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A NEW PROXIMITY FUNCTION GENERATING THE BEST KNOWN ITERATION BOUNDS FOR BOTH LARGE-UPDATE AND SMALL-UPDATE INTERIOR-POINT METHODS

KEYVAN AMINI[✉1] and ARASH HASELI[2]

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Abstract

Interior-Point Methods (IPMs) are not only very effective in practice for solving linear optimization problems but also have polynomial-time complexity. Despite the practical efficiency of large-update algorithms, from a theoretical point of view, these algorithms have a weaker iteration bound with respect to small-update algorithms. In fact, there is a significant gap between theory and practice for large-update algorithms. By introducing self-regular barrier functions, Peng, Roos and Terlaky improved this gap up to a factor of $\log n$. However, checking these self-regular functions is not simple and proofs of theorems involving these functions are very complicated. Roos *et al.* by presenting a new class of barrier functions which are not necessarily self-regular, achieved very good results through some much simpler theorems. In this paper we introduce a new kernel function in this class which yields the best known complexity bound, both for large-update and small-update methods.

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1. Introduction

We deal with the standard linear optimization problem

$$\min \left\{ c^{\mathsf{T}} x : Ax = b, x \geq 0 \right\} \tag{1.1}$$

where $A \in R^{m \times n}$ is a real $m \times n$ matrix with $\operatorname{rank}(A) = m$, $b \in R^m$ and $c \in R^n$. The dual problem of (1.1) is given by

$$\max \left\{ b^{\mathsf{T}} y : A^{\mathsf{T}} y + s = c, s \geq 0 \right\}. \tag{1.2}$$

[1]Department of Sciences, Razi University, Kermanshah, Iran; email: keyvanamini1353@yahoo.com or kamini@razi.ac.ir.
[2]Islamic Azad University, Kermanshah branch, Kermanshah, Iran; email: arhaseli@yahoo.com.

Kernel functions play an important role in defining new search directions in primal-dual interior point algorithms for solving linear optimization problems. A kernel function is a univariate strictly convex function which is defined for all positive reals t and is minimal at $t = 1$ where the minimal value equals zero. In other words $\psi(t) : D \to R_+$ with $R_{++} \subseteq D$, is a kernel function when it satisfies

$$\psi'(1) = \psi(1) = 0, \quad \psi''(t) > 0.$$

Moreover, $\psi(t)$ has the barrier property, that is, $\psi(t)$ goes to infinity if either $t \to 0$ or $t \to \infty$. Note that the above properties imply that $\psi(t)$ is completely determined by its second derivative

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi.$$

This kernel function may be extended to a positive n-dimensional vector μ by

$$\Psi(v) = \sum_{i=1}^{n} \psi(v_i), \tag{1.3}$$

yielding the scaled barrier function $\Psi(v)$. Note that the barrier function $\Psi(v)$ is nonnegative, and zero if and only if v is a vector of ones. Therefore, the value of the barrier function can be considered as a measure for the closeness of x and (y, s) to the μ-centres of (1.1) and (1.2). Hence some authors also call $\Psi(v)$ a proximity function. In the next section we briefly describe how any such barrier function defines a primal-dual interior-point method. The iteration bound for so-called large-update methods is obtained by showing that each iteration decreases $\Psi(v)$ by a sufficient amount. Table 1 gives some examples of kernel functions that have been analyzed so far, and the complexity results for the corresponding algorithms.

Note that all kernel functions in this table depend on a parameter and the mentioned iteration bounds in the table occur by choosing this parameter as indicated in the third column of the table. The first kernel function, the so-called self-regular function, was introduced and analyzed by Peng *et al.* Also the second kernel function is self regular and is the special case of the self regular function $\gamma_{p,q}(t)$, for $p = 1$. The third function is not self regular and has been proposed and analyzed by Bai *et al.* A surprising feature of this kernel function is that it is finite for $t = 0$, a property which separates it from self-regular functions, because self-regular functions become unbounded when t approaches zero. Also Bai *et al.* introduced the following kernel function in [2]:

$$\frac{t^2 - 1}{2} + \frac{e^{1/t} - e}{e}.$$

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TABLE 1. Kernel functions with the best known iteration bound for large-update methods. The iteration bound for all kernel functions is $O(\sqrt{n}\log n)\log(n/\epsilon)$.

