

ON NEAR-RING IDEMPOTENTS AND POLYNOMIALS ON DIRECT PRODUCTS OF Ω -GROUPS

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Abstract Let N be a zero-symmetric near-ring with identity, and let Γ be a faithful tame N -group. We characterize those ideals of Γ that are the range of some idempotent element of N . Using these idempotents, we show that the polynomials on the direct product of the finite Ω -groups V_1, V_2, \dots, V_n can be studied componentwise if and only if $\prod_{i=1}^n V_i$ has no skew congruences.

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1. Ideals that are the range of a near-ring element

For the basic notions of near-ring theory, such as *near-ring*, *zero-symmetric*, *left ideal*, *N -group*, *ideal*, we refer to [13].

Let N be a zero-symmetric near-ring with identity **id**. Then the unital N -group Γ is called *tame* [17] if and only if for all $\gamma, x \in \Gamma$ and $\mathbf{n} \in N$ there is an element $\mathbf{m} \in N$ such that

$$\mathbf{n} \cdot (\gamma + x) - \mathbf{n} \cdot \gamma = \mathbf{m} \cdot x.$$

We write near-ring elements in bold italic and N -group elements in plain italic. Every N -subgroup of a tame N -group Γ is an ideal of Γ . The lattice of ideals of Γ will be abbreviated by $\text{ld } \Gamma$; the sum of the ideals I and J is written as $I \vee J$, their intersection as $I \wedge J$. It is well known that for $A, B \in \text{ld } \Gamma$, $a \in A$, $b \in B$, $\mathbf{n} \in N$, the element $\mathbf{n} \cdot a + \mathbf{n} \cdot b - \mathbf{n} \cdot (a + b)$ lies in $A \wedge B$ [16, Proposition 1.3]. For two ideals $A, B \in \text{ld } \Gamma$ with $A \leq B$, we denote the interval $\{I \in \text{ld } \Gamma \mid A \leq I \leq B\}$ by $I[A, B]$. We write $A \prec B$ (A is a *subcover* of B) if and only if $A < B$ and there is no $I \in \text{ld } \Gamma$ with $A < I < B$. We call an ideal A of Γ *join irreducible* if and only if $A = B \vee C$ implies $A = B$ or $A = C$. In a lattice with both chain conditions, the join irreducible element A has a unique subcover; we abbreviate it by A^- . We say that two N -groups G and H are *N -isomorphic* if and only if there is a group isomorphism $\varphi : G \rightarrow H$ with $\mathbf{n} \cdot \varphi(g) = \varphi(\mathbf{n} \cdot g)$ for all $\mathbf{n} \in N$, $g \in G$. For A, B in $\text{ld } \Gamma$ with $A \leq B$, we define the set $B/A := \{b + A \mid b \in B\}$. On this

set we define addition by $(b_1 + A) + (b_2 + A) := (b_1 + b_2) + A$, and the operation of N by $\mathbf{n} \cdot (b + A) := (\mathbf{n} \cdot b) + A$. If Γ is a tame N -group and A, B are ideals of Γ with $A \leq B$, then the N -group B/A is tame, too. If, furthermore, $A \prec B$ in $\text{ld } \Gamma$, then the N -group B/A has no ideals except $0 = A/A$ and B/A . We write $\text{Ann}(B/A)$ for the annihilator of B/A , which means

$$\text{Ann}(B/A) = \{\mathbf{n} \in N \mid \mathbf{n} \cdot B \subseteq A\}.$$

The annihilator $\text{Ann}(B/A)$ is an ideal of the near-ring $(N, +, \circ)$. If the tame near-ring N has the descending chain condition (DCC) on left ideals (called DCCL from now on), and if $A \prec B$ in $\text{ld } \Gamma$, the near-ring $N/\text{Ann}(B/A)$ is 2-primitive. A 2-primitive near-ring R with identity and DCCL is either the full matrix-ring of $k \times k$ -matrices over a division ring, in which case it is simple [8, Theorem (3.1), p. 32], or, if it is not a ring, then it is a centralizer near-ring. By [3, 12], every 2-primitive near-ring with identity and DCCL which is not a ring is simple. Altogether, $N/\text{Ann}(B/A)$ is simple, and thus $\text{Ann}(B/A)$ is a maximal ideal of N .

Our first result characterizes those N -subgroups of Γ that are the range of an idempotent element of N .

