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ARTICLE

The optimal cyclical design for a target benefit pension plan

Lv Chen[1], Danping Li[2*], Yumin Wang[3] and Xiaobai Zhu[4]

[1]Academy of Statistics and Interdisciplinary Sciences, KLATASDS-MOE, East China Normal University, Shanghai, China, [2]School of Statistics, KLATASDS-MOE, East China Normal University, Shanghai, China, [3]Warren Centre for Actuarial Studies and Research, University of Manitoba, Winnipeg, Manitoba, Canada and [4]Department of Insurance, Finance and Actuarial Science, Southwestern University of Finance and Economics, Chengdu, China
*Corresponding author. Email: dpli@fem.ecnu.edu.cn

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Abstract
In this paper, we derive the optimal cyclical design of a target benefit (TB) pension plan that balances the sustainability and the benefit stability using the optimal control approach. The optimal design possesses a linear risk sharing structure with cyclical parameters. We observe that the optimal design should be pro-cyclical in the usual circumstances, but counter-cyclical when the pension plan is severely in deficit. We compare the TB plans with the defined benefit plans and conclude that a more aggressive investment strategy should be adopted for the TB plans. In the end, we provide a cautionary note on the optimal control approach in the study of the TB plans.

Key words: Hamilton–Jacobi–Bellman equation; optimal cyclical design; regime switching model; target benefit pension plan
JEL codes: J32

1. Introduction

Recent experience has shown that the traditional defined benefit (DB) plans threaten the solvency of the employers and do not deliver benefit security to the retirees when the default risk is considered. The generous benefits promised by the DB plans are hardly sustainable in a low interest rate environment (Greszler, 2017). There is a steady trend over the past two decades that the occupational plans are shifting toward alternative plans, particularly toward the defined contribution (DC) pension plans.[1] However, there is also a growing recognition that employees with only the DC plans are facing both the investment risks during the accumulation period, and the interest rate, longevity, and dissipation risks during the decumulation phase (Cooper, 2013). The inadequacy of both the pure DB and DC plans leads to the development of non-traditional pension designs; one example is the *target benefit* (TB) plans in Canada.

Canadian TB plans have risen in popularity during the past decades, and have already been adopted in the public sectors of New Brunswick, Alberta, and British Columbia. Similar pension schemes also exist in other countries, such as the 'defined ambition' plans in the Netherlands and the 'variable defined benefit' plans in the UK. A TB plan is essentially a collective type of pension scheme that allows for both intra-generational and inter-generational risk sharing. It shares many similarities with the traditional DB plans, and often adopts the same benefit formulae as the DB plans. However, these benefits are '*targeted*' instead of promised in the sense that the actual retirement benefits are adjustable according to the plans' funding status. Previous studies have demonstrated that a well-designed TB plan is able to deliver reasonably stable retirement incomes while ensuring the long-

[1]Thinking Ahead Institute, 2022, 'Global Pension Assets Study 2021'. Available at https://www.thinkingaheadinstitute.org

term sustainability of the pension plan. To name a few papers, Gollier (2008), Cui *et al.* (2011), Khorasanee (2012), Bovenberg and Mehlkopf (2014), Chen *et al.* (2017), Wang *et al.* (2018), Wang *et al.* (2019), Bégin (2020), Chen and Rach (2021), and Zhao and Wang (2022) construct different specifications of the risk sharing structures in the pension plans, and demonstrate their welfare improvement over both the DB and the DC plans. However, the models adopted in the aforementioned papers only reflect the risks under the regular market conditions. Whether and how a TB plan can be designed to survive under the extreme market conditions remain unexplored.

History has proved that a financial crisis may have disastrous effects on the healthiness of a pension plan. According to the Equable Institute, the value of the state-sponsored pension assets in the US has dropped by 30.8% during the 2007–8 financial crisis. The gap between the pension liability and the asset was 27.4% in 2019 compared with 6.2% before the Financial Crisis of 2008/09.[2] While the pension plans are not fully recovered yet from the previous financial crisis, the COVID-19 outbreak has brought even more challenges to the sponsors. The theoretical appealing features of risk sharing in the TB plans during the normal market conditions do not shed light on its performance during the financial crisis.

The flexible design of a TB plan allows the sponsors to construct a cyclical risk sharing structure in response to the financial crisis. Cyclical policies have already been established by the DB regulators for benefit security (discussions can be found in Antolín and Stewart (2009) and Yermo and Severinson (2010)). However, an unsolvable difficulty for the DB regulators is the trade-off between a loose regulation that the sponsors may act entirely against the retirees' interests and an over-regulation that puts more pressure on the sponsors' already weakened financial strength. In contrast, a TB design is able to automatically mitigate systematic risks without the reliance on regulation enforcement. We add to the literature by providing a theoretical justification on the benefits of adding the cyclical features to the risk sharing structure. More specifically, we explicitly solve the optimal TB design problem via the stochastic optimal control approach with a regime-switching model. Our objective takes into account intergenerational fairness by including both the sustainability of the pension plan and the income stability of the retirees.

Our main findings are in the following. First, we provide a transparent cyclical risk sharing design. We prove that the optimal risk sharing structure can be seen as a regime-dependent extension of the risk sharing plan proposed by Cui *et al.* (2011). The linearity of the surplus/deficit share is preserved, with regime-dependent parameters to be taken into account of the cyclicity. It is common to obtain the optimal structure as a linear form by setting the quadratic loss function as the objective; see, for example, Haberman and Sung (1994), Josa-Fombellida and Rincón-Zapatero (2001), Josa-Fombellida and Rincón-Zapatero (2004), and Wang *et al.* (2018), among others. However, those optimal results may involve overly complicated mathematical expressions. We have greatly simplified the theoretical work to construct a transparent cyclical design without losing its practicality. We further show that it is infeasible to construct a universal optimal risk sharing plan that is regime-independent. We also demonstrate that the percentage of the deficit share is mathematically equivalent to the intergenerational fairness defined in terms of income stability.

Second, our numerical experiments illustrate the optimal deficit sharing structure incorporating both pro- and counter-cyclical features. Specifically, if a plan stays relatively healthy, then a pro-cyclical structure can be adapted such that the retirement benefits will be higher under the tranquil periods, while a counter-cyclical structure should be applied if the plan is severely underfunded. We emphasize a key difference in the motivation for establishing a counter-cyclical policy between a DB and a TB plan: during the turbulent periods, the DB regulation aims to relieve the stress of the DB sponsors while a TB plan focuses on the relief of the tremendous deficit-sharing borne by the current retirees. The optimal structure indicates that although the sustainability of the pension plan has been taken into consideration, it does not mean that a complete sacrifice of the current

[2]Equable Institute, 'State of Pensions 2020: National Pension Funding Trends', https://equable.org/state-of-pensions-2020-national-pension-funding-trends/

and nearby generations should be allowed. In addition, by comparing with the DB plans, we verify with the literature that the retirees may not prefer a TB plan as they provide much more volatile incomes, but a TB does greatly enhance the sustainability of the plan, and thus better balances the intergenerational fairness.

Third, we investigate the relationship between the amount of risk sharing and the equity share. We suggest that to construct a stable risk sharing design, full investment in equities would be preferred, compared to the industrial standard of 60/40[3] or 70/30 for the DB pension funds. The intuition lies in the fact that the optimal structure is highly sensitive to the choice of the risk-free rate, and thus in order to construct a robust design, the sponsors should try to remove the risk-free asset from their portfolio.

In the end, we examine the common assumptions adopted by the stochastic control literature. Specifically, we demonstrate that a single regime model, which results in a highly counter-intuitive risk sharing scheme, is not enough in constructing a cyclical plan. Moreover, treating the equity share as a dynamic control variable, as often adopted in the literature, leads to highly impractical investment strategies. The benefit volatility can be fully hedged through dynamically balancing the portfolio, but with the weights of the equities fluctuating over ±100, 000%. Of course, our study relies on a rather stylized pension fund using a simplified economic model, but it does capture many realistic features that the results obtained are able to provide meaningful guidance to the practice.

The paper is organized as follows. Section 2 presents the models for the economy, the plan provisions, the dynamics of asset and liability, and the problem formulation. Section 3 summarizes our theoretical findings, and illustrates the optimal cyclical design with benchmark parameters. Section 4 discusses the robustness of the results and further examines commonly adopted assumptions in the literature. Section 5 concludes.

2. Model

Our study focuses on the optimal risk sharing design, and therefore, the risk allocation rule is the solution of our optimization problem rather than being pre-defined. In order to obtain explicit results, we adopt a simplified yet commonly used model under the continuous setting. We calibrate most of the parameters with the market data, and for others we will provide sensitivity tests in the later sections.