Kernel function	Parameter	References
$\frac{t^2-1}{2}+\frac{t^{1-q}-1}{q-1}, q>1$	$q=\frac{1}{2}\log n$	[5, 6]
$\frac{t^2-1}{2}+\frac{t^{1-q}-1}{q(q-1)}-\frac{q-1}{q}(t-1), q>1$		
$\frac{t^2-1}{2}+\frac{1}{\sigma}\left(e^{\sigma(1-t)}-1\right), \sigma>0$	$\sigma=O(\log n)$	[1]
$\frac{t^2-1}{2}+\int_1^t e^{q(1/\xi-1)}d\xi, q\geq 1$	$q=O(\log(1+n))$	[3]

They showed that the iteration bound for the corresponding algorithm is $O\left(\sqrt{n}(\log n)^2\log(n/\epsilon)\right)$. This bound is a factor $\log n$ worse than the bound in Table 1. In this paper we introduce a new kernel function as follows:

$$\psi(t)=\frac{t^2-1}{2}+\frac{e^{q(1/t-1)}-1}{q}, \quad q\geq 1. \tag{1.4}$$

We show that this kernel function yields the best-known iteration bound for large-update methods, that is, $O(\sqrt{n}\log n)\log(n/\epsilon)$. Figure 1 depicts the graph of $\psi(t)$.

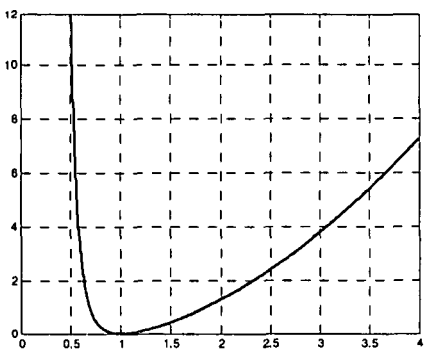

FIGURE 1. The graph of ψ for $q=4$.

The paper is organized as follows. In Section 2, we briefly recall how a given kernel function defines a primal-dual interior-point algorithm. In Section 3, we describe the

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simple conditions on the kernel function that define a general class of kernel functions introduced by Bai *et al.* in [2]. Then we show that the new kernel function introduced in (1.4) satisfies these conditions. In what follows, we use the general scheme for analyzing the generic algorithm, as presented in [2]. We obtain the iteration bounds for both large-update and small-update methods based on a new kernel function in Section 4 and finally, Section 5 contains some concluding remarks.

2. The generic primal-dual interior-point algorithm

It is well known that finding an optimal solution of (1.1) and (1.2) is equivalent to solving the following system:

$$\begin{cases} Ax = b, & x \geq 0, \\ A^{\mathsf{T}}y + s = c, & s \geq 0, \\ xs = 0. \end{cases}$$

The basic idea of the primal-dual IPMs is to replace the third equation, the so-called *complementarity condition*, by the parameterized equation $xs = \mu e$, with $\mu > 0$. Thus we consider the following system:

$$\begin{cases} Ax = b, & x \geq 0, \\ A^{\mathsf{T}}y + s = c, & s \geq 0, \\ xs = \mu e. \end{cases}$$

We assume that the primal and dual problem in (1.1) and (1.2) satisfies the interior-point condition (IPC), that is, there exists (x^0, s^0, y^0) such that

$$Ax^0 = b, \quad x^0 > 0, \quad A^{\mathsf{T}}y^0 + s^0 = c, \quad s^0 > 0.$$

It is well known that the IPC can be assumed without loss of generality. By using the self-dual embedding model, we will assume that $x^0 = s^0 = e$ [4, 7]. If $\operatorname{rank}(A) = m$ and the IPC holds, then for each $\mu > 0$ the above parameterized system has a unique solution. We denote this solution as $\big(x(\mu), y(\mu), s(\mu)\big)$ and call $x(\mu)$ the μ-centre of (1.1) and $\big(y(\mu), s(\mu)\big)$ the μ-centre of (1.2). The set of μ-centres (with μ running through all positive real numbers) gives a homotopy path, which is called the central path of (1.1) and (1.2). If $\mu \to 0$ then the limit of the central path exists and since the limit point satisfies the complementarity condition, the limit yields to optimal solutions for (1.1) and (1.2). If $\big(x(\mu), y(\mu), s(\mu)\big)$ is known for some positive μ, then we decrease μ to $\mu := (1 - \theta)\mu$, for some fixed $\theta \in (0, 1)$ and solve the following

