tilde{\sigma}$ is given at nonnegative $g \in M(0, \infty)$ by

$$\tilde{\sigma}(g) = \sup \tilde{\sigma}^0(g\chi_E), \tag{2.22}$$

the supremum being taken over all Lebesgue-measurable subsets E of $(0, \infty)$ with $m(E) < \infty$.

2. It is clear from the constructions of $\boldsymbol{\sigma}$ and $\tilde{\sigma}$ that, with respect to T, $L^{\boldsymbol{\sigma}}$ is the largest domain space having L^σ as range, while $L^{\tilde{\sigma}}$ is the smallest range space having L^σ as domain. Further, $L^{\boldsymbol{\sigma}} \subset L^\sigma$ and hence $L^\sigma \subset L^{\tilde{\sigma}}$. In particular, if $T \in [L^\sigma]$, then $L^{\boldsymbol{\sigma}} = L^{\tilde{\sigma}} = L^\sigma$, the norms being equivalent.

Definition 2.1. Let σ_1 and σ_2 be rearrangement invariant norms. The functional $\sigma_1 \wedge \sigma_2$ is given at nonnegative $f \in M(0, \infty)$ by

$$(\sigma_1 \wedge \sigma_2)(f) = \max[\sigma_1(f), \sigma_2(f)]$$

Remark. One readily verifies that $\sigma_1 \wedge \sigma_2$ is a rearrangement invariant norm and that, as sets, $L^{\sigma_1 \wedge \sigma_2} = L^{\sigma_1} \cap L^{\sigma_2}$. In view of this we will use the intersection notation for $L^{\sigma_1 \wedge \sigma_2}$.

In what follows, σ_α and σ'_α $(0 < \alpha \le 1)$ will denote the usual Lorentz norms for which $\phi(t) = t^{\alpha - 1}$; $\Lambda(\alpha)$, $M(\alpha)$ the corresponding Lorentz spaces. To keep notation uniform we will write $\Lambda(0)$ for L^∞ and $M(0)$ for L^1.

Theorem 2.3. *Suppose T is a positive operator of convolution-type with kernel (1.2). Let σ denote the usual norm on the Lorentz space $M(\phi)$, $\phi = T\chi_{(0,1)}$. Then, as a set, $L^{\boldsymbol{\sigma}}$ is equal to $\Lambda(p^{-1}) \cap \Lambda(q^{-1})$. In particular, if $q < \infty$, this is $\Lambda(\max[t^\epsilon, t^\eta])$, $\epsilon = 1/p - 1$, $\eta = 1/q - 1$.*

Proof. The boundedness of T follows once it is shown that for $u > 0$ a constant multiple of the norm of f (in $\Lambda(p^{-1}) \cap \Lambda(q^{-1})$) dominates

$$\int_0^\infty f^*(t) g_u(t)\, dt, \tag{2.23}$$

where

$$g_u(t) = [T'\chi_{(0,u)}](t)/\Phi(u) \tag{2.24}$$

Now, for the kernel $a(r) = \min(r^\epsilon, r^\eta)$ one easily sees that

$$a(rt) \le \max(t^\epsilon, t^\eta) a(r) \tag{2.25}$$

and so

$$g_u(t) \le \int_{t/u}^\infty a(r) \frac{dr}{r} \Big/ \int_{1/u}^\infty a(r) \frac{dr}{r}, \tag{2.26}$$

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since

$$\Phi(u) = \int_0^\infty \min(u, 1/r)a(r)\,dr \geq \int_{1/u}^\infty a(r)\frac{dr}{r}. \tag{2.27}$$

But,

$$\int_{1/u}^\infty a(rt)\frac{dr}{r} \Big/ \int_{1/u}^\infty a(r)\frac{dr}{r} \leq \max(t^\epsilon, t^{\mathfrak{S}}), \tag{2.28}$$

by (2.25). This completes the proof of the boundedness in case $q<\infty$ and, indeed, gives

$$\int_1^\infty f^*(t)g_u(t)\,dt \leq \|f\|_\epsilon \tag{2.29}$$

for all q. It is enough to show now that

$$\int_0^1 f^*(t)g_u(t)\,dt \leq \|f\|_\infty \int_0^1 g_u(t)\,dt = \|f\|_\infty. \tag{2.30}$$

However,

$$\int_0^1 [T'\chi_{(0,u)}](t)\,dt = \int_0^u [T\chi_{(0,1)}](t)\,dt = \Phi(u). \tag{2.31}$$

