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TEST MAP AND DISCRETENESS IN SL(2, $\mathbb{H}$)

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Abstract. Let $\mathbb{H}$ be the division ring of real quaternions. Let SL(2, $\mathbb{H}$) be the group of 2×2 quaternionic matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with quaternionic determinant $\det A = |ad - aca^{-1}b| = 1$. This group acts by the orientation-preserving isometries of the five-dimensional real hyperbolic space. We obtain discreteness criteria for Zariski-dense subgroups of SL(2, $\mathbb{H}$).

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1. Introduction. Let $\mathbf{H^{n+1}}$ be the $(n+1)$-dimensional (real) hyperbolic space and let $M(n)$ denotes the (orientation-preserving) Möbius group that acts on $\mathbf{H^{n+1}}$ by isometries. Given a subgroup G of $M(n)$, it is an interesting problem to ask when G is discrete. In particular, one asks when a two-generator subgroup of $M(n)$ is discrete. It has been seen in the literature, especially for $n = 2$, that the discreteness of the two-generator subgroups of G determine the discreteness of G. The linear group SL(2, $\mathbb{C}$) acts on $\partial\mathbf{H^3} \approx \mathbb{S}^2$ by linear fractional transformations, and this action identifies the group $M(2)$ with PSL(2, $\mathbb{C}$), e.g. see [**3**]. The Jørgensen inequality in SL(2, $\mathbb{C}$) gave a sufficient algorithm for discreteness of a two-generator subgroup. There have been many attempts in the literature to formulate generalizations of Jørgensen inequality in higher dimensions and to obtain discreteness criteria using two-generator subgroups, e.g. see [**9**, **13**, **17**, **18**, **21**] for some recent investigations in this direction.

A subgroup G of $M(n)$ is called *Zariski-dense* if it does not have a global fixed point and neither it preserves a proper totally geodesic subspace of $\mathbf{H^{n+1}}$. In [**1**], Abikoff and Haas proved that a Zariski-dense subgroup G of $M(n)$ is discrete if and only if every two-generator subgroup $\langle f, g\rangle$ of G is discrete. When n even, Abikoff and Haas proved a stronger result that says that a Zariski-dense subgroup G of $M(2m)$ is discrete if and only if every cyclic subgroup of G is discrete. This implies that the discreteness of a subgroup in $M(2m)$ is controlled by the cyclic subgroups. In [**7**], Chen obtained a discreteness criterion that uses a fixed (test) map to check discreteness of a Möbius subgroup. Chen proved that a Zariski-dense subgroup G of $M(n)$ is discrete if for any g in G, and a fixed non-trivial element f from $M(n)$, the group $\langle f, g\rangle$ is discrete, where f is not an irrational rotation (that is of infinite order) or if having finite order, it acts as a non-identity Möbius transformation on the minimal sphere containing the limit set of G. Chen's discreteness criterion involves two-generator subgroups of $M(n)$ with only one generator from G itself.

Motivated by Chen's work, it is natural to ask how far the test map f may be chosen outside G. This was the line of investigation of Yang who asked this problem for $\mathrm{SL}(2, \mathbb{C})$ in **[22]**. Yang gave a partial answer to this question and formulated a conjecture for the remaining cases. In **[4]**, Cao completed Yang's programme by solving Yang's conjecture. Yang and Zhao **[23]** gave another proof to the conjecture. Recently, Yang and Zhao **[25]** have obtained a discreteness criterion in $\mathrm{SL}(2, \mathbb{C})$ that says that a non-elementary subgroup G of $\mathrm{SL}(2, \mathbb{C})$ is discrete if every two generator subgroup $\langle g, fgf^{-1}\rangle$ is discrete, where g is a non-trivial element of G and f is an arbitrary but fixed element in $\mathrm{SL}(2, \mathbb{C})$. The work of Cao and Yang et al. shows that the discreteness of a subgroup G of $\mathrm{SL}(2, \mathbb{C})$ is completely determined by two-generator subgroups $\langle f, g\rangle$, where f is a test map and g is an element of G. However, given a test map f, it is not clear from these works that whether the elements g from G can be restricted to a smaller class.