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system:

$$\begin{cases} A\Delta x = 0, \\ A^{\top}\Delta y + \Delta s = 0, \\ s\Delta x + x\Delta s = \mu e - xs. \end{cases} \tag{2.1}$$

This system uniquely defines a search direction $(\Delta x, \Delta s, \Delta y)$. This direction approximates the next μ-centre. Hence all IPMs follow the central path approximately. Now we define

$$v = \sqrt{\frac{xs}{\mu}}.$$

One can easily check that System (2.1) which defines the search direction can be rewritten as follows:

$$\begin{cases} \bar{A}d_x = 0, \\ \bar{A}^{\top}\Delta y + d_s = 0, \\ d_x + d_s = v^{-1} - v, \end{cases} \tag{2.2}$$

where $d_x = v\Delta x/x$, $d_s = v\Delta s/s$ and $\bar{A} = AV^{-1}X$, with $V = \mathrm{diag}(v)$ and $X = \mathrm{diag}(x)$. The third equation in (2.2) is called the scaled centring equation. The right-hand side in the scaled centring equation equals minus the gradient of the basic logarithmic barrier function, that is,

$$\Psi_c(v) = \sum_{i=1}^{n} \left(\frac{v_i^2 - 1}{2} - \log v_i \right).$$

The basic idea in IPMs is to replace the scaled barrier function $\Psi_c(v)$ by an arbitrary strictly convex function $\Psi(v)$, such that $\Psi(v)$ is minimal at $v = e$ with $\Psi(e) = 0$. Thus System (2.2) converts to

$$\begin{cases} \bar{A}d_x = 0, \\ \bar{A}^{\top}\Delta y + d_s = 0, \\ d_x + d_s = -\Delta\Psi(v). \end{cases}$$

Since the vectors d_x and d_s belong to the null and row spaces of the matrix $\bar{A}$, these vectors are orthogonal. By taking a suitable step size $\alpha \in (0, 1)$, these search directions construct a new triple (x_+, y_+, s_+) with

$$x_+ = x + \alpha\Delta x, \quad y_+ = y + \alpha\Delta y, \quad s_+ = s + \alpha\Delta s.$$

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We repeat the procedure until we find an iterate in a certain neighbourhood of $\big(x(\mu), y(\mu), s(\mu)\big)$. Then μ is again reduced by the factor $1-\theta$ and we apply Newton's method targeting the new μ-centres, and so on. This process is repeated until μ is small enough and at this stage we have found an ϵ-solution of the problems (1.1) and (1.2). The generic form of the algorithm is shown in Figure 2.

Input:
A threshold parameter $\tau > 0$;
An accuracy parameter $\epsilon > 0$;
A fixed barrier update parameter θ, $0 < \theta < 1$;
begin
  $x := e$; $s := e$; $\mu := 1$;
  while $n\mu > \epsilon$ **do**
    begin
      $\mu := (1-\theta)\mu$;
      while $\Psi(v) > \tau$ **do**
        begin
          $x_+ := x + \alpha\Delta x$;
          $y_+ := y + \alpha\Delta y$;
          $v := \sqrt{\frac{xs}{\mu}}$;
        end
    end
  end

FIGURE 2. A generic primal-dual algorithm for Lo

3. A general class of the kernel functions

In [2] Bai, El Ghami and Roos introduced a general class of kernel functions by using the following conditions:

$$t\psi''(t) + \psi'(t) > 0, \quad t < 1, \tag{3.1}$$
$$t\psi''(t) - \psi'(t) > 0, \quad t > 1, \tag{3.2}$$
$$\psi'''(t) < 0, \quad t > 0, \tag{3.3}$$
$$2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0, \quad t < 1, \tag{3.4}$$
$$\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi'''(\beta t) > 0, \quad t > 1, \beta > 1. \tag{3.5}$$

The kernel function $\psi(t)$ is called eligible if it satisfies (3.1) and (3.3) to (3.5). It was shown in [2] that (3.2) and (3.3) imply (3.5). So a kernel function $\psi(t)$ is

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eligible if it satisfies conditions (3.1) to (3.4). First we show that the kernel function introduced in (1.4) satisfies these conditions. The first three derivatives of $\psi(t)$ are shown in the following table:

TABLE 2. ψ and its derivatives.