As the existing TB plans are mostly transitioned from the DB plans, actuarial practices in the DB plans are often kept the same for the TB plans (e.g., on the liability valuation). Our scheme is a stylized version of the real pension plan in Canada, with the contribution to the pension fund expressed as a fixed percentage of the employees' salary. Also, in order to reflect the fact that a TB plan provides a 'target' benefit level to the retirees, we include the target level into our objective function and minimize the deviations between the actual and the target incomes.

2.1. Overlapping generations

We adopt a multi-period overlapping generation model in the continuous-time setting. We assume homogeneous individuals in the sense that their employment entry age is fixed at A and their retirement age is fixed at R, with a maximum attainable age of ω. For the numerical experiments in the later sections, we set $A = 25$, $R = 65$, and $\omega = 115$.

Denote ${}_t p_x$ as the probability that an individual aged x will survive for at least t years, and $n_x(t)$ as the number of lives aged x at time t. We assume the population is stationary with 100 new employees every year entering at the age of 25 and retiring at the age of 65, and assume no pre-exit other than death.

We ignore the longevity risk, and assume the mortality risk is fully diversifiable, which means that the actual number of death is exactly the same as its expectation (i.e., $n_x(t) = 100 \times {}_{x-25}p_{25}$). Mortality

[3]60% in equities and 40% in risk-free assets.

rates are calculated based on the Makeham Law so the force of mortality at age x is defined as $m_x = A_m + B_m \times (c_m)^x$. Parameters are fitted to the Canadian Pensioners' Mortality Table 2014 (CPM2014, male), such that $A_m = 0.0015$, $B_m = 1.6605 \times 10^{-6}$, and $c_m = 1.1339$.

2.2. Economic assumption

Since the pioneer work of Merton (1969), the continuous-time model plays a key role in financial modeling. However, Merton's model is not suitable for the applications that require long-term modeling. Regime switching models, as initially proposed by Hamilton (1989), include a finite-state continuous-time Markov chain that represents the uncertainty of the market conditions, and thus grant a more accurate representation of the long-term financial market (see Hardy, 2001). In this paper, we follow the model that has been studied in Sotomayor and Cadenillas (2009), in which the regimes are modeled by an observable continuous-time Markov chain, and all coefficients of the market are regime-dependent.

Denote $(\Omega, \mathcal{F}, \mathbb{P})$ as a complete probability space with a right-continuous filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ generated by a two-dimensional standard Brownian Motion $(Z(t), \bar{Z}(t))$ and a continuous-time stationary Markov chain $\varepsilon(t)$ taking values in a finite state space $\mathcal{M} = \{1, 2, \ldots, m\}$. We further assume that $\varepsilon(t)$ is independent of $Z(t)$ and $\bar{Z}(t)$. The Markov chain $\varepsilon(t)$ has a generator $Q = (q_{ij})_{m\times m}$ (transition rate matrix) and stationary transition probabilities $p_{ij}(t) = \mathbb{P}(\varepsilon(t) = j|\varepsilon(0) = i) \geq 0$, for $t \geq 0$, and $i, j = 1, 2,.., m$. Moreover, we stipulate that $\bar{Z}(t)$ correlates with $Z(t)$ with a correlation coefficient $\rho_{\varepsilon(t)}$ at regime $\varepsilon(t)$.

The pension fund trustee allocates across two representative assets, one risk-free and one risky. The value evolution of the risk-free asset $S_0(t)$ and the risky asset $S_1(t)$ is given by

$$
\begin{aligned}
dS_0(t) &= r_{\varepsilon(t)} S_0(t)dt, \\
dS_1(t) &= \mu_{\varepsilon(t)} S_1(t)dt + \sigma_{\varepsilon(t)} S_1(t)dZ(t),
\end{aligned}
$$

with the initial prices $S_0(0) = 1$ and $S_1(0) = s > 0$, and the initial state $\varepsilon(0) = \varepsilon_0 \in \mathcal{M}$, where $r_{\varepsilon(t)}$, $\mu_{\varepsilon(t)}$, and $\sigma_{\varepsilon(t)}$ are $\varepsilon(t)$-dependent risk-free rate, equity return, and volatility, respectively. Note that each state of the Markov chain $\varepsilon(t)$ represents a certain market regime, and the condition $\mu > r$, which is often proposed in a single regime model, is no longer required.

In addition to the stochastic modeling of the equity price, we also model the uncertainty of the income process. Here we denote $L(t)$ as the inflation index and its dynamics is given by

$$
dL(t) = \alpha_{\varepsilon(t)} L(t)dt + \eta_{\varepsilon(t)} L(t)d\bar{Z}(t),
$$

where $\alpha_{\varepsilon(t)}$ and $\eta_{\varepsilon(t)}$ are the regime-dependent instantaneous growth rate and the volatility of the inflation index, respectively.

We assume the uncertainty in the salary growth is fully captured by the inflation index $L(t)$. The entry salary has been unitized to be the same as the inflation index, and we further stipulate a salary growth rate of γ_1 (e.g., 50bps) on top of the inflation index $L(t)$ for the active employees to represent the promotional increases. On the other hand, the retirement benefit is subjected to a depreciation rate of γ_2 (e.g., 100bps) to reflect the inflation adjustment (often referred to as the partial inflation indexation).

For our numerical experiments later, we adopt a two-regime model for the risky asset with its parameters calibrated to the monthly data of S&P 500 from January 2000 to October 2020, which is sufficient to capture the two most shocking events, the 2008–09 financial crisis and the recent COVID-19 pandemic. The two regimes represent the low-volatility regime (regime 1, regular market) and the high-volatility regime (regime 2, recession). Although the existing literature has investigated on modeling with more regimes (e.g., a three-regime model; Cerboni Baiardi *et al.*, 2020), a two-regime model

Table 1. The annualized parameters for a two-regime model

Regime ε	μ_ε	σ_ε	$p_{\varepsilon\varepsilon}$	$q_{\varepsilon\varepsilon}$
1	0.1403	0.0787	0.6486	−0.5828
2	−0.038	0.1979	0.6796	−0.5313

μ_ε and σ_ε are the equity return and the volatility of the risky asset, $p_{\varepsilon\varepsilon}$ and $q_{\varepsilon\varepsilon}$ are the transition probability and the transition rate between regimes, where $\varepsilon = 1$ represents the regular market and $\varepsilon = 2$ represents the recession.

remains popular in practice, and including additional regimes adds only marginal improvement to the overall likelihood (Hardy, 2001). The parameters are given in Table 1.

In addition, we take $\alpha_1 = \alpha_2 = \alpha = 0.03$, $\eta_1 = \eta_2 = \eta = 0.01$, and $\rho = 0.1$ for the salary process, and assume $r_1 = 0.04$ and $r_2 = 0.02$ for the risk-free asset. Note that the parameters are chosen based on the related literature (e.g., Sotomayor and Cadenillas, 2009; Yuen and Yang, 2009; Cui *et al.*, 2011; Wang *et al.*, 2018, etc.). Since the values are rather arbitrarily set, additional sensitivity tests are performed in the later sections.

Admittedly, it is impossible to actually detect the instantaneous moments of the switches between the market regimes. However, different indices in practice can be used as the signals for the changes in the underlying regimes. For example, Esteve *et al.* (2020) demonstrate that VIX is a good indicator for any structural changes in the market, and the National Financial Conditions Index (NFCI) is introduced to reflect the 'tightness' of the overall financial conditions. Cyclical risk sharing design can rely on these observable external financial stress indices to monitor the switches between different regimes.