The aim of this paper is to investigate the above problems in higher dimensions. We focus on the group $M(4)$ that provides the closest analogue of $\mathrm{PSL}(2, \mathbb{C})$ action on the Riemann sphere by Möbius transformations. Let $\mathbb{H}$ be the division ring of real quaternions. Let $\mathrm{SL}(2, \mathbb{H})$ be the group of 2×2 quaternionic matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with quaternionic determinant $\det A = |ad - aca^{-1}b| = 1$. The group $\mathrm{PSL}(2, \mathbb{H}) = \mathrm{SL}(2, \mathbb{H})/\{\pm I\}$ can be identified with the group of orientation-preserving isometries of the five-dimensional hyperbolic space using the quaternionic linear fractional transformations, see **[2, 14, 20]**. We investigate the discreteness of two-generator subgroups using this action.

To state our main results, we recall from **[11, 14]** that a parabolic element in $\mathrm{SL}(2, \mathbb{H})$ is conjugate to

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \ |\lambda| = 1, \ \lambda \in \mathbb{C}, \tag{1.1}$$

and upto conjugacy, an elliptic or hyperbolic element A is given by

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \tag{1.2}$$

where $\lambda, \mu \in \mathbb{C}$, and A is hyperbolic if and only if $|\lambda| \neq 1 \neq |\mu|$. If $|\lambda| = |\mu| = 1$ and λ is not similar to μ in $\mathbb{H}^*$, then A is called *2-rotatory elliptic*.

DEFINITION 1. Let A be an elliptic or hyperbolic element in $\mathrm{SL}(2, \mathbb{H})$ which is represented by (1.1) or (1.2) up to conjugacy. We define the *argument trace* of A by

$$\mathrm{argtr}(A) = \arg(\lambda) + \arg(\mu),$$

and the *absolute trace* of A by

$$\text{abstr}(A) = |\lambda| + |\mu|.$$

Note that an element of SL(2, $\mathbb{H}$) is hyperbolic if and only if $\text{abstr}(A) > 2$. Now we state our main result.

THEOREM 1.1. *Let G be a Zariski-dense subgroup of* SL(2, $\mathbb{H}$).

(1) *Let f be a* 2*-rotatory elliptic element of* SL(2, $\mathbb{H}$) *such that* $0 < \text{argtr}(f) < \frac{\pi}{3}$. *If the two generator subgroup $\langle f, g\rangle$ is discrete for every hyperbolic element g in G, then G is discrete.*
(2) *Let f be a hyperbolic element of* SL(2, $\mathbb{H}$) *such that*

$$\frac{1}{2}(\text{abstr}^2(f) - 3) < \cos(\text{argtr}(f)).$$

If the two generator subgroup $\langle f, g\rangle$ is discrete for every hyperbolic element g in G, then G is discrete.
(3) *Let f be a parabolic element of* SL(2, $\mathbb{H}$) *such that, up to conjugacy,*

$$f = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \ |\mu| \leq 1.$$

If the two generator subgroup $\langle f, g\rangle$ is discrete for every hyperbolic element g in G, then G is discrete.

After proving the above result, using similar methods, we have obtained the following.

THEOREM 1.2. *Let G be a Zariski-dense subgroup of* SL(2, $\mathbb{H}$).

(1) *Let f be a* 2*-rotatory elliptic element of* SL(2, $\mathbb{H}$) *such that* $0 < \text{argtr}(f) < \frac{\pi}{3}$. *If the two generator subgroup $\langle f, gfg^{-1}\rangle$ is discrete and non-elementary for every hyperbolic element g in G, then G is discrete.*
(2) *Let f be a hyperbolic element of* SL(2, $\mathbb{H}$) *such that*

$$\frac{1}{2}(\text{abstr}^2(f) - 3) < \cos(\text{argtr}(f)).$$

If the two generator subgroup $\langle f, gfg^{-1}\rangle$ is discrete for every hyperbolic element g in G, then G is discrete.
(3) *Let f be a parabolic element of* SL(2, $\mathbb{H}$) *such that, up to conjugacy,*

$$f = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \ |\mu| \leq 1.$$

If the two generator subgroup $\langle f, gfg^{-1}\rangle$ is discrete for every hyperbolic element g in G, then G is discrete.