$\psi(t)$	$\frac{t^2-1}{2}+\frac{e^{q/t}-e^q}{qe^q}$
$\psi'(t)$	$t-\frac{e^{q(1/t-1)}}{t^2}$
$\psi''(t)$	$1+\frac{q+2t}{t^4}e^{q(1/t-1)}$
$\psi'''(t)$	$-\frac{q^2+6qt+6t^2}{t^6}e^{q(1/t-1)}$

Since for $0 < t < 1$ and $q \geq 1$

$$t\psi''(t)+\psi'(t)=2t+\left[\frac{q+2t}{t^3}-\frac{1}{t^2}\right]e^{q(1/t-1)}=2t+\left(\frac{q+t}{t^3}\right)e^{q(1/t-1)}>0,$$

Condition (3.1) is satisfied. Furthermore, for $t > 1$

$$t\psi''(t)-\psi'(t)=\frac{q+3t}{t^3}e^{q(1/t-1)}>0.$$

So (3.2) is satisfied as well. Also from Table 2 it is obvious that (3.3) is satisfied. Finally for satisfying (3.4), it is seen that

$$2\psi''(t)^2-\psi'(t)\psi'''(t)$$
$$=\frac{2\left[t^4+(q+2t)e^{q(1/t-1)}\right]^2+e^{q(1/t-1)}(q^2+6qt+6t^2)\left(t^3-e^{q(1/t-1)}\right)}{t^8}.$$

If we set $A=e^{q(1/t-1)}$, after some elementary reduction we have

$$2\psi''(t)^2-\psi'(t)\psi'''(t)=\frac{A^2\left[2t^2+2qt+q^2\right]+A\left[14t^2+10qt+q^2t^3\right]+2t^8}{t^8}.$$

On the other hand, because $0 < t < 1$, it follows that $1/t-1>0$ and for $q \geq 1$ we have

$$A=e^{q(1/t-1)}>1.$$

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So we have $2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0$ and hence condition (3.4) holds. Therefore $\psi(t)$ lies in the general class introduced in [2]. Having such a kernel function, we can now construct the barrier function $\Psi(v)$ by (1.3). The function $\Psi(v)$ not only serves to define a search direction, but also as a measure of closeness of the current iterates to the μ-centre. We use the norm-based proximity measure $\delta(v)$ defined by

$$\delta(v) = \frac{1}{2}\|\nabla\Psi(v)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}(\psi'(v_i))}.$$

Since $\Psi(v)$ is strictly convex and minimal at $v = e$ we have

$$\Psi(v) = 0 \iff \delta(v) = 0 \iff v = e.$$

In other words, the proximity measure $\delta(v)$ is zero if and only if the current iterates are the μ-centres. For any kernel function in the mentioned class, iteration bounds for both small-update and large-update methods can be obtained by using the following scheme:

Step 1. Solve the equation $-\psi'(t)/2 = s$ to get $\rho(s)$, the inverse function of $-\psi'(t)/2$, $t \in (0, 1]$. If the equation is hard to solve, derive a lower bound for $\rho(s)$.

Step 2. Calculate the decrease of $\Psi(v)$ in terms of δ for the default step size $\tilde{\alpha} = 1/\psi''(\rho(2\delta))$ from $f(\tilde{\alpha}) = \delta^2/\psi''(\rho(2\delta))$.

Step 3. Solve the equation $\psi(t) = s$ to get $Q(s)$, the inverse function of $\psi(t)$, $t \geq 1$. If the equation is hard to solve, derive lower and upper bound for $Q(s)$.

