APPLICATION OF GENERATING FUNCTIONS TO A PROBLEM IN FINITE DAM THEORY*

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Summary

We consider the finite dam model due to Moran, in which the storage $\{Z_t\}$ is known to be a Markov chain. The method of generating functions is used to derive stationary distributions of Z_t in the two particular cases where the input is of geometric and negative binomial types.

1. Introduction

In considering a model for the discrete dam of finite capacity K, Moran (1954) makes the following assumptions: (1) The inputs X_t $(t = 0, 1, \cdots)$ which flow into the dam in the yearly intervals $(t, t+1)$ are independently and identically distributed. (2) If Z_t $(< K)$ is the storage at time t before an input X_t flows into the dam, then for $Z_t + X_1 > K$, an amount $Z_t + X_t - K$ will overflow, but for $Z_t + X_t < K$, there will be no overflow; the dam will then contain a quantity K or $Z_t + X_t$, whichever is the lesser. (3) At time $t+1$, the amount of water released is M if $Z_t + X_t > M$ or $Z_t + X_t$ if $Z_t + X_t \leqq M$, where $M < K$. It is then clear that $\{Z_t\}$ and $\{Z_t + X_t\}$ are both Markov chains, and for a given probability distribution of the input X_t, their stationary distributions may be studied. In particular, when X_t has a geometric distribution, the stationary distribution of Z_t has been obtained by Moran (1955) by an ingenious method, which, however, is not applicable to the more general class of input distributions. The case of the infinite dam $(K = \infty)$ is much simpler; for, as observed by Gani and Prabhu (1957), the Markov chain $\{Z_t + X_t\}$ also occurs in the theory of queues, and Bailey (1954) has obtained its stationary distribution by the method of probability generating functions (p.g.f.). In this paper Bailey's method is applied to the case of the dam of finite capacity K, to obtain the stationary distribution of the storage Z_t.

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2. Application of Bailey's Method to the Finite Dam

Let Δ be the unit of measurement, and $K = k\Delta$, $M = m\Delta$. For the sake of simplicity we take $m = 1$, for $m > 1$ our methods are applicable, but the resulting theory is much more complicated. Let $\{g_j\}$ be the probability distribution of X_t, so that

(1) $$Pr\{X_t = j\Delta\} = g_j, \quad (j = 0, 1, 2, \cdots).$$

Also, let $\{u_i\}$, $(i = 0, 1, \cdots k-1)$ be the stationary probability distribution of the dam storage Z_t; these satisfy Moran's (1954) fundamental equations

(2) $$\begin{aligned} u_0 &= u_0(g_0+g_1)+u_1g_0 \\ u_1 &= u_0g_2+u_1g_1+u_2g_0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_{k-2} &= u_0g_{k-1}+u_1g_{k-2}+\cdots+u_{k-1}g_0 \\ u_{k-1} &= u_0(g_k+\cdots)+u_1(g_{k-1}+\cdots)+\cdots+u_{k-1}(g_1+\cdots) \end{aligned}$$

together with $u_0 + u_1 + \cdots + u_{k-1} = 1$. Let

(3) $$G(z) = \sum_{j=0}^{\infty} g_j z^j, \; U(z) = \sum_{j=1}^{k-1} u_j z^j$$

be the g.f.'s of $\{g_j\}$ and $\{u_j\}$ respectively. Multiplying the equations (2) successively by $1, 0, z^2, \cdots z^{k-1}$ and adding up we obtain

$$U(z) = u_0 g_0 \frac{z-1}{z} + \frac{1}{z}U(z)G(z) - \frac{1}{z}\sum_{i=0}^{k-1} u_i z^i \sum_{j=k-i}^{\infty} g_j(z^j - z^{k-i}),$$

whence we see that the p.g.f. $U(z)$ is given by

(4) $$U(z) = \frac{u_0 g_0(z-1) - \sum_{i=0}^{k-1} u_i z^i \sum_{j=k-i}^{\infty} g_j(z^j - z^{k-i})}{z - G(z)}.$$

It may be noted that all the unknown quantities $u_0, u_1, \cdots u_{k-1}$ appear in the numerator of the last expression; however, in the particular cases considered below, these can be reduced to a finite number N of quantities which can be evaluated. This method is useful if $N < k$; if $N > k$ it seems advisable to proceed directly by solving the equations (2).

3. Stationary Distribution Arising from a Geometric Input

Consider first an input distribution of the geometric type,

(5) $$g_j = ab^j \quad (j = 0, 1, \cdots),$$

where $0 < a < 1$, and $b = 1 - a$; the p.g.f. of X_t is then

(6) $$G(z) = \frac{a}{1-bz},$$

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and the mean input is $\rho = b/a$. Substituting (5) and (6) in (4) we obtain, after some simplification,

$$U(z) = \frac{u_0(1 - bz) - (bz)^k \rho U(1/b)}{1 - \rho z}, \tag{7}$$

which gives the p.g.f. of the stationary distribution of Z_t except for the two unknown quantities u_0 and U $(1/b)$. To evaluate these, we first observe that the denominator of the expression on the right hand side of (7) has the zero $z = 1/\rho$, and since $U(z)$ is a polynomial, the numerator should also have $z = 1/\rho$ as a zero; thus

$$u_0 \left(1 - \frac{b}{\rho}\right) - \left(\frac{b}{\rho}\right)^k \rho U(1/b) = 0,$$

or, $U(1/b) = u_0/a^{k-1}$. Substituting this in (7) we have that

$$U(z) = u_0 \left\{a + b \frac{1 - (\rho z)^k}{1 - \rho z}\right\}. \tag{8}$$

Further, u_0 can be obtained by using the fact that $U(1) = 1$; we find that

$$u_0 = \frac{1 - \rho}{a(1 - \rho^{k+1})} \tag{9}$$

Thus, finally from (8),

$$u_i = \frac{(1 - \rho)\rho^{i+1}}{1 - \rho^{k+1}}, \quad (i = 1, 2, \cdots k - 1).$$