The above two theorems indicate that the discreteness of a Zariski-dense subgroup G of SL(2, $\mathbb{H}$), equivalently, $M(n)$, $n \leq 5$, is determined by the two-generator subgroups involving a test map and the hyperbolic elements of G. It is interesting to note that our choice of f in SL(2, $\mathbb{H}$) lies in a very nice region where one can choose uncountably

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many irrational rotations which are of infinite orders. Given the dynamical type of the test map, it belongs to a one parameter family where each element in the family may be chosen as a test map.

We note here that the restrictions on argtr(f) and abstr(f) in both the theorems are necessary. These quantities come from the Jørgensen type inequalities in [**10**] and cannot be relaxed. In part (1) of both the theorems, the quantity argtr(f) cannot be zero, as in that case, f will reduce to a 1-rotatory elliptic. If argtr(f) $= \frac{\pi}{3}$, then the arguments we give here become inconclusive. Similarly in part (2), equality of the given inequality would imply that f is an elliptic of order at least seven, by [**10**, Corollary 8]. This would contradict the hypothesis that f is hyperbolic.

Plan of the paper is as follows. In Section 2, we recall some preliminary results that include Jørgensen type inequalities for two generator subgroups of SL(2, $\mathbb{H}$) as obtained in [**10**], also see [**12**, **19**]. We apply these results to prove Theorems 1.1 and 1.2 in Section 3.

2. Preliminaries.

2.1. The quaternions. Let $\mathbb{H}$ denote the division ring of quaternions. Recall that every element of $\mathbb{H}$ is of the form $a_0 + a_1 i + a_2 j + a_3 k$,where $a_0, a_1, a_2, a_3 \in \mathbb{R}$, and i, j, k satisfy relations: $i^2 = j^2 = k^2 = -ijk = -1$. Any $a \in \mathbb{H}$ can be uniquely written as $a = a_0 + a_1 i + a_2 j + a_3 k$. We define $\Re(a) = a_0 =$ the real part of a and $\Im(a) = a_1 i + a_2 j + a_3 k =$ the imaginary part of a. Also, define the conjugate of a as $\overline{a} = \Re(a) - \Im(a)$. The norm of a is $|a| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$. Two quaternions a, b are said to be *similar* if there exists a non-zero quaternion c such that $b = c^{-1}ac$ and we write it as $a \backsim b$. It is easy to verify that $a \backsim b$ if and only if $\Re(a) = \Re(b)$ and $|a| = |b|$. Thus, the similarity class of every quaternion a contains a pair of complex conjugates with absolute value $|a|$ and real part equal to $\Re(a)$. Let a be similar to $re^{i\theta}$, $\theta \in (-\pi, \pi]$. We shall adopt the convention of calling $|\theta|$ as the *argument* of a and will denote it by arg(a).

2.2. Quaternionic matrices. Let M(2, $\mathbb{H}$) denote the group of all 2×2 quaternionic matrices. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ M(2, $\mathbb{H}$), define the 'quaternionic determinant' of M by

$$\det M = |ad - aca^{-1}b|.$$

THEOREM 2.1 ([**12**, **14**]). *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ M(2, $\mathbb{H}$) *be such that* $\det M \neq 0$. *Then, M is invertible and*

$$M^{-1} = \begin{pmatrix} d^{\sim} & -b^{\sim} \\ -c^{\sim} & a^{\sim} \end{pmatrix}, \textit{where}$$

$$d^{\sim} = l_{11}^{-1} d, \qquad c^{\sim} = l_{21}^{-1} c, \qquad b^{\sim} = l_{12}^{-1} b, \qquad a^{\sim} = l_{22}^{-1} a;$$

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$$l_{11} = da - dbd^{-1}c \qquad l_{12} = bdb^{-1}a - bc$$
$$l_{21} = cac^{-1}d - cb \qquad l_{22} = ad - aca^{-1}b.$$

Let

$$\mathrm{SL}(2,\mathbb{H}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}(2,\mathbb{H}) : \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = |ad - aca^{-1}b| = 1 \right\}.$$

The group SL(2, $\mathbb{H}$) acts by the orientation-preserving isometries of the hyperbolic 5-space $\mathbf{H}^5$, see [**14**] for more details. We identify the extended quaternionic line $\widehat{\mathbb{H}} = \mathbb{H} \cup \{\infty\}$ to the conformal boundary $\mathbb{S}^4$ of the hyperbolic 5-space. The group SL(2, $\mathbb{H}$) acts on $\widehat{\mathbb{H}}$ by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : Z \mapsto (aZ + b)(cZ + d)^{-1}.$$

The action is extended over $\mathbf{H}^5$ by Poincaré extensions. Under this action, the group of orientation-preserving isometries of $\mathbf{H}^5$ is PSL(2, $\mathbb{H}$) = SL(2, $\mathbb{H}$)/$\{+I, -I\}$. However, often we will not distinguish between an isometry of $\mathbf{H}^5$ and its linear representation in SL(2, $\mathbb{H}$).