Theorem 1.1. *Let N be a zero-symmetric near-ring with identity, and let Γ be a tame N -group. We assume that N has the DCC on left ideals, and that the ideal lattice of Γ satisfies both the ascending chain condition (ACC) and the DCC. Let H be an N -subgroup of Γ . Then the following are equivalent.*

- (1) *There is an element $e \in N$ with $e \cdot h = h$ for all $h \in H$, and $e \cdot \Gamma \subseteq H$.*
- (2) *If A and D are join irreducible ideals in $\text{ld } \Gamma$, and if $A \leq H$ and the N -groups A/A^- and D/D^- are N -isomorphic, then $D \leq H$.*

Proof. (1) \Rightarrow (2). We fix two join irreducible ideals $A, B \in \text{ld } \Gamma$ with $A \leq H$ such that A/A^- and B/B^- are N -isomorphic. We have $(e - \mathbf{id}) \cdot a = 0$ for all $a \in A$, and therefore $(e - \mathbf{id}) \cdot (a + A^-) = 0 + A^-$ for all $a \in A$. Since A/A^- and B/B^- are N -isomorphic, $(e - \mathbf{id}) \cdot (b + B^-) = 0 + B^-$ for all $b \in B$. Thus for every $b \in B$, b is congruent to $e \cdot b$ modulo B^- . Since $e \cdot b$ lies in H , we get

$$\text{for all } b \in B : b \in B^- \vee H.$$

Hence $B \leq B^- \vee H$, and therefore $B = B \wedge (B^- \vee H)$, and, by the modularity of the lattice $\text{ld } \Gamma$, $B = B^- \vee (B \wedge H)$. Since B is join irreducible, we obtain $B \wedge H = B$, and therefore $B \leq H$. This completes the proof of (1) \Rightarrow (2).

(2) \Rightarrow (1). We may assume $H \neq 0$. We define a subset \mathcal{B} of $\text{ld } \Gamma$ by

$$\mathcal{B} := \{B \in \text{ld } \Gamma \mid \text{there is no } \mathbf{p} \in N \text{ with } (\mathbf{p} - \mathbf{id}) \cdot H = 0 \text{ and } \mathbf{p} \cdot B \subseteq H\}. \quad (1.1)$$

Our endeavour will be to show $\mathcal{B} = \emptyset$, because then $\Gamma \notin \mathcal{B}$, which yields a $\mathbf{p} \in N$ with $\mathbf{p} \cdot h = h$ for all $h \in H$, and $\mathbf{p} \cdot \Gamma \subseteq H$. Seeking a contradiction, we suppose $\mathcal{B} \neq \emptyset$. Since $\text{ld } \Gamma$ has the DCC, \mathcal{B} has a minimal element, say D . Since $0 \notin \mathcal{B}$ (take $\mathbf{p} := \mathbf{id}$ in (1.1)), we have $D \neq 0$. We will now show that

$$D \text{ is join irreducible.} \quad (1.2)$$

Suppose $D = D_1 \vee D_2$ with $D_1 < D, D_2 < D$. By the minimality of D , we have $D_1 \notin \mathcal{B}, D_2 \notin \mathcal{B}$. Hence there are $\mathbf{p}_1, \mathbf{p}_2 \in N$ with

$$\begin{aligned} \mathbf{p}_1 \cdot D_1 &\subseteq H, \quad \forall h \in H : \mathbf{p}_1 \cdot h = h, \\ \mathbf{p}_2 \cdot D_2 &\subseteq H, \quad \forall h \in H : \mathbf{p}_2 \cdot h = h. \end{aligned}$$

We define $\mathbf{p}_3 := \mathbf{p}_2 \circ \mathbf{p}_1$. We have $\mathbf{p}_3 \cdot D_1 \subseteq H$ and $\mathbf{p}_3 \cdot D_2 \subseteq H$. Let $\mathbf{p}_4 := \mathbf{p}_3 \circ \mathbf{p}_3$. We show

$$\mathbf{p}_4 \cdot D \subseteq H. \tag{1.3}$$

We fix $d \in D$. Since $D_1 \vee D_2 = D$, there are $d_1 \in D_1, d_2 \in D_2$ with $d = d_1 + d_2$. We first compute $\mathbf{p}_3 \cdot (d_1 + d_2)$. Since $\mathbf{p}_3 \cdot (d_1 + d_2)$ is congruent to $\mathbf{p}_3 \cdot d_1 + \mathbf{p}_3 \cdot d_2$ modulo $D_1 \wedge D_2$, there is an element $d_3 \in D_1 \wedge D_2$ such that

$$\mathbf{p}_3 \cdot (d_1 + d_2) = (\mathbf{p}_3 \cdot d_1 + \mathbf{p}_3 \cdot d_2) + d_3.$$

