Canad. Math. Bull. Vol. **18** (2), 1975

LIMIT DISTRIBUTIONS FOR SUMS OF WEIGHTED RANDOM VARIABLES

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ABSTRACT. Let $X_1, X_2, X_3, \ldots$ be i.i.d., S_n their nth partial sum with $S_0=0$; Suppose that

$$B_n^{-1}(S_n-nc_\alpha) \xrightarrow{\mathscr{D}} Y_\alpha(1); \qquad c_\alpha = \begin{cases} E(X_k), & \alpha > 1 \\ 0, & \alpha < 1 \end{cases}; \quad \alpha \neq 1$$

LEMMA. *$B_n^{-1}(S_{[nt]}-[nt]c_\alpha)\to Y_\alpha(t)$, a stable process whose one-dimensional distributions are characterized by $Y_\alpha(t)\overset{\mathscr{D}}{=}t^{1/\alpha}Y_\alpha(1)$.*

THEOREM 1. *The second characteristic of $Y_\alpha(t)$ is $\lambda(u)|u|^\alpha t$ with $\lambda(u)$ linear in* sgn(u).

COROLLARY. *The second characteristic of $Y_\alpha(1)$ is $\lambda(u)|u|^\alpha$; i.e., if the X_k are suitably centered, then so is $Y_\alpha(1)$.*

THEOREM. *Put $c_\alpha=0$, σ_n the Cesàro sum of index* 1. *Then*

$$B_n^{-1}\sigma_n \xrightarrow{\mathscr{D}} Y_\alpha\left(\frac{1}{1+\alpha}\right)$$

(This was obtained in a different fashion than its generalization in the Note; i.e., a different sort of functional was used.)

Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed random variables which belong to the domain of attraction of a stable law of exponent $\alpha\neq 1$. The purpose of this note is to obtain limit distributions for sums of the form

$$(1) \qquad T_n = \sum_{k=1}^{n} \mathfrak{f}(n^{-1}k)X_k,$$

where $\mathfrak{f}$ is non-negative and continuous on [0, 1]. As a special case, we obtain limit distributions for Cesàro sums of general index, r. This extends the work of Beuerman [1], whose notation we follow. In particular, $Y_\alpha(t)$, $0\leq t\leq 1$, is a stable process of exponent α whose one-dimensional distributions are characterized by $Y_\alpha(t)\overset{\mathscr{D}}{=}t^{1/\alpha}Y_\alpha(1)$, $Y_\alpha(1)$ being the corresponding stable random variable. Our main result is the following. Put $S_n=\sum_{k=1}^{n} X_k$, $S_0=0$.

THEOREM. *Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed variables. Suppose there exist norming constants B_n and centering constants*

https://doi.org/10.4153/CMB-1975-055-7 Published online by Cambridge University Press

nc_α such that

$$B_n^{-1}(S_n - nc_\alpha) \xrightarrow{\mathscr{D}} Y_\alpha(1), \tag{2}$$

a stable random variable of exponent $\alpha \neq 1$. If T_n is given by (1), *with* $\mathfrak{f}$ *non-negative and continuous on* [0, 1] *then*

$$B_n^{-1}\left(T_n - c_\alpha \sum_{k=1}^{n} \mathfrak{f}(n^{-1}k)\right) \xrightarrow{\mathscr{D}} Y_\alpha\left(\int_0^1 (\mathfrak{f}(t))^\alpha dt\right). \tag{3}$$

Proof. From the Lemma of [1] and Theorem 5.1 of Billingsley [2], we have

$$\int_0^1 \mathfrak{f}(t)\, d(B_n^{-1}(S_{[nt]} - [nt]c_\alpha)) \xrightarrow{\mathscr{D}} \int_0^1 \mathfrak{f}(t)\, dY_\alpha(t). \tag{4}$$

Now, Stieltjes integration yields

$$\int_0^1 \mathfrak{f}(t)\, d(B_n^{-1}(S_{[nt]} - [nt]c_\alpha)) = B_n^{-1}\left(T_n - c_\alpha \sum_{k=1}^{n} \mathfrak{f}(n^{-1}k)\right). \tag{5}$$

From Lemma 1 of Laha and Lukacs [4], the second characteristic of $\int_0^1 \mathfrak{f}(t)\, dY_\alpha(t)$ is

$$\int_0^1 \psi(u\mathfrak{f}(t))\, dt, \tag{6}$$

$\psi(u)$ being the second characteristic of $Y_\alpha(1)$. As in [1] we may write $\psi(u)$ in the form

$$\lambda(u)\, |u|^\alpha \tag{7}$$

where $\lambda(u)$ is linear in $\mathrm{sgn}(u)$. Thus, from (6) and (7), the second characteristic of $\int_0^1 \mathfrak{f}(t)\, dY_\alpha(t)$ is

$$\int_0^1 \lambda(u)\, |u|^\alpha (\mathfrak{f}(t))^\alpha\, dt = \lambda(u)\, |u|^\alpha \int_0^1 (\mathfrak{f}(t))^\alpha\, dt,$$

From Theorem 1 of [1], this is the second characteristic of

$$Y_\alpha\left(\int_0^1 (\mathfrak{f}(t))^\alpha\, dt\right).$$

Hence, (4), (5) and (6) yield (3). Q.E.D.

An interesting special case of (3) is for the Cesàro sums of index r, $C_n^{(r)}$; cf. Hobson [3]. Here we may write

$$\mathfrak{f}(n^{-1}k) = A_{n-k}^{(r)}/A_n^{(r)}, \qquad A_n^{(r)} = \frac{\Gamma(n+r+1)}{\Gamma(n+1)\Gamma(r+1)}.$$

From Stirling's approximation, $A_n^{(r)} \sim n^r/\Gamma(r+1)$, so that here $\mathfrak{f}(n^{-1}k) \sim (1-n^{-1}k)^r$ and (3) becomes

$$B_n^{-1}(C_n^{(r)} - (nc_\alpha/(r+1))) \xrightarrow{\mathscr{D}} Y_\alpha(1/(1+r\alpha)) \tag{8}$$

since $\sum_{k=1}^{n}(1-n^{-1}k)^r \sim n/(r+1)$, $\int_0^1 (1-t)^{r\alpha}\,dt = 1/(1+r\alpha)$. Theorem 2 of [1] is (8) for $r=1$, $c_\alpha=0$.

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