2.3. Classification of isometries. Every isometry of $\mathbf{H}^5$ has a fixed point on the closure of the hyperbolic space $\overline{\mathbf{H}}^5$ and this gives us the usual classification of elliptic, parabolic, and hyperbolic (or loxodromic) elements in the isometry group. Further, it follows from the Lefschetz fixed point theorem that every isometry has a fixed point on the conformal boundary. Up to conjugacy, we can take that fixed point to be ∞. It follows that every element in SL(2, $\mathbb{H}$) is conjugate to an upper-triangular matrix. For more details of the classification and algebraic criteria to detect them, see [**5**, **11**, **15**], also see [**8**].

2.4. Jørgensen inequality. The following result is a Jørgensen type inequality for two-generator subgroups of SL(2, $\mathbb{H}$) when one of the generators is either elliptic or hyperbolic.

THEOREM 2.2. [**10**] *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, *$\lambda$ is not similar to μ, generate a discrete non-elementary subgroup $\langle S, T \rangle$ of* SL(2, $\mathbb{H}$). *Then,*

$$\{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2\}(1 + |bc|) \geq 1.$$

This gives the following.

COROLLARY 2.3 ([**10**, **12**]). *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \in$ SL(2, $\mathbb{H}$), *λ is not similar to μ, generate a discrete non-elementary subgroup $\langle S, T \rangle$ of* SL(2, $\mathbb{H}$). *Then,*

$$2(\cosh\tau - \cos(\alpha + \beta))(1 + |bc|) \geq 1,$$

where $\alpha = arg(\lambda)$, $\beta = arg(\mu)$, $\tau = 2\log|\lambda|$.

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Observe that with the above expression of τ, we have that $2\cosh\tau = |\lambda|^2 + |\lambda|^{-2}$. When one of the generators is a translation, we have the following result.

COROLLARY 2.4 ([**19**, **12**]). *If* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $T = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ *generate a non-elementary discrete subgroup in* $\mathrm{SL}(2, \mathbb{H})$*, then* $|c|.|\lambda| \geq 1$.

2.5. Limit sets. Let $L(G)$ be the limit set of a subgroup G of $M(n)$, see [**16**] for basic properties of limit sets. The limit set $L(G)$ is a closed G-invariant subset of $\mathbb{S}^n$. The group G is elementary if $L(G)$ is finite. If G is elementary, $L(G)$ consists of at most two points. If G is non-elementary, then $L(G)$ is an infinite set and every non-empty, closed G-invariant subset of $\mathbb{S}^n$ contains $L(G)$. We note the following lemma, for a proof see [**16**, Chapter 12].

LEMMA 2.5. *Let G be a subgroup of $M(n)$. Let $a \in \partial\mathbf{H}^{\mathbf{n+1}}$ be a fixed point of a non-elliptic element of G. Then a is a limit point of G.*

Let F be the set of fixed points of all non-elliptic elements of G. The above lemma implies that F is G-invariant. Further if G is non-elementary, then F contains at least three points. We will use these facts while proving the theorems. Another crucial result to be used in the next section is the following.

THEOREM 2.6. [**6**, Corollary 4.5.1] *Let G be a subgroup of* $\mathrm{SL}(2, \mathbb{H})$ *that does not leave invariant a point in* $\overline{\mathbf{H}}^5$ *or a proper totally geodesic submanifold in* $\mathbf{H}^5$ *which is invariant under G. Then G is either discrete or dense in* $\mathrm{SL}(2, \mathbb{H})$.

3. Discreteness using a test map.

3.1. Proof of Theorem 1.1. By hypothesis, G is a Zariski-dense subgroup of $\mathrm{SL}(2, \mathbb{H})$. Therefore, G is non- elementary. In the sequel, we suppose that G is not discrete and derive contradictions when considering the cases (1)–(3) in the statement of the theorem.