2.3. Plan provisions

The TB pension plan can be either funded by the active members, the sponsor, or both, via contributing a fixed percentage of the active workers' salary c continuously into the fund until they are either retired or deceased. The retirement benefit, which will be adjusted according to the financial conditions, is expressed as a percentage $b(t)$ of their pre-retirement income (inflation adjusted). Therefore, $b(t)$ can be viewed as the instantaneous replacement ratio at time t. In a TB plan, the actual benefit $b(t)$ may fluctuate around a target level $\bar{b}$. Since we are considering a homogeneous population without the mortality risk, the aggregate contribution $C_{\varepsilon(t)}(t)$ and the aggregate benefit $B(t)$ at time t can be expressed as:

$$C_{\varepsilon(t)}(t) = \underbrace{\int_A^R}_{\text{Active Workers}} c_{\varepsilon(t)} \times \underbrace{L(t) \times e^{(x-A)\gamma_1}}_{\text{Current Salary}} \times \underbrace{n_x(t)}_{\text{\# of Age x}} \, dx = c_{\varepsilon(t)} \times L(t) \times \mathcal{A}(t), \tag{1}$$

$$\begin{aligned} B(t) &= \underbrace{\int_R^\omega}_{\text{Retirees}} b(t) \times \underbrace{L(t) \times e^{(R-A)\gamma_1} \times e^{-(x-R)\gamma_2}}_{\text{Salary Adjustment}} \times n_x(t) dx = b(t) \times L(t) \times \mathcal{R}(t), \\ \mathcal{A}(t) &= \int_A^R e^{(x-A)\gamma_1} \times n_x(t) dx, \\ \mathcal{R}(t) &= \int_R^\omega e^{(R-A)\gamma_1} \times e^{-(x-R)\gamma_2} \times n_x(t) dx, \end{aligned} \tag{2}$$

where $e^{(x-A)\gamma_1}$ is the promotion adjustment to the salary for the age x, and since the mortality risk is ignored, the salary immediately before the retirement at time t is simply $L(t)e^{(R-A)\gamma_1}$. When we ignore the salary growth rate and the retirement benefit adjustment rates such that $\gamma_1 = \gamma_2 = 0$, then $\mathcal{A}(t)$ and

$\mathcal{R}(t)$ will represent the population size of the active workers and the retirees at time t, respectively. Furthermore, if the population becomes stationary, i.e., $n_x(t) = n_x$, then we have $\mathcal{A}(t) = \mathcal{A}$ and $\mathcal{R}(t) = \mathcal{R}$.

In addition to the contribution and the retirement benefit, we assume that a death benefit is payable immediately as the member deceases. Here we denote $D_x(t)$ as the death benefit for an aged x worker at time t, and assume it is simply a multiple δ of the member's current annualized salary if s/he is an active worker, or the target retirement benefit if s/he is a retiree. Specifically,

$$D_x(t) = \begin{cases} \delta \times L(t) \times e^{(R-A)\gamma_1}, & \text{for active employees,} \\ \delta \times \bar{b} \times L(t) \times e^{(R-A)\gamma_1 - (x-R)\gamma_2}, & \text{for retirees.} \end{cases}$$

Therefore, the aggregate death benefit payment made by the sponsor at time t is:

$$\begin{aligned} D(t) &= \int_A^{\omega} n_x(t) \times m_x(t) \times D_x(t) dx \\ &= \delta \times L(t) \times \mathcal{D}_A(t) + \delta \times \bar{b} \times L(t) \times \mathcal{D}_R(t) \\ &= L(t) \times \mathcal{D}(t), \end{aligned}$$

where $m_x(t)$ is the force of mortality for workers aged $x + t$, $\mathcal{D}_A(t) = {}_A^R\, n_x(t) \times m_x(t) \times e^{(x-A)\gamma_1} dx$, $\mathcal{D}_R(t) = \int_R^{\omega} n_x(t) \times m_x(t) \times e^{(R-A)\gamma_1 - (x-R)\gamma_2} dx$, and $\mathcal{D}(t) = \delta \times \mathcal{D}_A(t) + \delta \times \bar{b} \times \mathcal{D}_R(t)$. Again, if $\gamma_1 = \gamma_2 = 0$, then $\mathcal{D}_A(t)$ and $\mathcal{D}_R(t)$ represent the number of death for the active workers and the retirees at time t, respectively.

2.4. Pension assets

Let $\pi_{\varepsilon(t)}$ be the portfolio weight invested in the risky asset when the market is in regime $\varepsilon(t)$, then the pension asset $X(t)$ has the following SDE:

$$\begin{aligned} dX(t) &= \pi_{\varepsilon(t)} X(t) \frac{dS_1(t)}{S_1(t)} + (1 - \pi_{\varepsilon(t)}) X(t) \frac{dS_0(t)}{S_0(t)} + (C_{\varepsilon(t)}(t) - B(t) - D(t)) dt \\ &= [X(t)(r_{\varepsilon(t)} + \pi_{\varepsilon(t)}(\mu_{\varepsilon(t)} - r_{\varepsilon(t)})) + (C_{\varepsilon(t)}(t) - B(t) - D(t))] dt + \sigma_{\varepsilon(t)} \pi_{\varepsilon(t)} X(t) dZ(t), \end{aligned} \tag{3}$$

where $X(0) = x_0$. Here we assume the trustee of the pension fund adopts the constant weighting portfolio. In contrast to the common assumption in the optimal control literature where a dynamic asset allocation is often used, the pension portfolio in practice is often closer to a constant weighting strategy where the asset allocation is closely tied to a target mix.[4] The traditional approach for the portfolio mix is 60/40 in Anglo-American countries, and 70/30 in the UK (Franzen, 2010). In addition, it has also been demonstrated by, for example, Graf (2017) and Forsyth and Vetzal (2019), that it is not easy to out-perform the constant weight portfolio. We do include the equity share as an additional control variable in Section 4 and illustrate the limitations of the market friction-less model.

2.5. Target benefit and pension liability

Since most of the TB plans are transitioned from the existing DB plans, the benefit accrual formulae, as well as the liability valuation methods, are often kept the same. The TB level under a TB plan is often set as the promised benefit level in the original DB plans. Therefore, we assume the target replacement rate $\bar{b}$ is externally given[5] such that the TB is $\bar{b} \times L(t)$ for a newly retired person at

[4]Pension sponsors do adjust their portfolio mix over time, but the adjustments are rather slow. For example, over the past decade, the target mix for Canadian Pension Plan is roughly 60% invested in equities, and 40% invested in fixed income and real assets.

[5]The choice of $\bar{b}$ is often between 40% and 70% for employees that spent their entire career in the company.

time t and is $\bar{b} \times L(t) \times e^{(R-A)\gamma_1-(x-R)\times\gamma_2}$ for a retiree aged x. The aggregate amount of target retirement benefit can be simplified as:

$$\bar{B}(t) = \bar{b} \times L(t) \times \mathcal{R}(t).$$

Based on the TB level, we adopt the traditional unit credit (TUC) approach for the liability valuation.[6] When early leaves are not allowed (except in case of death), the liability can be expressed as:

$$\begin{aligned}\text{Liability}(t) &= \int_A^\omega n_x \times \underbrace{\frac{\min(x,R)-A}{R-A}}_{\text{Accrued Benefit}} \times \underbrace{\int_{\max(x,R)}^{\omega} v^{u-x} \times {}_{u-x}p_x \times \bar{b} \times L(t)du\,dx}_{\text{Present Value of Future Retirement Benefit}} \\ &+ \int_A^\omega n_x \times \underbrace{\int_x^\omega {}_{u-x}p_x \times m_u \times v^{u-x} \times D_u(t)du\,dx}_{\text{Present Value of Death Benefit}},\end{aligned} \tag{4}$$

where $v = (1 + r_d)^{-1}$ is the discount factor with r_d being the discount rate. Although the choice of the discount rate should depend on both the current interest rate and the expected return of the pension funds, in practice, changes in the discount rate are rather slow to avoid volatile liability values between the adjacent years. Therefore, in this paper, the discount rate is chosen to be a regime-independent constant, and the liability value will also be regime-independent.

2.6. Problem formulation

One of the main purposes of introducing risk sharing structure in a pension plan is to postpone the pension deficits of the unfortunate cohorts, or to spread excess profits from the windfall cohorts. Therefore, a successful TB design should be able to limit the variability of benefits across all generations. Specifically, the objective is to minimize the variation between the actual and the TB level. In this paper, we choose a quadratic function as the measure to gauge the income stability.[7] In light of the above, the income stability at time t is measured as

$$(\text{Aggregate Benefit} - \text{Aggregate Target Benefit})^2 = (b(t) \times L(t) \times \mathcal{R}(t) - \bar{b} \times L(t) \times \mathcal{R}(t))^2.$$

Notice that changing the TB level is equivalent to adding a penalty term. For example, in Wang *et al.* (2018), the authors add a linear penalty term $\varrho \times [\bar{b} \times L(t) \times \mathcal{R}(t) - b(t) \times L(t) \times \mathcal{R}(t)]$, which is equivalent to changing the target replacement rate to $\bar{b} + \varrho/2$. For most DB plans, the promised benefit level will be kept the same regardless of the market regimes. Therefore, the TB is selected to be regime-independent in this paper.

From both the numerical and the practical aspects, the projection period is often a finite number (e.g., T is chosen between 20 and 100 years). Thus, some restrictions must be made at the terminal time T to avoid excessive deficits being transferred to the future generations. Similar to the income

[6]For the DB liability valuation, TUC is often used for solvency purposes, and the projected unit credit (PUC) is often used for going-concern purposes. The technical difference is that the PUC method includes future salary growth while the TUC method does not. Some discussions on the TB liability valuation from the industrial perspectives can be found in Ma (2017a, 2017b, 2018).