Since $\mathbf{p}_3 \cdot d_1 \in H$ and $\mathbf{p}_3 \cdot d_2 \in H$, there is an element $h \in H$ such that $\mathbf{p}_3 \cdot (d_1 + d_2) = h + d_3$. Applying \mathbf{p}_3 again, we obtain

$$\begin{aligned} (\mathbf{p}_3 \circ \mathbf{p}_3) \cdot (d_1 + d_2) &= \mathbf{p}_3 \cdot (h + d_3) \\ &= (\mathbf{p}_3 \cdot h + \mathbf{p}_3 \cdot d_3) + d_4, \end{aligned}$$

where d_4 lies in $H \wedge (D_1 \wedge D_2)$. Therefore every summand of $\mathbf{p}_3 \cdot h + \mathbf{p}_3 \cdot d_3 + d_4$ lies in H , which proves (1.3). Altogether, we have $\mathbf{p}_4 \cdot h = h$ for all $h \in H$ and $\mathbf{p}_4 \cdot D \subseteq H$. This shows that D is not in \mathcal{B} , a contradiction to the choice of D , and completes the proof of (1.2).

The next step is to show that

$$\text{for all } \mathbf{p} \in N : \mathbf{p} \cdot H = 0 \Rightarrow \mathbf{p} \cdot D \subseteq D^-. \tag{1.4}$$

We consider the ideals $\text{Ann}(H) = \{\mathbf{n} \in N \mid \mathbf{n} \cdot H = 0\}$ and $\text{Ann}(D/D^-) = \{\mathbf{n} \in N \mid \mathbf{n} \cdot D \subseteq D^-\}$. Since $D^- \prec D$ in $\text{ld } \Gamma$, the N -group D/D^- has no proper N -subgroups; therefore $N/\text{Ann}(D/D^-)$ is 2-primitive on $\text{Ann}(D/D^-)$, and thus simple. Hence $\text{Ann}(D/D^-)$ is a maximal ideal of N . Supposing that (1.4) fails, we have $\text{Ann}(H) \not\subseteq \text{Ann}(D/D^-)$. By the maximality of $\text{Ann}(D/D^-)$, this implies

$$\text{Ann}(H) + \text{Ann}(D/D^-) = N.$$

Thus there are $\mathbf{a} \in \text{Ann}(H), \mathbf{i} \in \text{Ann}(D/D^-)$ with $\mathbf{a} + \mathbf{i} = \mathbf{id}$. The element \mathbf{i} satisfies $\mathbf{i} \cdot D \subseteq D^-$ and $\mathbf{i} \cdot h = (-\mathbf{a} + \mathbf{id}) \cdot h = -0 + h = h$ for all $h \in H$. Since D is minimal in \mathcal{B} , we have $D^- \notin \mathcal{B}$. So there exists $\mathbf{p}_5 \in N$ with $\mathbf{p}_5 \cdot h = h$ for all $h \in H$ and $\mathbf{p}_5 \cdot D^- \subseteq H$. We consider the element $\mathbf{p}_6 \in N$ defined by $\mathbf{p}_6 := \mathbf{p}_5 \circ \mathbf{i}$. It is easy to see that $\mathbf{p}_6 \cdot h = h$ for all $h \in H$. We also have $\mathbf{p}_6 \cdot D = \mathbf{p}_5 \cdot (\mathbf{i} \cdot D) \subseteq \mathbf{p}_5 \cdot D^- \subseteq H$. The existence of such a $\mathbf{p}_6 \in N$ implies $D \notin \mathcal{B}$; this contradiction proves (1.4). We defer the rest of the proof of Theorem 1.1 to p. 384 because we first need some facts about near-rings. \square

Let N be a near-ring which is tame on Γ . In the sequel, we state two conditions on ideals A, B, C, D with $A \leq B$ and $C \leq D$ that imply that the N -groups B/A and D/C are N -isomorphic. We express the first condition using notions from lattice theory. For $A, B, C, D \in \text{Id } \Gamma$ with $A \leq B, C \leq D$, we say that the interval $I[A, B]$ *projects up* to $I[C, D]$ if and only if $A = B \wedge C$ and $D = B \vee C$, and we write $I[A, B] \nearrow I[C, D]$ or $I[C, D] \searrow I[A, B]$. The smallest equivalence relation that contains \nearrow will be abbreviated by \leftrightarrow . If $I[A, B] \leftrightarrow I[C, D]$, we say that the two intervals are *projective*.