Suppose G is not discrete. Then G is a dense subgroup of $\mathrm{SL}(2, \mathbb{H})$. It is a well-known fact, e.g. see [**24**], that the set of all hyperbolic elements is open in $\mathrm{SL}(2, \mathbb{H})$. Hence, we may choose a hyperbolic element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G such that it fixes a point other than $0, \infty$.

Let $z_0 \neq 0, \infty$ be a fixed point of g. Consider the element $h = \begin{pmatrix} z_0^{-1} & -1 \\ 0 & z_0 \end{pmatrix}$. It is easy to see that $h^{-1} = \begin{pmatrix} z_0 & 1 \\ 0 & z_0^{-1} \end{pmatrix}$. Note that $h(z_0) = 0$. Since G is dense in $\mathrm{SL}(2, \mathbb{H})$, so there exists a sequence $\{h_n\} \subseteq G$ such that $h_n \to h$. We can choose h_n such that $h_n(z_0) \neq 0 \neq h_m(z_0)$ for large n, m.

(1) Suppose f is 2-rotatory elliptic. We can assume, up to conjugacy that,

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \ \lambda, \mu \in \mathbb{C},$$

$|\lambda| = |\mu| = 1$, λ is not similar to μ. Further assume $0 < \arg\mathrm{tr}(f) = \arg\lambda + \arg\mu < \frac{\pi}{3}$. Let $\arg\lambda = \alpha$, $\arg\mu = \beta$.

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Let $h_ngh_n^{-1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. By hypothesis, each two generator subgroup $\langle f, h_ngh_n^{-1}\rangle$ is discrete. For large n, it follows from Lemma 2.5 that $\langle f, h_ngh_n^{-1}\rangle$ has at least three limit points, and hence, it is non-elementary. By Theorem 2.2, for sufficiently large n,

$$2(1-\cos(\alpha+\beta))(1+|b_nc_n|) \geq 1.$$

Now note that

$$\begin{aligned} hgh^{-1} &= \begin{pmatrix} z_0^{-1} & -1 \\ 0 & z_0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_0 & 1 \\ 0 & z_0^{-1} \end{pmatrix} \\ &= \begin{pmatrix} z_0^{-1}az_0 & z_0^{-1}a+z_0^{-1}bz_0^{-1}-c-dz_0^{-1} \\ z_0cz_0 & z_0c+z_0dz_0^{-1} \end{pmatrix}. \end{aligned}$$

Since z_0 is a fixed point of g, we have

$$(az_0+b)(cz_0+d)^{-1} = z_0$$
$$\text{that is, } (z_0^{-1}a+z_0^{-1}bz_0^{-1}-c-dz_0^{-1})z_0cz_0 = 0.$$

Since $0 < \alpha+\beta < \frac{\pi}{3}$, this implies

$$\begin{aligned} &2(1-\cos(\alpha+\beta))(1+|(z_0^{-1}a+z_0^{-1}bz_0^{-1}-c-dz_0^{-1})z_0cz_0|) \\ &= 2(1-\cos(\alpha+\beta)) < 1. \end{aligned}$$

By Theorem 2.2, this contradiction completes the proof of (1).

(2) Let f be hyperbolic. Using the hypothesis, we can assume up to conjugacy that

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \ |\lambda| \neq |\mu|, \ |\lambda\mu| = 1,$$
$$\arg\lambda = \alpha, \ \arg\mu = \beta, \ 2\cos(\alpha+\beta) > |\lambda|^2+|\mu|^2-1.$$

Let $h_ngh_n^{-1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. By hypothesis and using Corollary 2.3, we have for sufficiently large n,

$$2(\cosh\tau-\cos(\alpha+\beta))(1+|b_nc_n|) \geq 1, \tag{3.1}$$

where $\tau = 2\log|\lambda|$. But, we have

$$\begin{aligned} hgh^{-1} &= \begin{pmatrix} z_0^{-1} & -1 \\ 0 & z_0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_0 & 1 \\ 0 & z_0^{-1} \end{pmatrix} \\ &= \begin{pmatrix} z_0^{-1}az_0 & z_0^{-1}a+z_0^{-1}bz_0^{-1}-c-dz_0^{-1} \\ z_0cz_0 & z_0c+z_0dz_0^{-1} \end{pmatrix}. \end{aligned}$$