[7]Alternative objectives often used in the literature are the utility functions. However, commonly used utility functions may achieve an overall higher utility by providing large benefits to certain generations while sacrificing the rest, which is inconsistent with the aim of a TB plan, namely providing adequate and sustainable benefits. On the contrary, minimizing the income instability will penalize on overly generous benefit payment that threatens the long-term funding levels, and avoid small benefit payment that sacrifices the interests of the current retirees.

stability, here we impose a quadratic terminal condition at the end of the projection period to penalize the instability of the terminal asset value. Specifically, we minimize the quadratic difference between the actual and the target asset levels:

$$\text{Penalty Weight} \times (\text{Terminal Asset Level} - \text{Target Asset Level})^2 = \lambda_{\varepsilon(T)} \cdot (X(T) - X^*_{\varepsilon(T)}(T))^2,$$

where $\lambda_{\varepsilon(T)}$ is the penalty weight, and $X^*_{\varepsilon(T)}(T)$ is the target asset level at time T. The target asset level is decided at the plan inception, and here we define $X^*_{\varepsilon(T)}(T) = X^*_{\varepsilon(T)} \times L(T)$ where $X^*_{\varepsilon(T)}$ is the target asset level expressed in the real terms, and $L(T)$ is the inflation adjustment. Note that λ and X^* are chosen to be regime-dependent to reflect the changes in the risk appetites and the regulatory requirements under different market conditions.

Given the objective function above, we solve the following optimization problem:

$$\inf_{b \in \Pi} \mathbb{E}_{t,x,l,\varepsilon}\{ \int_t^T e^{-r_0 s}(b(s)L(s)\mathcal{R}(s) - \bar{b}L(s)\mathcal{R}(s))^2 ds + \lambda_{\varepsilon(T)} e^{-r_0 T}(X(T) - X^*_{\varepsilon(T)} \times L(T))^2\}, \tag{5}$$

where Π denotes the set of all admissible strategies with its mathematical definition given in the online appendix (Definition Appendix A.1).

Notice that all measures of the income stability are multiplied by an exponential discounting function $e^{-r_0 t}$, where $r_0 > 0$ is referred to as the time-preference rate which is not necessarily the same as the risk-free rate $r_{\varepsilon(t)}$. The choice of r_0 and λ reflects a balance between the current and the future generations. More specifically, a large value of r_0 or a small value of λ essentially means that the sponsor has no interest in delivering secured retirement benefits to the future generations.

3. Optimal cyclical design

Problem (5) can be solved explicitly via dynamic programming, by deriving the associated Hamilton–Jacobi–Bellman (HJB) equation. The details of the derivation can be found in the online appendix and the solution is summarized in the following remark.

Remark 1. *The optimal benefit structure can be written as:*

$$Actual\,Benefit_{\varepsilon(t)}(t) = Target\,Benefit_{\varepsilon(t)}(t) + \beta_{\varepsilon(t)}(t)\left(\frac{Asset(t) - \psi_{\varepsilon(t)}(t) \times Liability(t)}{\mathcal{R}(t)}\right), \tag{6}$$

where $Liability(t)$ is the liability valuation at time t, $\psi_{\varepsilon(t)}(t)$ is a threshold in funding level (i.e., $\frac{Asset(t)}{Liability_{\varepsilon(t)}(t)}$) that triggers the deficit/profit sharing, and $\beta_{\varepsilon(t)}(t)$ is the percentage of surplus/deficit that distributes to each individual. In addition, the surplus/deficit share is equivalent to the penalty weight at the end of projection period, $\beta_{\varepsilon(T)}(T) = \lambda_{\varepsilon(T)}$.

The resulting optimal structure is a linear cyclical risk sharing design where the risk sharing rules are based on the current market regime. The interpretation of this risk sharing structure is straightforward and similar to the one proposed by Cui *et al.* (2011). If the asset level is below a benchmark value, then the deficit will be evenly distributed to all retirees in the form of reducing their benefit payments. The surplus share has a similar interpretation. Here we express the benchmark as a certain funding level ψ of the liability value at time t to align with the market practice.[8]

Although the allocation rule (6) is interpretable, it is both regime- and time-dependent with complicated mathematical expressions. Therefore, it is not as transparent as the ones proposed in the literature (such as Cui *et al.*, 2011; Gollier, 2008; Bégin, 2020, etc.). One common feature with the academic proposals and the existing designs is that all risk sharing parameters are fixed constants instead of being time-varying, i.e., the design is ought to be stable within each market regime. The following section discusses how to construct a regime-stable/transparent TB design.

[8]For example, $\psi = 130\%$ is being used for surplus share for the collective DC plans in the Netherlands.

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3.1. Transparent risk allocation rules

To construct a stable risk sharing structure based on (6), some restrictions must be applied to the parameters. Often in an optimal control problem, these constraints are applied to the economic parameters and are too strict to be realistic. However, in our case, constraints can be applied to the parameters that are subjectively chosen by the sponsor, making a stable design practically feasible.

Remark 2. *A regime-stable design such that*

$$Actual\,Benefit_{\varepsilon(t)}(t) = Target\,Benefit_{\varepsilon(t)}(t) + \beta_{\varepsilon(t)}\left(\frac{Asset(t) - \psi_{\varepsilon(t)} \times Liability(t)}{\mathcal{R}(t)}\right), \tag{7}$$

can be constructed if the pension sponsor is restricted to the choices of π_ϵ, β_ϵ, and X^*_ϵ (see Equations (A.12) and (A.13)). Furthermore, we have $\beta_{\epsilon(t)} = \lambda_{\epsilon(t)}$ and $\psi_{\epsilon(t)} \times \text{Liability}_{\epsilon(t)}(t) = X^*_{\epsilon(t)} \times L(t)$.

The simplicity of this regime-stable TB structure enhances the transparency of the risk sharing mechanism. The plan consists of only two sets of linear risk allocation rules with only regime-dependent parameters. Practically, once the market regime is determined through some publicly available indices such as NFCI, the members should easily understand how their actual benefit is being calculated. It is important to strengthen again that the constraints are applied to subjective chosen parameters π, β, and X^*, and all economic parameters are externally given.

Recall that λ_ε measures the discontinuity risk, and is defined as the penalty weight for the difference between the actual and the target terminal asset levels. Although this definition is qualitatively intuitive, it fails to provide any quantitative justification for specific choices of its values. For example, $\lambda =$ 10% is a common choice in the literature, but the question is what would be the practical meaning by saying that the stability of future liability $(X(t) - X^* \times L(T))^2$ is weighted only 10% compared to the stability of the retirees' income $(b(t) \times L(t) - \bar{b} \times L(t))^2$. We are essentially comparing apples and oranges in one equation.

Remark 2 indicates that λ_ε now represents the proportion of deficit/surplus shared to the retirees. For example, $\lambda = 10\%$ simply represents that 10% of the deficit/surplus will be distributed. The new interpretation also helps restrict the range of λ that should be used for the sensitivity tests. For example, $\lambda > 100\%$ corresponds to distribute more deficit than the pension plan has, which immediately loses its practical meaning. We would emphasize that based on our numerical analysis later, even for a non-transparent plan as Equation (6), the value of $\beta_{\epsilon(t)}(t)$ is still close to the value of $\lambda_{\epsilon(t)}$, and therefore the natural boundary for β should apply to λ as well.

Furthermore, the terminal condition $X^*_{\epsilon(t)}$ which represents the target asset level (in real terms) is now equivalent to the benchmark level that decides the risk allocation. Once the equity share is chosen, it also decides, implicitly, the optimal asset transfer to the future generation. Therefore, to construct a regime-stable risk sharing design, the sponsor is essentially facing the trade-off between the flexible asset allocation and the amount of intergenerational deficit transfer. In fact, the regime-stable structure is the solution of an infinite time problem, where ψ and X^* will not be directly involved in the objective function.

The following remark states that the regime-stable structure (7) is indeed a solution to an infinite-time problem, where ψ and X^* represent the income stability of all future generations that are taken into consideration.