Lemma 1.2 (cf. [1]). *Let N be a zero-symmetric near-ring, let Γ be a tame N -group, and let A, B, C, D be ideals of Γ with $A \leq B, C \leq D$ such that the intervals $I[A, B]$ and $I[C, D]$ are projective. Then the N -groups B/A and D/C are N -isomorphic.*

Proof. We assume $I[A, B] \nearrow I[C, D]$. Then every element in $d \in D$ can be written as $d = b + c$ with $b \in B, c \in C$. The mapping $\varphi : D/C \rightarrow B/A$ with $\varphi((b + c) + C) = b + A$ is an isomorphism. \square

The result is actually well-known as the homomorphism theorem $(B + C)/B \cong_N B/(C \cap B)$ [11, Theorem 2.28].

The next method to find isomorphic N -groups is a generalization of the known fact that for a finite simple ring with unit R , all faithful simple unital R -modules are isomorphic (cf. [14, Proposition 2.1.15, p. 154], [3, Theorem 4.3]). We need the following lemma.

Lemma 1.3 (cf. Proposition 2.2 of [1], [2] and Theorem 4.56(a) of [13]). *Let N be a zero-symmetric near-ring, let I be an ideal of N , and let Γ be an N -group that satisfies $\text{Ann}(\Gamma) = I$ and $N \cdot \gamma = \Gamma$ for all $\gamma \in \Gamma$ with $\gamma \neq 0$. We assume that we have a left ideal L of N such that $L > I$ and there is no left ideal L' of N with $L > L' > I$. Then the N -group Γ is N -isomorphic to the N -group L/I .*

Proof. Since $L \not\leq \text{Ann}(\Gamma)$, we have elements $\mathbf{l}_0 \in L, \gamma_0 \in \Gamma$ with $\mathbf{l}_0 \cdot \gamma_0 \neq 0$. We define a mapping φ by

$$\varphi : L \rightarrow \Gamma, \quad \mathbf{l} \mapsto \mathbf{l} \cdot \gamma_0.$$

It is easy to see that φ is an N -homomorphism from the N -group L to the N -group Γ . Since $\mathbf{l}_0 \cdot \gamma_0 \neq 0$, the assumptions on Γ yield $N \cdot \mathbf{l}_0 \cdot \gamma_0 = \Gamma$. Since $N \cdot \mathbf{l}_0 \subseteq L$, we get $L \cdot \gamma_0 = \Gamma$, and hence φ is surjective. We take L' to be the kernel of φ , i.e.

$$L' = \{\mathbf{l} \in L \mid \mathbf{l} \cdot \gamma_0 = 0\}.$$

We check that L' is a left ideal of N . Furthermore, every element of $I = \text{Ann}(\Gamma)$ lies in L' . So we have

$$I \leq L' \leq L.$$

Since by the assumptions L covers I , L' has to be either L or I . The element \mathbf{l}_0 shows $L' < L$, and so $L' = I$. The homomorphism theorem yields that the N -group $L/\ker \varphi = L/I$ is N -isomorphic to Γ . \square

If two N -groups are N -isomorphic, they have the same annihilators. Sometimes, the converse is true.

Corollary 1.4 (cf. [1]). *Let N be a zero-symmetric near-ring with identity, let Γ be a tame N -group, and let A, B, C, D be ideals of Γ with $A \prec B, C \prec D$, and $\text{Ann}(B/A) = \text{Ann}(D/C)$. If the near-ring $N/\text{Ann}(B/A)$ has a minimal left ideal, the N -groups B/A and D/C are N -isomorphic.*

Proof. Since $A \prec B$, and since Γ is a tame N -group, B/A has no N -subgroups. For every $\beta \in B/A$, the set $N \cdot \beta$ forms an N -subgroup of B/A , and since for the identity of \mathbf{id} of N we have $\mathbf{id} \cdot \beta = \beta$, we see $N \cdot \beta = B/A$ for every non-zero $\beta \in B/A$. The fact that $N/\text{Ann}(B/A)$ has a minimal left ideal gives us the left ideal L required in the assumptions of Lemma 1.3. Now the result follows from this Lemma. \square

Lemma 1.5 (cf. [2]). *Let N be a zero-symmetric near-ring with identity and the DCC on left ideals, let Γ be a tame N -group such that $\text{ld } \Gamma$ satisfies both the ACC and the DCC, and let $A, B, C, D \in \text{ld } \Gamma$ such that $C \leq D, A \prec B$. We assume that each $\mathbf{p} \in N$ with $\mathbf{p} \cdot D \subseteq C$ satisfies $\mathbf{p} \cdot B \subseteq A$. Then there are ideals $C', D' \in \text{ld } \Gamma$ with $C \leq C' \prec D' \leq D$ such that D'/C' and B/A are N -isomorphic.*

Proof. We take I to be the ideal $\text{Ann}(B/A)$ of N . Since $A \prec B$ in $\text{ld } \Gamma$, the near-ring N/I is 2-primitive on B/A , and thus, as a 2-primitive near-ring with DCCL, simple. Hence I is a maximal ideal of N .