Note that $(z_0^{-1}a+z_0^{-1}bz_0^{-1}-c-dz_0^{-1})z_0cz_0 = 0$. It follows that

$$\begin{aligned} &2(\cosh\tau-\cos(\alpha+\beta))(1+|(z_0^{-1}a+z_0^{-1}bz_0^{-1}-c-dz_0^{-1})z_0cz_0|) \\ &\qquad = 2(\cosh\tau-\cos(\alpha+\beta)). \end{aligned}$$

Since $2\cos(\alpha+\beta) > |\lambda|^2 + |\mu|^2 - 1$, this implies

$$2(\cosh\tau - \cos(\alpha+\beta)) < 1.$$

This is a contradiction to (3.1). Hence, part (2) of the theorem follows.

(3) Consider the parabolic element $u = \begin{pmatrix} 1 & 0 \\ -z_0{}^{-1} & 1 \end{pmatrix}$. Note that $u(0) = 0$. It is easy to see that $u^{-1} = \begin{pmatrix} 1 & 0 \\ z_0{}^{-1} & 1 \end{pmatrix}$. Since G is dense in $\mathrm{SL}(2,\mathbb{H})$, there exists a distinct sequence $\{g_n\} \subseteq G$ such that $g_n \to u$. We may choose g_n such that for large n, $g_n(z_0) \neq \infty$, and hence, having $\langle f, g_n g g_n^{-1}\rangle$ non-elementary. By hypothesis, these groups are all discrete. Hence, by Corollary 2.4,

$$|c_n|.|\mu| \geq 1,$$

where $g_n g g_n^{-1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. By computations, we see that

$$\begin{aligned} ugu^{-1} &= \begin{pmatrix} 1 & 0 \\ -z_0{}^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z_0{}^{-1} & 1 \end{pmatrix} \\ &= \begin{pmatrix} a + bz_0{}^{-1} & b \\ -z_0{}^{-1}(a + bz_0{}^{-1}) + (c + dz_0{}^{-1}) & -z_0{}^{-1}b + d \end{pmatrix}. \end{aligned}$$

Since z_0 is a fixed point of g, so we have

$$c_\infty = -z_0{}^{-1}(a + bz_0{}^{-1}) + (c + dz_0{}^{-1}) = 0.$$

Since $|\mu| \leq 1$, this implies

$$|c_n| \geq \frac{1}{|\mu|} \geq 1.$$

But we see that $c_n \to c_\infty = 0$ as $n \to \infty$, which gives a contradiction. This proves (3). This completes the proof.

3.2. Proof of Theorem 1.2. By similar arguments as used at the beginning of the proof of Theorem 1.1, we can choose h_n such that $h_n(z_0) \neq 0 \neq h_m(z_0)$ for large n, m. Let $h_n g h_n^{-1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$.

(1) For all n, consider

$$\begin{aligned} L_n &= h_n g h_n^{-1} f h_n g^{-1} h_n^{-1} \\ &= \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} d_n^{\sim} & -b_n^{\sim} \\ -c_n^{\sim} & a_n^{\sim} \end{pmatrix} \\ &= \begin{pmatrix} a_n\lambda d_n^{\sim} - b_n\mu c_n^{\sim} & -a_n\lambda b_n^{\sim} + b_n\mu a_n^{\sim} \\ c_n\lambda d_n^{\sim} - d_n\mu c_n^{\sim} & -c_n\lambda b_n^{\sim} + d_n\mu a_n^{\sim} \end{pmatrix} \\ &= \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}. \end{aligned}$$

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As $n \to \infty$, let $L_n \to L_\infty$, where

$$L_\infty = hgh^{-1}fhg^{-1}h^{-1} = \begin{pmatrix} A_\infty & B_\infty \\ C_\infty & D_\infty \end{pmatrix}.$$

Now we see that

$$\begin{aligned} |B_nC_n| &\leq |a_nb_nc_nd_n||\lambda - a_n^{-1}b_n\mu\tilde{a_n}\tilde{b_n}^{-1}||\lambda - c_n^{-1}d_n\mu\tilde{c_n}\tilde{d_n}^{-1}| \\ &= \{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2\}(1 + |b_nc_n|)|b_nc_n|. \end{aligned}$$