Remark 3. *The regime-stable structure* (7) *is the solution of the following infinite time problem:*

$$\inf_{b\in\Pi} \mathbb{E}\left\{\left[\underbrace{\int_0^T e^{-r_0\times s} \times (B(s) - \bar{B}(s))^2 ds}_{\text{Generations getting benefit before T}}\right] + E\left[e^{-r_0\times T} \times \underbrace{\int_T^\infty e^{-r_0\times(s-T)} \times (B(s) - \bar{B}(s))^2 ds}_{\lambda_{\epsilon(T)}\times(X(T) - X^*_{\epsilon(T)}\times L(T))^2}\right]\right\}. \tag{8}$$

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The regime-stable TB structure implicitly balances the income stability between the current and the future generations. The weight given to the stability of a benefit payment at time t is the time-preference factor $e^{-r_0 \times t}$. A higher discount rate simply represents that the pension design sacrifices the income security of the future generations for the benefits of the nearby generations.

Clearly, the parameters λ_ϵ and X^*_ϵ are not externally given, and implicitly represent the cumulative income uncertainty of all future generations.

We end this section by presenting an additional remark that relates our problem to Cui *et al.* (2011). By constraining ψ_ϵ to be fixed constants that are externally given, the problem becomes determining the optimal equity share and the optimal risk distribution percentage β_ϵ.

Remark 4. *If ψ_1 and ψ_2 are pre-determined, then the regime-stable structure (7) is indeed the solution of the following problem:*

$$\inf_{\boldsymbol{\beta},\boldsymbol{\pi}} \mathbb{E}\left\{\int_t^T e^{-r_0 s}(b(s)L(s)\mathcal{R} - \bar{b}L(s)\mathcal{R})^2 ds + \beta_{\varepsilon(T)} e^{-r_0 T}(X(T) - \psi_{\varepsilon(T)}\text{Liability}_{\varepsilon(T)}(T))^2\right\}.$$

The problem is similar to Cui *et al.* (2011) except that they work on a single regime model and aim to maximize an individual's lifetime utility.

3.2. Cyclicity

Based on the benchmark parameters, here we illustrate the cyclicity of the optimal risk allocation rules. We first examine each of the risk sharing parameters separately, and then discuss their cumulative effects.

3.2.1 Counter-cyclical risk sharing of λ

Figure 1 displays the relationship between the level of risk sharing λ and the equity share π for different time-preference rates r_0 in two market regimes. In this figure, the curves in blue correspond to the regular market case and the curves in red to the market recession. Notice that λ ranges from 0 to 1 in this figure. As discussed in the previous section, λ represents the level (percentage) of risk sharing in the regime-stable structure. In all scenarios, λ lies between 0% and 10%, which means a deficit will be smoothly distributed for more than 10 years (deficit recovery period $\approx \frac{1}{\beta}$).

We first observe that the higher the proportion of risky investment is, the higher risk sharing will be required. This is not surprising as the risk sharing mechanism is meant to offset the risks embedded in the pension assets. It is also clear that the risk sharing level λ_1 in the regular market (blue) is always

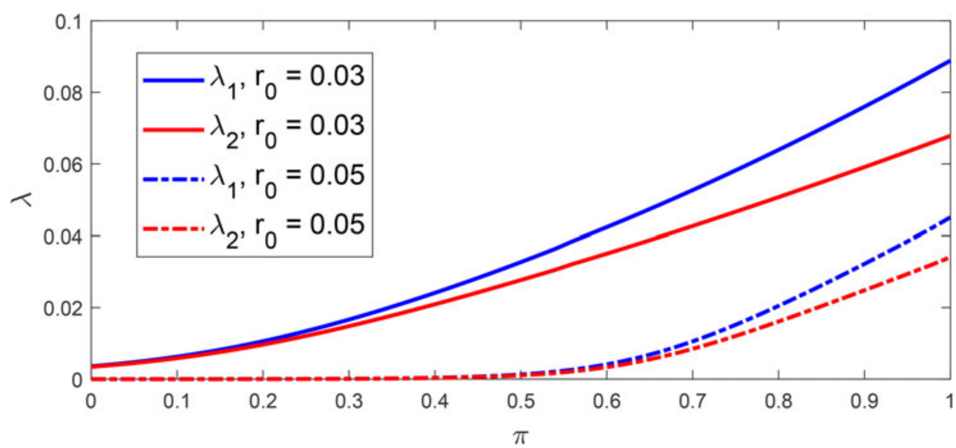

Figure 1. The relationship between the optimal surplus/deficit sharing (λ) and the equity share (π), for different time-preference rates r_0.

higher than λ_2 in the recession (red). For the same amount of deficit, the pension fund actually distributes less during a recession, which indicates a *counter-cyclical* risk sharing policy with respect to the sustainability of the plan. In this case, current generations' income stability outweighs the sustainability of the fund. During the recession, to lessen the income volatility, risk sharing must be lowered to safeguard the retirees' benefits.

Another highlight drawn from Figure 1 is that the risk sharing level λ has a negative relationship with the time-preference rate of r_0. This is because the lower the time-preference rate is, the higher weight is given to the stability of terminal asset level and thus the asset is required to be closer to the target asset level, which can be achieved through a higher level of risk sharing.

Under the circumstances that the time-preference rate r_0 is large (which favors the nearby generations) and the investment strategy is conservative (i.e., small π), the TB plan degenerates to the traditional DB plan (where $\lambda = 0$) in which risk-sharing simply becomes risk-transferring to the future generations.

3.2.2 Pro-cyclical risk sharing threshold (ψ_ϵ)

Before we proceed with the cyclicity of the risk allocation implied by the parameter ψ, we stress that the liability valuation in our paper is not regime-dependent. Despite the fact that a cyclical policy is often adopted for the traditional DB plans, the regulation always recommends liability valuation assumptions that are robust with respect to the changes in the market regimes (see, e.g., Yermo and Severinson, 2010).[9] In this spirit, we assume that the liability valuation is independent of the underlying regime. Recall that X_ε^* is $\psi_\varepsilon \times$ Liability (in real terms) as shown in Remark 2, and thus we may compare ψ_ε through the relationship $X_2^*/X_1^* = \psi_2/\psi_1$. We further assume that $\psi_1 = 1$, i.e., TB shares the surplus/deficit when the pension asset deviates from the liability, to better compare risk sharing thresholds in different market regimes.

Figure 2 presents the relationship between the risk sharing threshold ψ_2 during the market recession and the equity share when $\psi_1 = 1$ under different values of time-preference rates r_0. As can be seen in this figure, ψ_2 is larger than ψ_1 by a factor between 1.02 and 1.14. The higher values of ψ_2 mean that during the recession, the benefits will be reduced to a greater degree than the regular time. Hence, contradicting with λ_ε, the values of ψ_ε indicate a *pro-cyclical* policy which means that during the recession, more surplus will be held by the fund and more deficit will be distributed.

Notice that the discount rate r_d only indirectly affects ψ through the liability values. However, the ratio $\frac{\psi_2}{\psi_1}$ and the values of β_1 and β_2 will be irrelevant to the choice of r_d.

3.2.3 Overall cyclicity

The analysis above has shown that the optimal risk sharing parameters λ_ε and ψ_ε have counter- and pro-cyclical features, respectively. Therefore, the overall cyclicity of the deficit sharing remains unclear.

Heuristically, the pro-cyclical regulation in a DB plan requires an increase in the pension contribution during the economic downturn and allows the contribution holidays if both the market and the pension fund perform well, and the counter-cyclical policy refers to the opposite. For a TB plan, we refer the pro-cyclical deficit sharing in the sense that more deficit will be distributed during a market crash. In other words, for the same level of deficit, Actual Benefit$_1(t) >$ Actual Benefit$_2(t)$. Counter-cyclical, on the other hand, corresponds to Actual Benefit$_1(t) <$ Actual Benefit$_2(t)$. Since λ is counter-cyclical for the deficit sharing but ψ is pro-cyclical, the TB design will exhibit both pro- and counter-cyclicity features. Due to the linearity of the optimal risk allocation rules, pro- and counter-cyclical policy will be applied depending on the funding ratio. Mathematically, the pro-

[9]Even though the discount rate is related to the interest rate, it is often defined as the average over those of several years in the past; therefore, the valuation assumption will not have a significant impact for the adjacent years even though they may belong to different regimes.

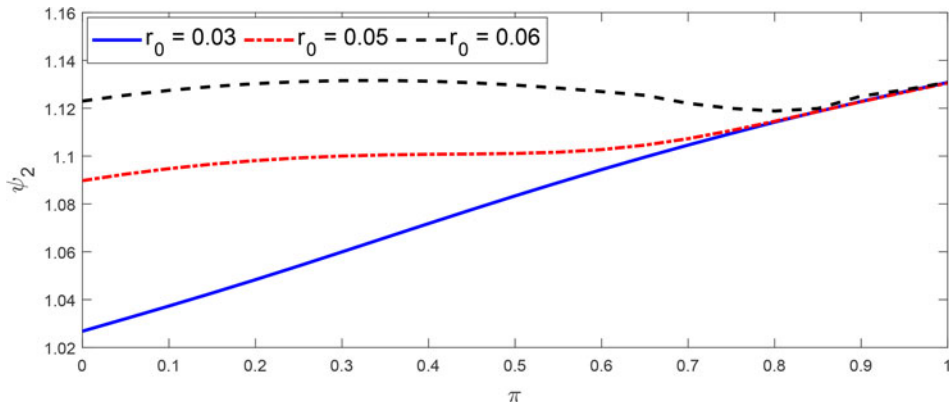

Figure 2. The risk sharing threshold ψ_2 during recession ($\psi_1 = 1$), for different equity share π and different time-preference rates r_0.