We thus see from (8) that a geometric input results in the stationary distribution of the geometric type which is truncated at $Z = k - 1$ and has a modified initial term. This result is implied in Moran's (1955) solution for the general case $m > 1$, although it is not explicitly mentioned by him; for $m = 1$ this solution is given by the formula

$$\pi_{k-r}/\pi_k = {}^rS_1 a - {}^{r-1}S_2 a^2 + {}^{r-2}S_3 a^3 - \cdots,$$

where

$$ {}^nS_p = \binom{n-1}{p-1} b^{-n} - \binom{n-2}{p-1} b^{1-n}, \tag{10}$$

$$u_0 = \pi_0 + \pi_1, \; u_i = \pi_{i+1}, \quad (i = 1, 2, \cdots k - 1).$$

From (10) we obtain (9) after some simple reduction.

For the mean and variance of the stationary distribution we have the expressions

$$\begin{aligned} \bar{u} &= \frac{\rho}{1 - \rho} - \frac{\rho(1 + k\rho^k)}{1 - \rho^{k+1}} \\ \sigma_u^2 &= \frac{\rho}{(1 - \rho)^2} - \rho \frac{1 + k^2 \rho^k}{1 - \rho^{k+1}} - \rho^2 \frac{(1 + k\rho^k)^2}{(1 - \rho^{k+1})^2}. \end{aligned} \tag{11}$$

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4. Stationary Distribution Arising from an Input of the Negative Binomial Type

Next we consider the input distribution given by

(12) $$g_j = (j+1)a^2 b^j \quad (j = 0, 1, \cdots),$$

where $0 < a < 1$ and $b = 1 - a$; in this case we have

(13) $$G(z) = a^2(1 - bz)^{-2}.$$

Further, we have

(14) $$\sum_{j=k-i}^{\infty} g_j z^j = \frac{a^2(bz)^{k-i}}{(1-bz)^2}\{1 + (k-i)(1-bz)\},$$

or, for $z = 1$,

(15) $$\sum_{j=k-i}^{\infty} g_j = b^{k-i}\{1 + (k-i)a\}.$$

Substituting (13), (14) and (15) in (4) we obtain, after some simplification,

(16) $$U(z) = a^2 u_0 \frac{(1-bz)^2 - z^k(w_0 + w_1 z)}{b^2(z_1 - z)(z_2 - z)},$$

where u_0, w_0 and w_1 are unknown quantities, and z_1 and z_2 are the roots (other than unity) of the equation $z(1-bz)^2 - a^2 = 0$. To evaluate w_0 and w_1, we argue as in the previous sections; we find that these must satisfy the equations

$$z_1^{k+1} w_0 + z_1^{k+2} w_1 = a^2$$
$$z_2^{k+1} w_0 + z_2^{k+2} w_2 = a^2.$$

we obtain readily

(17) $$w_0 = b^2 \frac{[k+2]}{(z_1 z_2)^k[1]}, \quad w_1 = b^2 \frac{[k+1]}{(z_1 z_2)^k[1]},$$

where $[r] = z_1^r - z_2^r$, $(r = 1, k+1, k+2)$. Substituting (17) in (16) and simplifying, we obtain

(18) $$\begin{aligned} U(z) &= a^2 u_0 \left\{1 + \frac{1}{[1]}\frac{z_1}{z_2^k}\frac{z_2^k - z^k}{z_2 - z} - \frac{1}{[1]}\frac{z_2}{z_1^k}\frac{z_1^k - z^k}{z_1 - z}\right\} \\ &= a^2 u_0 \left\{1 + \frac{\alpha^2}{\alpha - \beta}\frac{1 - (\alpha z)^k}{1 - \alpha z} - \frac{\beta^2}{\alpha - \beta}\frac{1 - (\beta z)^k}{1 - \beta z}\right\}, \end{aligned}$$

where $\alpha = 1/z_2$, $\beta = 1/z_1$. Finally, u_0 is given by

(19) $$u_0 = a^{-2}\left\{1 + \frac{\alpha^2}{\alpha - \beta}\frac{1 - \alpha^k}{1 - \alpha} - \frac{\beta^2}{\alpha - \beta}\frac{1 - \beta^k}{1 - \beta}\right\}^{-1}.$$

From (19) we see that the stationary distribution of the dam storage is a linear combination of distributions of the type obtained in the previous section. The mean and variance of the distribution can be obtained from (18).

References

[1] Bailey, N. T. J., 'On queueing processes with bulk service,' *J. Roy. Stat. Soc.*, B Vol. 16 (1954), pp. 80–87.

[2] Gani, J. and Prabhu, N. U., 'Stationary distributions of the negative exponential type for the infinite dam,' *J. Roy. Stat. Soc.*, B Vol. 19 (1957), 342–351.

[3] Moran, P. A. P., 'A probability theory of dams and storage systems,' *Aust. J. Appl. Sci.*, Vol. 5 (1954), 116–124.

[4] Moran, P. A. P., 'A probability theory of dams and storage systems: modification of release rules,' *Aust. J. Appl. Sci.*, Vol. 6 (1955), 117–130.

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