Let

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = hgh^{-1} = \begin{pmatrix} z_0^{-1}az_0 & z_0^{-1}a + z_0^{-1}bz_0^{-1} - c - dz_0^{-1} \\ z_0cz_0 & z_0c + z_0dz_0^{-1} \end{pmatrix}.$$

Since z_0 is a fixed point of g, we have seen that

$$(z_0^{-1}a + z_0^{-1}bz_0^{-1} - c - dz_0^{-1})z_0cz_0 = 0,$$

which shows that $b_0c_0 = 0$.
By a similar calculations above in the case L_n, we see that

$$|B_\infty C_\infty| \leq \{(\Re\lambda - \Re\mu)^2 + (|\Im\lambda| + |\Im\mu|)^2\}(1 + |b_0c_0|)|b_0c_0| = 0,$$

and therefore we have $B_\infty C_\infty = 0$. This shows that $B_nC_n \to 0$. Now we see that by hypothesis, each two generator subgroup $\langle f, L_n\rangle$ is discrete and non-elementary. So by Theorem 2.2,

$$2(1 - \cos(\alpha + \beta))(1 + |B_nC_n|) \geq 1. \tag{3.2}$$

Since $0 < \alpha + \beta < \frac{\pi}{3}$, this implies for sufficiently large n,

$$2(1 - \cos(\alpha + \beta))(1 + |B_nC_n|) = 2(1 - \cos(\alpha + \beta)) < 1.$$

This is a contradiction to (3.2) which completes the proof of (1).

(2) For this part, the proof follows from similar calculations as in the proof of (1) and the fact that

$$\begin{aligned} 2(\cosh\tau - \cos(\alpha + \beta))(1 + |B_\infty C_\infty|) \\ = 2(\cosh\tau - \cos(\alpha + \beta)). \end{aligned}$$

Since $2\cos(\alpha + \beta) > |\lambda|^2 + |\mu|^2 - 1$, this implies

$$2(\cosh\tau - \cos(\alpha + \beta)) < 1.$$

This leads to a contradiction. Hence, part (2) of the theorem follows.

(3) Consider the parabolic element $h = \begin{pmatrix} 1 & 0 \\ -z_0^{-1} & 1 \end{pmatrix}$. Note that $h(0) = 0$. It is easy to see that $h^{-1} = \begin{pmatrix} 1 & 0 \\ z_0^{-1} & 1 \end{pmatrix}$. Since G is dense in SL(2, ℍ), there exists a sequence $\{h_n\} \subseteq G$ such that $h_n \to h$. We may choose distinct h_n such that for large n, $h_n(z_0) \neq \infty$.

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Let

$$\begin{aligned} L_n &= h_n g h_n^{-1} f h_n g^{-1} h_n^{-1} \\ &= \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_n^{\sim} & -b_n^{\sim} \\ -c_n^{\sim} & a_n^{\sim} \end{pmatrix} \\ &= \begin{pmatrix} a_n d_n^{\sim} - a_n \mu c_n^{\sim} - b_n c_n^{\sim} & -a_n \mu a_n^{\sim} \\ -c_n \mu c_n^{\sim} & -c_n b_n^{\sim} + c_n \mu a_n^{\sim} + d_n a_n^{\sim} \end{pmatrix} \\ &= \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}, \text{ say.} \end{aligned}$$

Now as $n \to \infty$, $L_n \to L_\infty$, where

$$\begin{aligned} L_\infty &= h g h^{-1} f h g^{-1} h^{-1} \\ &= \begin{pmatrix} A_\infty & B_\infty \\ C_\infty & D_\infty \end{pmatrix}, \text{ say.} \end{aligned}$$

It is clear that for large values of n, $\langle f, L_n \rangle$ are non-elementary and by hypothesis, these groups are also discrete. Hence, by Corollary 2.4, $|C_n|.|\mu| \geq 1$. Let

$$hgh^{-1} = \begin{pmatrix} a + b{z_0}^{-1} & b \\ -{z_0}^{-1}(a + b{z_0}^{-1}) + (c + d{z_0}^{-1}) & -{z_0}^{-1} b + d \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}.$$

We have seen that since z_0 is a fixed point of g, so

$$c_0 = -{z_0}^{-1}(a + b{z_0}^{-1}) + (c + d{z_0}^{-1}) = 0.$$

Thus, it follows that $C_\infty = 0$. So $C_n \to 0$, as $n \to \infty$. Since $|\mu| \leq 1$, this implies

$$|C_n| \geq \frac{1}{|\mu|} \geq 1,$$

which leads to a contradiction. This completes the proof.