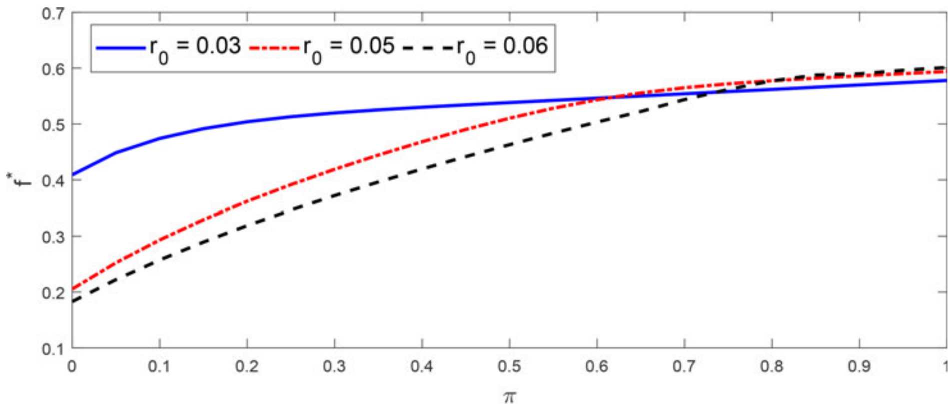

Figure 3. The benchmark funding level f^* for different equity share π, for different time-preference rates r_0.

cyclical policy will be applied if the funding ratio falls into the set:

$$\left\{\frac{Asset}{Liability}|\lambda_1\left(\frac{Asset}{Liability}-\psi_1\right)>\lambda_2\left(\frac{Asset}{Liability}-\psi_2\right)\right\}.$$

It is easy to see that we may find a benchmark funding level f^* such that pro-cyclical deficit sharing will apply to any level higher than f^*.

Figure 3 illustrates the value of f^*. Clearly, counter-cyclicity only applies to the extremely under-funded pension plans, which surprisingly aligns with the market practice for the DB plans. However, we should emphasize that the motivation for the counter-cyclical TB design is to relieve the risk sharing obligations borne by the retirees. On the contrary, counter-cyclicity in the DB regulation is to relieve the stress of the DB sponsors who are facing extreme financial difficulties.

One thing we would like to highlight from Figures 2 and 3 is the convergence of different curves when the equity weight increases. As the percentage of equity investment is larger than 80%, the choice of the time-preference rate has an insignificant impact on the risk sharing structure.

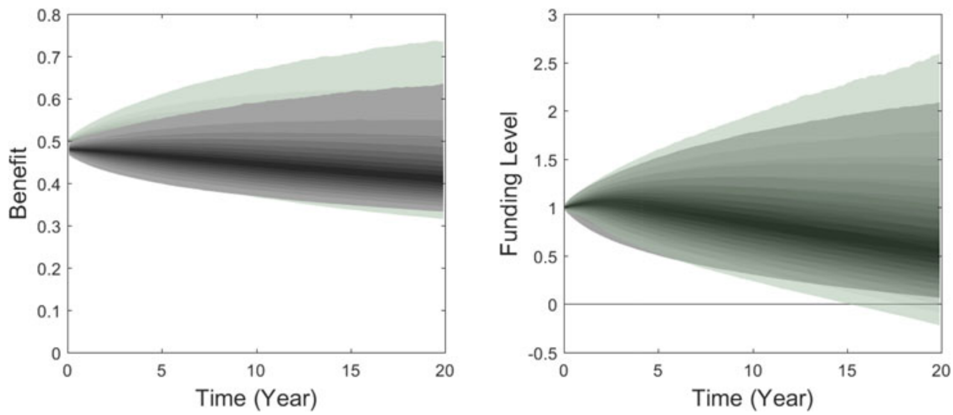

Figure 4. The 95% fan-chart for the income security and the sustainability of the pension plans. The left panel displays the income security of a TB plan during the regular market (light/green) and the recession (dark/grey). The right panel displays the funding levels of both DB (light/green) and TB (dark/grey) plans.

Contradicting to the standard 60/40 portfolio for a DB plan, the sponsor may choose more aggressive investment strategies for a TB plan if the time-preference rate is hard to decide.

3.3. Comparison to DB plan

Until now we have not yet compared a TB plan with a DB plan. In our model, since the traditional DB plan is only a special case of a TB plan when $\beta_\varepsilon(t) = 0$, it is undoubted that a DB plan cannot over-perform a TB plan when both income security and fund sustainability are considered jointly. Since no risk sharing exists in the DB plan, it would be clear that DB is preferable to TB for short-term income security at the expense of sacrificing long-term sustainability. The question is on the magnitude of the trade-off.

The left panel of Figure 4 displays the 95% fan-chart of actual benefit payments during the regular market (light) and the recession (dark), and the right panel displays the 95% fan-chart of funding level for both TB (grey) and DB (green) plans. We use the base case parameters with $r_0 = 0.05$. We also use an aggressive strategy of 80/20 for the regular market, and 60/40 for the recession. The initial pension asset is assumed to be fully funded in the regular market regime (i.e., $X_0 = X_1^*$).

Here we emphasize that if the default risk is considered, then the income variation does exist in a DB plan. Once a DB plan is terminated, the promised retirement income would no longer be deliverable, and a large income reduction is expected. TB plans essentially break down those large reductions into a stream of smaller shares. The default risk (Asset < 0) for a DB plan which initially is fully funded is 14% over a 20-year horizon which is significantly large, and this figure jumps to 47% if it is 70% funded initially. On the other hand, these numbers are only 0.57% and 0.43% for a TB plan. Despite the fact that the TB looks volatile for the retirees at the first sight, risk sharing effectively avoids the situation where the current generation will be left empty-handed when they retire. Cui *et al.* (2011) illustrates that TB plan may be the solution in the sense that the new cohort joining an underfunded risk sharing scheme may not be significantly worse-off in the welfare terms; here we reinstate this argument, but in terms of the income security.

Notice also that although we did not constrain on $b_t > 0$, the possibility of getting a negative benefit is negligible in our benchmark study. We also want to emphasize that when the 'optimal' benefit is negative, it simply reflects the fact that the asset becomes way below the liability value. In such a scenario, any pension plan will become insolvent, and the risk sharing structure is no exception. However, as shown in the right panel, the sustainability of the pension fund has been greatly improved when risk sharing is involved.

3.4. Time- and regime-independent structure

The transparent plan consists of two sets of linear risk sharing rules, one for each market regime. Therefore, it is natural to consider a universal risk sharing rule that is regime-independent. Technically, it only requires more restrictions on the subjectively chosen parameters as shown in the following remark.

Remark 5. *Given the penalty λ, if the investment strategies under different regimes satisfy the following system of quadratic functions:*

$$\pi_1^2\sigma_1^2 + \pi_1 \times 2(\mu_1 - r_1) + 2(r_1 - r_0) - \lambda = 0,$$

$$\pi_2^2\sigma_2^2 + \pi_2 \times 2(\mu_2 - r_2) + 2(r_2 - r_0) - \lambda = 0,$$

then the risk sharing parameter $\lambda_\varepsilon(t) = \lambda$ will be both regime- and time-independent. In addition, to obtain a time-invariant ψ, the terminal target asset level X^* must satisfy:

$$X^* = \frac{c_1\mathcal{A} - \mathcal{D} - \bar{b}\mathcal{R}}{r_1 + \pi_1(\mu_1 - r_1) - 2r_0 + \alpha_1 + \pi_1\sigma_1\eta_1\rho_1 - \lambda}.$$

Notice immediately that $\pi_1 \neq \pi_2$, and π_ε is a function of $\lambda + 2r_0$.

Given different values of $\lambda + 2r_0$, we observe the corresponding investment strategies in Figure 5. For example, by setting $\lambda = 0.05$ and $r_0 = 0.05$, the asset allocation used in the regular market is close to the rule of thumb 60/40 strategy (indicated by the blue circle in the figure). However, we need to short the equity during the recession (see the corresponding red line).