Let C_1 be any ideal of Γ in $I[C, D]$. Let J_1, J_2 be the ideals of N defined by

$$J_1 := \text{Ann}(C_1/C), \quad J_2 := \text{Ann}(D/C_1).$$

We show that

$$J_1 \subseteq I \quad \text{or} \quad J_2 \subseteq I. \tag{1.5}$$

Seeking a contradiction, we suppose that both inclusions fail. Since I is a maximal ideal, we get $I + J_1 = N$ and $I + J_2 = N$. Thus there are $\mathbf{i}_1, \mathbf{i}_2 \in I$ and $\mathbf{j}_1 \in J_1, \mathbf{j}_2 \in J_2$ with

$$\mathbf{i}_1 + \mathbf{j}_1 = \mathbf{id} \quad \text{and} \quad \mathbf{i}_2 + \mathbf{j}_2 = \mathbf{id}. \tag{1.6}$$

We observe that $\mathbf{j}_1 \circ \mathbf{j}_2$ lies in $\text{Ann}(D/C)$ because $\mathbf{j}_1 \cdot (\mathbf{j}_2 \cdot D) \subseteq \mathbf{j}_1 \cdot C_1 \subseteq C$. By the assumption, $\mathbf{j}_1 \circ \mathbf{j}_2$ therefore lies in I . By (1.6), \mathbf{j}_1 and \mathbf{j}_2 are both congruent to \mathbf{id} modulo I . This implies $\mathbf{id} \circ \mathbf{id} \in I$, yielding the contradiction $\mathbf{id} \in I$, and thus completing the proof of (1.5). Since every chain in $\text{ld } \Gamma$ is finite, repeating this process allows us to obtain $C', D' \in \text{ld } \Gamma$ with $C \leq C' \prec D' \leq D$ and

$$\text{Ann}(D'/C') \leq I.$$

Since $C' \prec D'$, $N/\text{Ann}(D'/C')$ is 2-primitive with DCCL, and thus simple, and so $\text{Ann}(D'/C')$ is a maximal ideal of N , and therefore equal to I . Since N has the DCCL, Corollary 1.4 yields that the N -groups D'/C' and B/A are N -isomorphic. \square

We can now resume the proof of Theorem 1.1 suspended on p. 381.

Continuation of the proof of Theorem 1.1. Having proved (1.4), we can use Lemma 1.5 to produce ideals $0 \leq C_1 \prec C_2 \leq H$ such that C_2/C_1 and D/D^- are N -isomorphic. We take A to be minimal in $\text{ld } \Gamma$ with $A \leq C_2$, $A \not\leq C_1$. Then A is join irreducible and the interval $I[A^-, A]$ projects up to $I[C_1, C_2]$ in $\text{ld } \Gamma$. Thus, by Lemma 1.2, A/A^- and C_2/C_1 are N -isomorphic. Since $A \leq C_2 \leq H$, A lies in $I[0, H]$. Every ideal in $I[0, H]$ does not lie in \mathcal{B} (take $\mathbf{p} := \mathbf{id}$ in (1.1)), and since $D \in \mathcal{B}$, we get $D \not\leq H$. So A and D are two join-irreducible ideals of Γ such that A/A^- and D/D^- are N -isomorphic, $A \leq H$ and $D \not\leq H$, contradicting the assumptions. This completes the proof of (2) \Rightarrow (1) of Theorem 1.1. \square

2. Polynomials on a direct product without skew congruences

We shall now investigate the polynomial functions on an Ω -group V which is a direct product of *similar* (cf. [10, p. 13]) Ω -groups; we do so by applying Theorem 1.1 to the near-ring $P_0(V)$ of zero-preserving polynomial functions on V . For the notion of Ω -group, we refer to [7]; polynomial functions are defined, for example, in [9] or [10, Definition 4.4]. The near-ring operations of $P_0(V)$ are pointwise addition of functions, and functional composition. If V is a group (without further operations), then this near-ring is more commonly known as the inner automorphism near-ring $I(V)$.