Step 4. Derive a lower bound for δ in term of $\Psi(v)$ by using $\delta(v) \geq \psi'\big(Q(\Psi(v))\big)/2$.

Step 5. Using the result of Steps 3 and 4 find positive constants κ and γ, with $\gamma \in (0, 1]$, such that $f(\tilde{\alpha}) \leq -\kappa\Psi(v)^{1-\gamma}$.

Step 6. Calculate the upper bound for Ψ_0 from

$$\Psi_0 \leq L_\psi(n, \theta, \tau) = n\psi\left(\frac{Q(\tau/n)}{\sqrt{1-\theta}}\right).$$

Step 7. Derive an upper bound for the total number of iterations from

$$\frac{\Psi_0^\gamma}{\kappa\gamma}\frac{1}{\theta}\log\frac{n}{\epsilon}.$$

Step 8. To calculate a complexity bound for large-update algorithms set $\tau = O(n)$ and $\theta = \Theta(1)$ and for small-update method algorithms set $\tau = O(1)$ and $\theta = \Theta(1/\sqrt{n})$.

In the next section, we apply this scheme to obtain an iteration bound for the algorithm generated by the kernel function introduced in (1.4).

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4. Iteration bounds of the algorithm

Since we are unable to get explicit expressions for the inverse functions ρ and Q in the above steps, we recall two lemmas from [2].

LEMMA 4.1. *Let $Q : [0, \infty] \to [1, \infty]$ be the inverse function of $\psi(t)$ for $t \geq 1$. Then we have*

$$\sqrt{1+2s} \leq Q(s) \leq 1 + \sqrt{2s}, \quad s \geq 0.$$

LEMMA 4.2. *Let $\psi_b(t)$ be the barrier term of $\psi(t)$ ($\psi(t) = (t^2-1)/2 + \psi_b(t)$) and let $\underline{\rho} : [0, \infty) \to (0, 1]$ be the inverse function of the restriction of $-\psi_b'(t)$ to the interval $(0, 1]$. Then one has*

$$\rho(s) \geq \underline{\rho}(1+2s).$$

Now by using these two lemmas, we derive some bounds for ρ and Q.

Step 1. From the equation $-\psi_b'(t) = s$ we have

$$-\psi_b'(t) = \frac{e^{q(1/t-1)}}{t^2} = s,$$

$$e^{q(1/t-1)} = st^2 \quad \Longleftrightarrow \quad q\left(\frac{1}{t} - 1\right) = \log s + 2\log t.$$

From $0 < t \leq 1$ we find that $\log t$ and as a result

$$q\left(\frac{1}{t} - 1\right) \leq \log s \quad \Longleftrightarrow \quad t = \underline{\rho}(s) \geq \frac{1}{1 + q^{-1}\log s}.$$

Now by using Lemma 4.2, since $\rho(s) \geq \underline{\rho}(1+2s)$, we derive a lower bound for $\rho(s)$ as follows:

$$\rho(s) \geq \frac{1}{1 + q^{-1}\log(1+2s)}, \quad s \geq 0. \tag{4.1}$$

Step 2. The function $\psi''(t)$ is monotonically decreasing, hence

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi''(\rho(2\delta))} \leq -\frac{\delta^2}{\psi''(\underline{\rho}(1+4\delta))}.$$

Putting $t = \underline{\rho}(1+4\delta)$, we have $t \leq 1$ and can write

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi''(t)} \leq -\frac{\delta^2}{1 + (q+2t)t^{-4}e^{q(1/t-1)}}. \tag{4.2}$$

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Note that

$$t = \underline{\rho}(1+4\delta) \quad \Longleftrightarrow \quad (1+4\delta) = -\psi_b'(t) = \frac{e^{q(1/t-1)}}{t^2}.$$

Substituting this equation in (4.2), we have

$$f(\tilde{\alpha}) \le -\frac{\delta^2}{1+(q+2t)t^{-2}(1+4\delta)} \le -\frac{\delta^2}{1+3qt^{-2}(1+4\delta)}.$$

On the other hand,

$$\frac{1}{t^2} = \frac{1}{\underline{\rho}(1+4\delta)^2} \le \frac{1}{\rho(2\delta)^2} \le \left(1+q^{-1}\log(1+4\delta)\right)^2.$$