Indeed, there is no set of optimal strategies that hold positive equity in both market regimes. In most countries, pension regulators set strict rules on the asset mix where the equity weight is often required to be positive. In addition, in response to the financial turmoil, many regulators impose short-selling bans that further limit the sponsors' ability in short-selling. Apparently, a stable TB design that is independent of both time and regime leads to practically infeasible investment strategies.

4. Robustness check and model discussion

We now perform the robustness check to verify if our results remain valid under different parameters. Since the effect of the time-preference rate r_0 has already been demonstrated in Section 2 (which is stable when an aggressive investment strategy has been adopted), here we fixed $r_0 = 0.05$ and examine other parameters such as the risk-free rate, the salary growth, and the contribution policy.

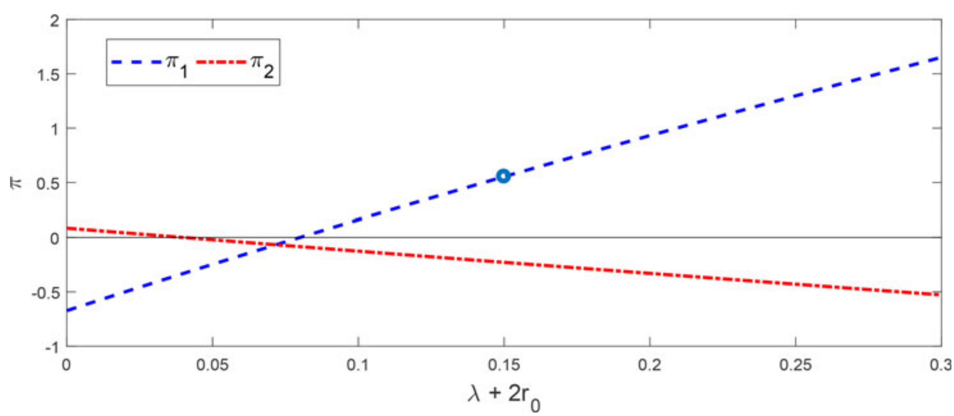

Figure 5. The equity exposure π given different time-preferences (r_0) and surplus deficit sharing (λ).

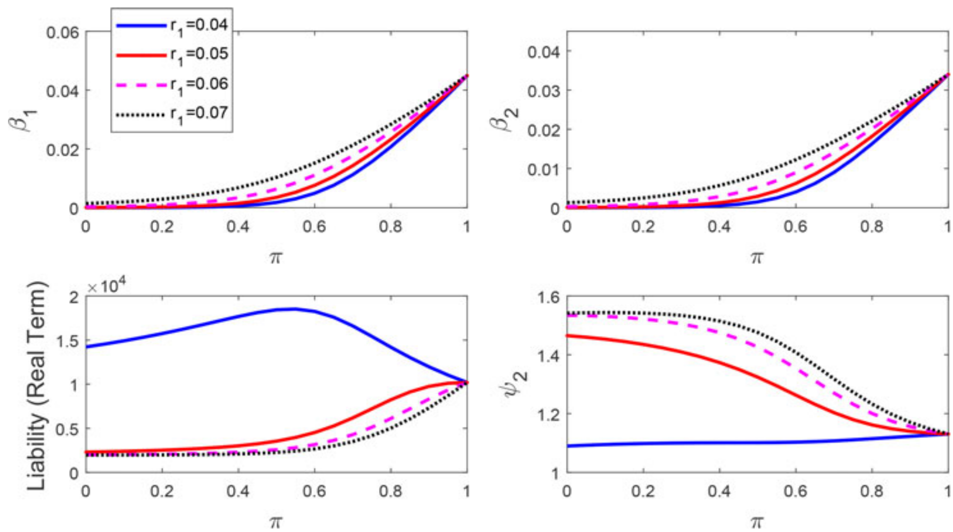

Figure 6. The impact of the risk-free rate during the regular market r_1, given the time-preference rate $r_0 = 0.05$ and the risk-free rate during the recession $r_2 = 2\%$. The figure displays the risk sharing parameters during the regular market β_1 (left top), during the recession β_2 (right top), the liability value (left bottom), and the risk sharing threshold ψ_2 (right bottom). Each variable is plotted against different equity exposure π.

We further investigate our model assumptions to see whether a cyclical risk allocation rule can be adopted without a regime-switching model. In the end, we also discuss whether the equity share should be treated as a control variable to align with the literature.

4.1. Risk-free rate

Most economic parameters are direct calibrated from the historical data; however, the choice of the risk-free rate r_ε is rather arbitrary. Here we perform its sensitivity test to examine its impacts on the values of β and ψ.

In Figure 6, we present the impact of the risk-free rate r_ε on the risk sharing parameters β_ε and ψ_ε as well as the liability (or the risk sharing benchmark) X^*_ε. More specifically, we fix the risk-free rate during the market recession ($r_2 = 2\%$) and explore the impact of the risk-free rate in the regular market r_1. In this figure, the blue curve in each panel corresponds to the base case scenario.

Overall, a similar pattern on the risk sharing parameter β has been observed for different choices of r_ε. For the effect on the risk sharing benchmark X^*_ε, we have noticed that the relationship between r_1 and r_0 is a crucial factor. The benchmark level X^*_ε is rather stable when $r_1 \geq r_0$, but becomes sensitive when $r_1 < r_0$. Nonetheless, when the portfolio leans toward fully investing in the equity (i.e., when the risk-free asset will not be involved), the risk-free rate no longer has any impact on the TB design.

While Figure 6 is based on an expansionary fiscal policies where $r_1 > r_2$, we run additional tests for the contractionary policies during the recession where $r_1 < r_2$ in Figure 7. Conclusions obtained from the previous observations remain valid.

4.2. Salary assumption

In the base case scenario, we omit the effect of the promotional salary growth γ_1 and the partial inflation indexation γ_2. In fact, they impact the optimal TB design only through the terms $\mathcal{A}$, $\mathcal{R}$, and $\mathcal{D}$. Immediately from Equation (A.12), we notice that β_ε is independent of the choice of γ_ε.

Figure 8 displays the impact of salary growth γ_1 on the overall risk sharing weight as well as the terminal target asset X^*_1.

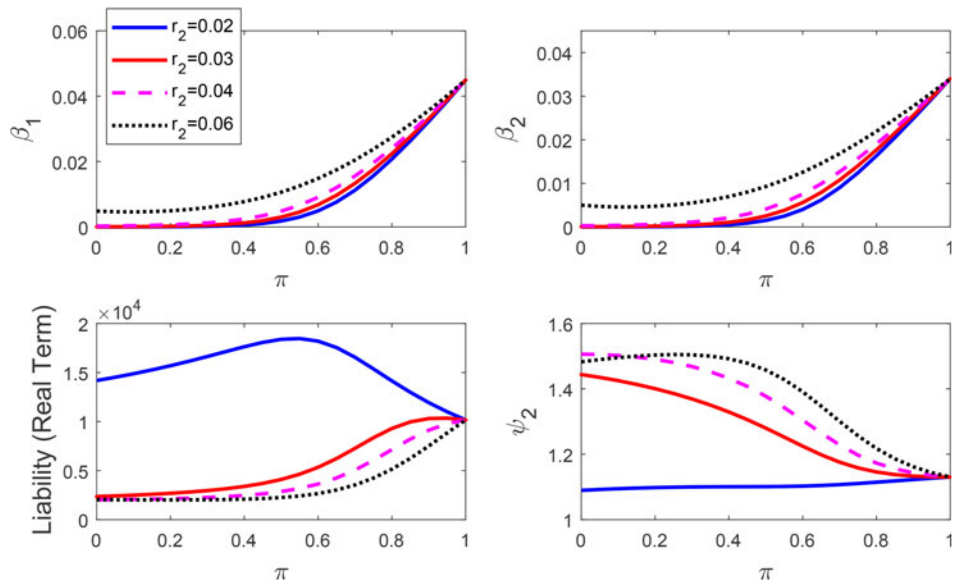

Figure 7. The impact of the risk-free rate in recession r_2, given the time-preference rate $r_0 = 0.05$ and the risk-free rate during the regular market $r_1 = 0.04$. The figure displays the risk sharing parameters during the regular market β_1 (left top), during the recession β_2 (right top), the liability value (left bottom), and the risk sharing threshold ψ_2 (right bottom). Each variable is plotted against different equity exposure π.