Let R_1 and R_2 be two commutative rings with unit. It is well known that $P_0(R_1 \times R_2)$ is isomorphic to $P_0(R_1) \times P_0(R_2)$ via a (natural) isomorphism. The same is not true for groups: let $G_1 = G_2 = \mathbb{Z}_2$. We know that $|I(\mathbb{Z}_2 \times \mathbb{Z}_2)| = |I(\mathbb{Z}_2)| = 2$, making the desired isomorphism impossible. However, using Scott's concept of length [15], it is possible to show that for finite groups G_1, G_2 we have $I(G_1 \times G_2) \cong I(G_1) \times I(G_2)$ if and only if $G_1 \times G_2$ has no skew congruences. This is the case if and only if every normal subgroup N of $G_1 \times G_2$ is equal to $N_1 \times N_2$ for some normal subgroups N_1 of G_1 and N_2 of G_2 . The following theorem tells that all Ω -groups behave like groups.

Theorem 2.1. *Let V_1, V_2, \dots, V_k be similar finite Ω -groups, and let $V := \prod_{i=1}^k V_i$. Then the following statements are equivalent.*

- (1) *For every ideal I of V there are ideals I_1 of V_1, I_2 of V_2, \dots, I_k of V_k such that $I = \prod_{j=1}^k I_j$.*
- (2) *For every $j \in \{1, 2, \dots, k\}$, the function*

$$e_j : V \rightarrow V, e_j(v_1, \dots, v_k) = (0, \dots, 0, v_j, 0, \dots, 0)$$

(with v_j standing on the j th place) is in $P_0(V)$.

Proof. Let W_j be the ideal $0 \times \dots \times 0 \times V_j \times 0 \times \dots \times 0$ (V_j on the j th place), i.e. W_j is the intersection of the kernels of the projections to all but the j th component. Condition (1) is equivalent to

$$\text{For every ideal } I \text{ of } V: \quad I \leq \bigvee_{j=1}^k (I \wedge W_j). \quad (2.1)$$

We first prove (2) \Rightarrow (1): we show that (2.1) holds. To this end, let $i \in I$. Since $i = \sum_{j=1}^k e_j(i)$ and $e_j(i)$ lies in both W_j and I , i also lies in $\bigvee_{j=1}^k (I \wedge W_j)$.

(1) \Rightarrow (2). We have to show that e_j is in $P_0(V)$. For simplicity we assume that $j = 1$. Let $W_1 := V_1 \times 0 \times \dots \times 0$, and let $K_1 := 0 \times V_2 \times V_3 \times \dots \times V_k$. We first construct a polynomial function e with $e(V) \subseteq W_1$ and $e(w_1) = w_1$ for all $w_1 \in W_1$. By Theorem 1.1, there is such an $e \in P_0(V)$ if for all join irreducible ideals A, D of V such that A/A^- and D/D^- are $P_0(V)$ -isomorphic and $A \leq W_1$, the ideal D satisfies

$$D \leq W_1. \tag{2.2}$$

Suppose $D \not\leq W_1$. By the fact that V has no skew congruences, we know that

$$D = (D \wedge W_1) \vee (D \wedge K_1).$$

Since $D \not\leq W_1$, we have $D \wedge W_1 < D$. But D is join irreducible, and thus we get $D = D \wedge K_1$, which implies $D \leq K_1$. Let $\bar{\delta}$ be the $P_0(V)$ -isomorphism from A/A^- to D/D^- . Then we can define a mapping δ from A to D such that $\delta(a) + D^- = \bar{\delta}(a + A^-)$ for all $a \in A$. We have

$$\mathbf{p}(\delta(a)) \equiv \delta(\mathbf{p}(a)) \pmod{D^-} \quad \text{for all } \mathbf{p} \in P_0(V), a \in A,$$

and also

$$\delta(a_1 + a_2) \equiv \delta(a_1) + \delta(a_2) \pmod{D^-} \quad \text{for all } a_1, a_2 \in A.$$