So

$$f(\tilde{\alpha}) \le -\frac{\delta^2}{1+3q(1+4\delta)\left(1+q^{-1}\log(1+4\delta)\right)^2}. \tag{4.3}$$

Step 3. By Lemma 4.1 the inverse function of $\psi(t)$ for $t \in [1,\infty)$ satisfies

$$\sqrt{1+2\psi(t)} \le Q(\psi(t)) \le 1+\sqrt{2\psi(t)}. \tag{4.4}$$

Step 4. Using $\delta(\nu) \ge \psi'\big(Q(\Psi(\nu))\big)/2$, we have

$$\delta \ge \frac{\psi'\big(Q(\Psi(\nu))\big)}{2} \ge \frac{1}{2}\left(\sqrt{1+2\Psi} - \frac{e^{q(1/\sqrt{1+2\Psi}-1)}}{(1+2\Psi)^2}\right). \tag{4.5}$$

Note that $\Psi \ge 1$, so $(1+2\Psi)^{3/2} \ge e^{q(1/\sqrt{1+2\Psi}-1)}$ and $\sqrt{1+2\Psi} \le \sqrt{3\Psi}$. So we can write

$$\delta \ge \frac{1}{2}\left(\sqrt{1+2\Psi} - \frac{1}{\sqrt{1+2\Psi}}\right) = \frac{\Psi}{\sqrt{1+2\Psi}} \ge \frac{\Psi}{\sqrt{3\Psi}} \ge \sqrt{\frac{\Psi}{3}}. \tag{4.6}$$

Step 5. Let $\Psi_0 \ge \Psi \ge \tau \ge 3$. We deduced that $\delta \ge 1$ and $\sqrt{\Psi} \le \sqrt{3}\delta \le 2\delta$. Now by using (4.3) we have

$$f(\tilde{\alpha}) \le -\frac{\delta^2}{16q\delta\left(1+q^{-1}\log(1+4\delta)\right)^2} \le -\frac{\delta}{16q\left(1+q^{-1}\log(1+4\delta)\right)^2},$$

that is,

$$f(\tilde{\alpha}) \le -\frac{\sqrt{\Psi}}{48q\left(1+q^{-1}\log(1+\sqrt{\Psi_0})\right)^2}. \tag{4.7}$$

Thus it follows that

$$\Psi_{k+1} \le \Psi_k - \kappa(\Psi_k)^{1-\gamma}, \quad k = 0, 1, \ldots, K-1, \tag{4.8}$$

$$\text{where} \quad \kappa = \frac{1}{48q\left(1+q^{-1}\log(1+\sqrt{\Psi_0})\right)^2}, \quad \gamma = \frac{1}{2}$$

and K denotes the number of inner iterations.

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Step 6. From Lemma 4.1 we have $Q(\tau/n) \le 1 + \sqrt{2\tau/n}$. As a consequence

$$\Psi_0 \le L_\psi(n, \theta, \tau) = n\psi\left(\frac{Q(\tau/n)}{\sqrt{1-\theta}}\right) \le n\psi\left(\frac{1+\sqrt{2\tau/n}}{\sqrt{1-\theta}}\right).$$

Since $\psi(t) \le (t^2-1)/2$ for $t \ge 1$,

$$\Psi_0 \le \frac{n^2}{2}\frac{\sqrt{2\tau/n} + 2\tau/n}{1-\theta} = \frac{\tau + \sqrt{2n\tau}}{1-\theta}. \tag{4.9}$$

Step 7. By inequality (4.8), the number of inner iterations is bounded by

$$K \le \frac{\Psi_0^\gamma}{\kappa\gamma} = 96q\left(1 + q^{-1}\log\left(1 + \sqrt{\Psi_0}\right)\right)^2 \Psi_0^{1/2}.$$

Substituting (4.9) in this inequality gives

$$K \le 96q\left(1 + q^{-1}\log\left(1 + \sqrt{\frac{\tau+\sqrt{2n\tau}}{1-\theta}}\right)\right)^2 \left(\frac{\tau+\sqrt{2n\tau}}{1-\theta}\right)^{1/2}.$$