The methods of [7, Theorem 4.7] readily show $\Lambda(p^{-1}) \cap \Lambda(q^{-1})$ is the largest space having range $M(\phi)$ under T. Indeed, suppose, if possible, that $f \in M(0,\infty)$, $Tf^* \in M(\phi)$, but $f \notin \Lambda(p^{-1})$. *From*

$$\lim_{u\to\infty} g_u(t) = -(1+\epsilon^{-1})t^\epsilon, \tag{2.32}$$

we conclude, using Fatou's lemma on (2.23), that

$$\|Tf\|_{M(\phi)} \geq -(1+\epsilon^{-1})\,\|f\|_\epsilon = \infty, \tag{2.33}$$

a contradiction. Similar considerations show that when $q<\infty$, one must have $f \in \Lambda(q^{-1})$ whenever $Tf^* \in M(\phi)$. Assume, then, if possible, that $f \in M(0,\infty)$, $(P_p + Q_\infty)f^* \in M(\phi)$, but $f \notin L^\infty$. Given $B>0$ there must exist $b>0$ such that $f^*(t) \geq B$ when $0<t\leq b$. For $u \leq t \leq b$, the expression (2.23) is no smaller than

$$B(u\ln(b/u)) \tag{2.34}$$

which approaches B as $u \to 0+$. Since B was arbitrary, a contradiction has been reached.

In view of the second remark following Theorem 2.2′, the proof is complete.

THEOREM 2.3′. *Supose T is a positive operator of convolution-type with kernel* (1.2). *Let σ denote the usual norm on the Lorentz space $\Lambda(\psi)$, $\psi = T'\chi_{(0,1)}$. Then, as a set, $L^{\bar\sigma}$ is equal to $M(1-1/p)+M(1-1/q)$. In particular, if $p>1$, it is* $M(\max[t^{-p^{-1}}, t^{-q^{-1}}])$.

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REMARKS. 1. The mappings of Theorems 2.2 and 2.2′ need not invert one another. Thus, if $-1<\eta<\epsilon<0$, Theorem 2.2 shows that both $L^1\cap L^\infty$ and $\Lambda(p^{-1})\cap\Lambda(q^{-1})(L^1\cap L^\infty\subsetneq\Lambda(p^{-1})\cap\Lambda(q^{-1}))$ must have $M(\phi)$ as their minimal range space under T. The other assertion follows by duality.

The above example leads to the conjecture that the mappings of Theorems 2.2 and 2.2′ applied succesively to $L^\sigma\cap\Lambda(\psi)$ yield the space $(L^\sigma\cap\Lambda(\psi))+L^\rho$, ρ being the usual norm of $M(\phi)$. This would just be L^σ when that space is intermediate between $\Lambda(p^{-1})$ and $\Lambda(q^{-1})$; that is,

$$\Lambda(p^{-1})\cap\Lambda(q^{-1})\subset L^\sigma\subset\Lambda(p^{-1})+\Lambda(q^{-1}). \tag{2.35}$$

In view of [4, Lemma 2] and [1, Theorem 13.VII], then the conjecture would not hold if L^σ were intermediate but not such that all operators in $[\Lambda(p^{-1})\cap [\Lambda(q^{-1})]$ were in $[L^\sigma]$. But, for $p=2$, $q=4$, $L^\sigma\equiv L^2\cap\Lambda(\frac{1}{3})$ satisfies (2.35), while the mapping that sends f to

$$\left(\int_0^\infty \min(t^{-1/2}, t^{-3/4})f(t)\,dt\right)\chi_{(0,1)} \tag{2.36}$$

is in $[\Lambda(1/2)]\cap[\Lambda(1/4)]$, though not in $[L^\sigma]$.

2. It is easily seen that the mappings of Theorems 2.2 and 2.2′ do invert each other when restricted in the domain spaces to the L^σ or in the range spaces to the $L^{\tilde\sigma}$.

3. If T has kernel

$$\max[t^{1/p-1}, t^{1/q-1}], \qquad 1<p\le q<\infty, \tag{2.37}$$

then it will be bounded between every reasonable pair of rearrangement invariant spaces; more precisely,

$$T\in[\Lambda(\psi), M(\phi)]. \tag{2.38}$$

Indeed, (2.38) will be true for a general T of form (1.1) if and only if

$$\Phi(rs)\le c\Phi(r)\Phi(s), \tag{2.39}$$

$c>0$ being independent of $r, s>0$.

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