We define a subset S of V by

$$S := \{a + \delta(a) + a_1 + d_1 \mid a \in A, a_1 \in A^-, d_1 \in D^-\}.$$

We will now prove that S is an ideal of V . To this end, we check that for $s_1, s_2 \in S$ and $\mathbf{p} \in P_0(V)$, we have $s_1 + s_2 \in S$ and $\mathbf{p}(s_1) \in S$. Let $a + \delta(a) + a_1 + d_1$ and $a' + \delta(a') + a'_1 + d'_1$ ($a, a' \in A, a_1, a'_1 \in A^-, d_1, d'_1 \in D^-$) be elements of S . We know that

$$a + \delta(a) + a_1 + d_1 + a' + \delta(a') + a'_1 + d'_1 \tag{2.3}$$

is congruent to $a + \delta(a) + a' + \delta(a')$ modulo $A^- \vee D^-$. Since $A \wedge D = 0$, and since for all ideals I, J of V and $i \in I, j \in J$, we have $i + j \equiv j + i \pmod{I \wedge J}$, this is equal to $a + a' + \delta(a) + \delta(a')$, which is congruent to $a + a' + \delta(a + a')$ modulo D^- . These congruences allow us to write expression (2.3) as $a + a' + \delta(a + a') + e$, where $e \in A^- \vee D^-$, which proves that (2.3) lies in S ; thus S is closed under $+$. For computing

$$\mathbf{p}(a + \delta(a) + a_1 + d_1),$$

we observe that this is congruent to $\mathbf{p}(a + \delta(a))$ modulo $A^- \vee D^-$. Since $A \wedge D = 0$, we have $\mathbf{p}(a + \delta(a)) = \mathbf{p}(a) + \mathbf{p}(\delta(a))$, which is congruent to $\mathbf{p}(a) + \delta(\mathbf{p}(a))$ modulo D^- . So we can write $\mathbf{p}(a + \delta(a) + a_1 + d_1)$ as $\mathbf{p}(a) + \delta(\mathbf{p}(a)) + e'$, where e' is in $A^- \vee D^-$, which proves that S is closed under the application of zero-preserving unary polynomial functions. By [13, Theorem 7.123], S is an ideal of V .

We compute $S \wedge K_1$. Let $s = a + \delta(a) + a_1 + d_1$ ($a \in A$, $a_1 \in A^-$, $d_1 \in D^-$) be an element of $S \wedge K_1$. Since s , $\delta(a)$ and d_1 lie in K_1 , we obtain $a + a_1 \in K_1$. Since a and a_1 are elements of W_1 , we have $a + a_1 \in K_1 \wedge W_1 = 0$. Thus $a \in A^-$, and therefore $\delta(a) \in D^-$. So all four summands a , $\delta(a)$, a_1 , d_1 lie in $A^- \vee D^-$, which implies

$$S \wedge K_1 \leq A^- \vee D^-.$$

Now we compute $S \wedge W_1$. We assume $a + \delta(a) + a_1 + d_1 \in W_1$. Since a and a_1 are in A , which is less than or equal to W_1 , we obtain $\delta(a) + d_1 \in W_1$. Thus $\delta(a) + d_1 \in W_1 \wedge K_1 = 0$, which implies $\delta(a) \in D^-$. Since $\bar{\delta}$ is an isomorphism, we get $a \in A^-$. Again, we see that all four summands are in $A^- \vee D^-$, and so we get

$$S \wedge W_1 \leq A^- \vee D^-.$$

By the fact that V has no skew congruences, we have $S = (S \wedge W_1) \vee (S \wedge K_1)$, and therefore $S \leq A^- \vee D^-$. We will infer the contradiction $A \leq A^-$ from this fact. We fix $a \in A$. Since $a + \delta(a) \in A^- \vee D^-$, we get $a \in (A^- \vee D^-) \vee D = A^- \vee D$. Thus, we have $A \leq A^- \vee D$. We know that the lattice of ideals of V is modular; by this fact and $D \leq K_1$, we have $A \leq (A^- \vee D) \wedge W_1 = A^- \vee (D \wedge W_1) = A^- \vee 0 = A^-$, a contradiction. This proves (2.2). So, by Theorem 1.1, there is an $e \in P_0(V)$ with $e(w_1) = w_1$ for all $w_1 \in W_1$ and $e(V) \subseteq W_1$. What remains for us to determine is the behaviour of e on $V - W_1$. To this end, we compute $e(w_1 + k_1)$ with $w_1 \in W_1$, $k_1 \in K_1$. The difference $e(w_1 + k_1) - e(w_1)$ lies in K_1 , and since $e(V) \subseteq W_1$, it also lies in W_1 . From $W_1 \wedge K_1 = 0$, we obtain

$$e(w_1 + k_1) = e(w_1) = w_1.$$

Thus e is the required polynomial e_1 . □

We will now study the polynomials on a direct product without skew-congruences. By $\text{Pol}_n V$, we denote all n -ary polynomial functions from V^n to V .