Thus the total number of iterations is bounded above by

$$\frac{K}{\theta}\log\frac{n}{\epsilon} \le 96q\left(1 + q^{-1}\log\left(1 + \sqrt{\frac{\tau+\sqrt{2n\tau}}{1-\theta}}\right)\right)^2 \left(\frac{\tau+\sqrt{2n\tau}}{1-\theta}\right)^{1/2} \frac{1}{\theta}\log\frac{n}{\epsilon}. \tag{4.10}$$

Step 8. For large-update methods set $\tau = O(n)$ and $\theta = \Theta(1)$. As a consequence,

$$\Psi_0 \le \frac{\tau + \sqrt{2n\tau}}{1-\theta} = O(n).$$

By choosing

$$q = \log\left(1 + \sqrt{\frac{\tau+\sqrt{2n\tau}}{1-\theta}}\right) = O(\log n)$$

the total iteration bound in (4.10) becomes

$$O\left(\sqrt{n}(\log n)\right)\log\frac{n}{\epsilon}.$$

Setting $\tau = O(1)$ and $\theta = \Theta(1/\sqrt{n})$ for small-update methods, we can obtain the best known bound as follows.

By Lemma 2.4 in [2], we have $\psi(t) < \psi''(1)(t-1)^2/2$, $t > 1$. Since $\psi''(1) = q + 3$, we achieve an upper bound for Ψ_0 of

$$\Psi_0 = \frac{n\psi''(1)}{2}\left(\frac{1+\sqrt{2\tau/n}}{\sqrt{1-\theta}} - 1\right)^2 \le \frac{n\psi''(1)}{2}\left(\frac{\theta+\sqrt{2\tau/n}}{\sqrt{1-\theta}}\right)^2$$
$$= \frac{q+3}{2}\frac{\left(\sqrt{2\tau} + \theta\sqrt{n}\right)^2}{1-\theta},$$

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where we also use that $1 - \sqrt{1-\theta} \leq \theta$. Now $\Psi_0 = O(q)$ and the iteration bound becomes

$$O\left(q\sqrt{qn}\right)\log\frac{n}{\epsilon}.$$

By choosing the parameter q as a constant which is independent of n, this is the best bound for small-update methods.

5. Concluding remarks

In this paper, we introduced a new kernel function and showed that it generates the best possible iteration bounds, both for small-update and for large-update methods. This paper was inspired as a result of the work on primal-dual interior-point methods (IPMs) for linear optimization based on kernel functions and the scheme for analyzing such methods. The new kernel function is a parameterized version of a kernel function introduced in [2].

References

[1] Y. Q. Bai, M. El Ghami and C. Roos, "A new efficient large-update primal-dual interior-point method based on a finite barrier", *SIAM J. Optim.* **13** (2003) 766–782 (electronic).

[2] Y. Q. Bai, M. El Ghami and C. Roos, "A comparative study of new barrier functions for primal-dual interior-point algorithms in linear optimization", *SIAM J. Optim.* **15** (2004) 101–128.

[3] Y. Q. Bai, J. Guo and C. Roos, "A new kernel function yielding the best known iteration bounds for primal-dual interior-point algorithms", 2006, working paper, available at http://www.isa.ewi.tudelft.nl/~roos/wpapers.html.

[4] N. Megiddo, "Pathways to the optimal set in linear programming", in *Progress in Mathematical Programming: Interior Point and Related Methods* (ed. N. Megiddo), (Springer Verlag, New York, 1989), 131–158. (Identical version in: Proceedings of the 6th Mathematical Programming Symposium of Japan, Nagoya, Japan, pages 1-35, 1986.)

[5] J. Peng, C. Roos and T. Terlaky, "Self-regular functions and new search directions for linear and semidefinite optimization", *Math. Program.* **93** (2002) 129–171.

[6] J. Peng, C. Roos and T. Terlaky, *Self-Regularity: A New Paradigm for Primal-Dual Interior-Point Algorithms* (Princeton University Press, Princeton, New Jersey, 2002).

[7] C. Roos, T. Terlaky and J.-Ph. Vial, *Theory and Algorithms for Linear Optimization. An Interior Point Approach* (John Wiley & Sons, Chichester, 1997).