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REFERENCES

1. W. Abikoff and A. Haas, Nondiscrete groups of hyperbolic motions, *Bull. London Math. Soc.* **22**(3) (1990), 233–238.

2. C. Bisi and G. Gentili, Möbius transformations and the Poincaré distance in the quaternionic setting, *Indiana Univ. Math. J.* **58**(6) (2009), 2729–2764.

3. A. F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, vol. 91 (Springer-Verlag, New York, 1995), Corrected reprint of the 1983 original.

4. W. Cao, Discreteness criterion in SL(2, ℂ) by a test map, *Osaka J. Math.* **49**(4) (2012), 901–907.

5. W. Cao, On the classification of four-dimensional Möbius transformations, *Proc. Edinb. Math. Soc., II. Ser.* **50**(1) (2007), 49–62.

6. S. S. Chen and L. Greenberg, Hyperbolic spaces, in *Contributions to analysis (a collection of papers dedicated to Lipman Bers)* (Academic Press, New York, 1974), 49–87.

7. M. Chen, Discreteness and convergence of Möbius groups, *Geom. Dedicata* **104** (2004), 61–69.

8. B. Foreman, Conjugacy invariants of $Sl(2, \mathbb{H})$, *Linear Algebra Appl.* **381** (2004), 25–35.

9. A. Fang and B. Nai, On the discreteness and convergence in n-dimensional Möbius groups, *J. London Math. Soc. II. Ser.* **61**(3) (2000), 761–773.

10. K. Gongopadhyay and A. Mukherjee, Extremality of quaternionic Jørgensen inequality, *Hiroshima Math. J.* **47**(2) (2017), 113–137.

11. K. Gongopadhyay, Algebraic characterization of the isometries of the hyperbolic 5-space, *Geom. Dedicata* **144** (2010), 157–170.

12. R. Kellerhals, Quaternions and some global properties of hyperbolic 5-manifolds, *Canad. J. Math.* **55**(5) (2003), 1080–1099.

13. L.-L. Li and X.-T. Wang, Discreteness criteria for Möbius groups acting on $\overline{\mathbb{R}}^n$ II, *Bull. Aust. Math. Soc.* **80**(2) (2009) 275–290.

14. J. R. Parker, *Hyperbolic spaces*, Jyväskylä Lectures in Mathematics, vol. 2 (University of Jyväskylä, Finland, 2008).

15. J. R. Parker and I. Short, Conjugacy classification of quaternionic Möbius transformations, *Comput. Methods Funct. Theory* **9**(1) (2009), 13–25.

16. J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Mathematics, vol. 149 (Springer-Verlag, New York, 2nd edition, 2005).

17. P. Tukia and X. Wang, Discreteness of subgroups of SL(2,C) containing elliptic elements, *Math. Scand.* **91** (2002), 214–220.

18. X. Wang, L. Li and W. Cao, Discreteness criteria for Möbius groups acting on $\mathbb{R}^n$, *Israel J. Math.* **150** (2005), 357–368.

19. P. Waterman, Möbius groups in several dimensions, *Adv. Math.* **101** (1993), 87–113.

20. J. B. Wilker, The quaternion formalism for Möbius groups in four or fewer dimensions, *Linear Algebra Appl.* **190** (1993), 99–136.

21. S. Yang, Elliptic elements in Möbius groups, *Israel J. Math.* **172** (2009), 309–315.

22. S. Yang, Test maps and discrete groups in SL(2, C), *Osaka J. Math.* **46**(2) (2009), 403–409.

23. S. Yang and T. Zhao, Test maps and discrete groups in $SL(2, \mathbb{C})$ II, *Glasg. Math. J.* **56**(1) (2014), 53–56.

24. S. Yang, On geometric convergence of discrete groups, *Czech. Math. J.* **64**(139) (2014), 305–310.

25. S. Yang and T. Zhao, Conjugacy class and discreteness in $SL(2, \mathbb{C})$, *Osaka J. Math.* **53**(4) (2016), 1047–1053.

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