Corollary 2.2. *Let V_1, \dots, V_k be similar finite Ω -groups such that $V = \prod_{j=1}^k V_j$ has no skew-congruences. Then there is a bijective mapping*

$$\Phi : \text{Pol}_n \prod_{j=1}^k V_j \rightarrow \prod_{j=1}^k \text{Pol}_n V_j.$$

Proof. Let e_1, \dots, e_k be the functions in $P_0(V)$ produced by Theorem 2.1. We observe that V_i is isomorphic to V/β_i , where β_i is the congruence corresponding to $V_1 \times \dots \times V_{i-1} \times 0 \times V_{i+1} \times \dots \times V_k$. If α is a congruence of V and $v \in V$, we write v/α for the congruence class of v modulo α . For a polynomial $\mathbf{p} \in \text{Pol}_n V$ and a congruence α of V , $[\mathbf{p}]_\alpha$ denotes the function in $\text{Pol}_n V/\alpha$ defined by $[\mathbf{p}]_\alpha(v_1/\alpha, \dots, v_n/\alpha) := \mathbf{p}(v_1, \dots, v_n)/\alpha$. We define Φ by

$$\begin{aligned} \Phi : \text{Pol}_n V &\rightarrow \prod_{i=1}^k \text{Pol}_n V/\beta_i \\ \mathbf{p} &\mapsto ([\mathbf{p}]_{\beta_1}, \dots, [\mathbf{p}]_{\beta_k}). \end{aligned}$$

Since $\beta_1 \wedge \dots \wedge \beta_k$ is the equality relation on V , the mapping Φ is injective. In order to show that Φ is surjective, we fix

$$(\mathbf{q}_1, \dots, \mathbf{q}_k) \in \prod_{i=1}^k \text{Pol}_n V / \beta_i.$$

Let $\mathbf{p}_1, \dots, \mathbf{p}_k \in \text{Pol}_n V$ be such that $[\mathbf{p}_i]_{\beta_i} = \mathbf{q}_i$. We observe that $[\mathbf{e}_i]_{\beta_j} = [\mathbf{0}]_{\beta_j}$ for $i \neq j$, whereas $[\mathbf{e}_i]_{\beta_i} = [\mathbf{id}]_{\beta_i}$. We define

$$\mathbf{p} := \sum_{i=1}^k (\mathbf{e}_i \circ \mathbf{p}_i).$$

Then for $i \in \{1, \dots, k\}$, we obtain

$$[\mathbf{p}]_{\beta_i} = \left[\sum_{j=1}^k \mathbf{e}_j \circ \mathbf{p}_j \right]_{\beta_i}.$$

For every congruence α of V , the mapping $\mathbf{p} \mapsto [\mathbf{p}]_{\alpha}$ is a homomorphism with respect to $+$ and \circ , and thus we get

$$\begin{aligned} [\mathbf{p}]_{\beta_i} &= \sum_{j=1}^k [\mathbf{e}_j]_{\beta_i} \circ [\mathbf{p}_j]_{\beta_i} \\ &= [\mathbf{e}_i]_{\beta_i} \circ [\mathbf{p}_i]_{\beta_i} \\ &= [\mathbf{id}]_{\beta_i} \circ [\mathbf{p}_i]_{\beta_i} \\ &= [\mathbf{p}_i]_{\beta_i} \\ &= \mathbf{q}_i. \end{aligned}$$

This proves that Φ is indeed surjective. □

Remarks

For every Ω -group W , the set of mappings $\text{Pol}_n W$ is closed under the pointwise application of the operations of W . To these pointwise operations, we add the $n + 1$ -ary operation \circ of composition defined by

$$\mathbf{f}_0 \circ (\mathbf{f}_1, \dots, \mathbf{f}_n)(w_1, \dots, w_n) := \mathbf{f}_0(\mathbf{f}_1(w_1, \dots, w_n), \dots, \mathbf{f}_n(w_1, \dots, w_n)).$$

Under the assumptions of Corollary 2.2, the bijective mapping Φ is then even an isomorphism with respect to these operations.

For the case of groups (without further operations), Theorem 2.1 can also be derived from Scott’s results about polynomial functions on groups in [15]. In contrast to the methods applied in the present paper, Scott makes use of the term representation of a polynomial function, and uses this term representation to define a concept of ‘degree’ for polynomial functions on groups, called *length*. The concept of length was generalized to Ω -groups in [4], but we do not see how to derive Theorem 2.1 from their results.

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