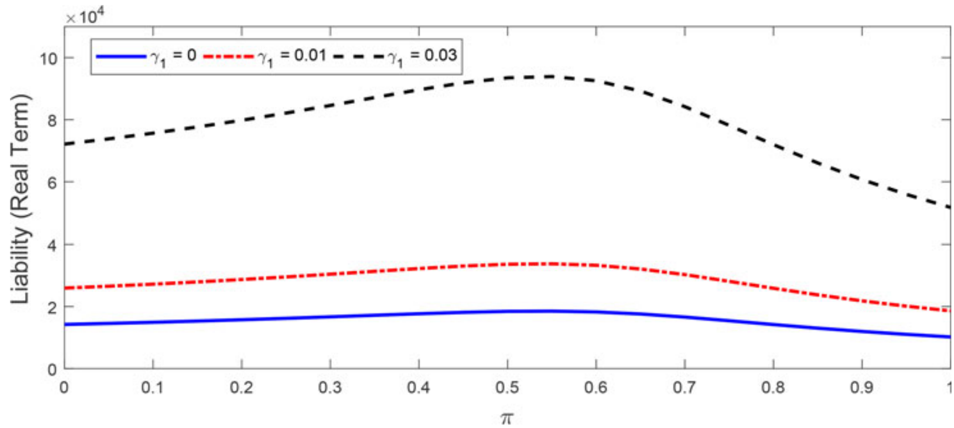

Figure 8. The impact of the salary growth γ on liability value (real term), given different equity weights (x-axis), and time-preference rate $r_0 = 0.05$.

Here we stress that although it is expected that the liability value can be significantly different under different salary growth assumptions, the ratio between X_2^*/X_1^* (which is ψ_2 if $\psi_1 = 1$) can be shown to have the following expression based on Equation (A.13):

$$\frac{X_2^*}{X_1^*} = \frac{\zeta_1 \times [c_2\mathcal{A}(t) - \mathcal{D}(t) - \bar{b}\mathcal{R}(t)] - \dfrac{q_{21}\lambda_1}{\lambda_2} \times [c_1\mathcal{A}(t) - \mathcal{D}(t) - \bar{b}\mathcal{R}(t)]}{\zeta_2 \times [c_1\mathcal{A}(t) - \mathcal{D}(t) - \bar{b}\mathcal{R}(t)] - \dfrac{q_{12}\lambda_2}{\lambda_1} \times [c_2\mathcal{A}(t) - \mathcal{D}(t) - \bar{b}\mathcal{R}(t)]}.$$

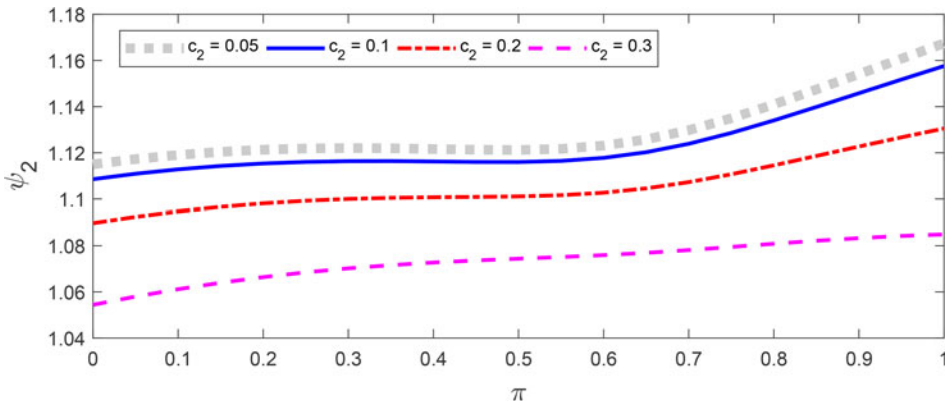

Figure 9. The effect of contribution during the recession (c_2) on the risk sharing threshold during the recession ψ_2, given the contribution during the regular regime (c_1) remains the same, the time-preference rate $r_0 = 0.05$ and the risk sharing threshold during the regular market $\psi_1 = 1$.

It is clear that when $c_1 = c_2$ (i.e., contribution rate is independent of the market regimes), γ_1 and γ_2 have no impact on ψ_2. Therefore, the risk sharing mechanism is stable for different salary growth assumptions.

Similar to the effect of γ_ε, parameters such as the contribution level c_ε and the target replacement rate $\bar{b}$ affect the optimal plan design only through $\mathcal{A}$, $\mathcal{D}$, and $\mathcal{R}$, and therefore are irrelevant to the optimal values of ψ_2 and β_ε.

4.3. Effect of contribution policy

DB regulations in many OECD countries allow contribution relief during the recession to avoid putting further pressures on a company's profitability. Therefore, we include the scenarios when $c_1 \Leftrightarrow c_2$ in Figure 9 to illustrate the effect of different contribution patterns.

When a counter-cyclical contribution policy is adopted (i.e., decreasing contribution during a recession), it increases the pro-cyclicality in the deficit sharing. Although we have chosen a large range of c_2 from 5% to 30%, the risk sharing threshold is relatively stable in the sense that ψ_2 is always higher than ψ_1 and lies between 1.05 and 1.18.

4.4. Single-regime model

To demonstrate the necessity of introducing a regime-switching model, we compare our result with that of a single regime model (similar to Wang *et al.*, 2018). With the same objective function but all parameters being regime-independent, it is not difficult to establish the following remark:

Remark 6. *The optimal benefit structure is*

$$\textit{Actual Benefit}(t) \quad = \bar{b} \times L(t) + \beta(t) \times \left(\frac{X(t) - \psi(t) \times L(t)}{\mathcal{R}(t)} \right), \tag{9}$$

where $\beta(t)$ and $\psi(t)$ can be derived from Equation (A.6). Similarly, a time-invariant strategy can be obtained by constraining the value of π and λ:

$$\pi = \frac{\sqrt{(\mu - r)^2 - 2(r - r_0)\sigma^2 + \lambda\sigma^2} - (\mu - r)}{\sigma^2}, \tag{10}$$

and we have $\beta(t) = \lambda$.

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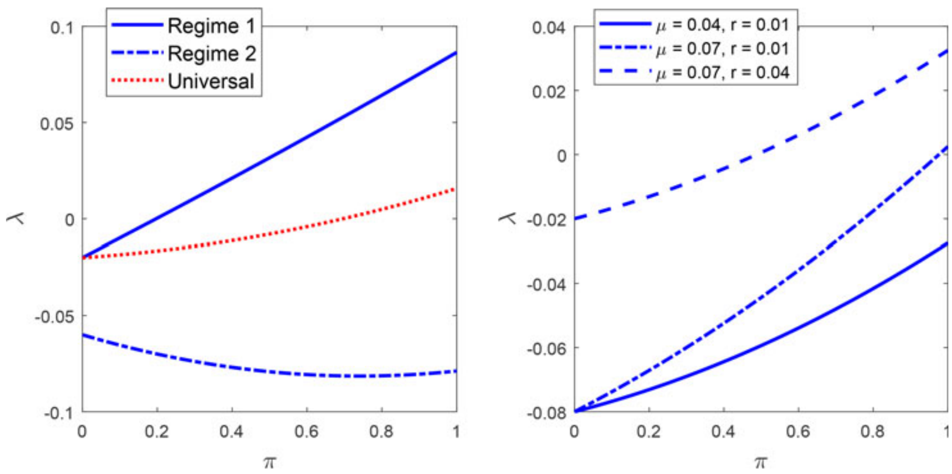

Figure 10. Time-invariant strategy in a single regime model. The left panel displays the optimal risk sharing λ using the benchmark parameters, and the right panel displays the universal risk sharing under different parameter sets. Both panels are plotted against different equity weights π.

The left panel in Figure 10 demonstrates the cyclical design using a single-regime model. Specifically, the TB design is derived from Equation (9) using the parameters from the base case (where the market parameters differ in different regimes). We observe a negative risk sharing parameter (β_2) during the recession regardless of the asset allocation. This design completely loses its practical meaning since it increases the retirees' benefits for an under-funded plan and reduces the benefits for an over-funded plan. The right panel in Figure 10 further illustrates that this counter-intuitive behavior remains an issue for a wider range of reasonable market parameters.

4.5. Equity share as control variable

In this paper, we assume that the pension fund trustees adopt constant weighting strategies for investment in each regime. However, for most stochastic control problems in pension literature, e.g., Wang *et al.* (2018), equity weight is often treated as a control variable. Therefore, it would be also interesting to consider the dynamic investment strategies. In this subsection, we take both the benefit payment and the portfolio weight as the control variables, and minimize the same objective function. Mathematically, we have

$$\inf_{b,\pi} \mathbb{E}_{t,x,l,\varepsilon}\left\{\int_t^T e^{-r_0 s}(b(s)L(s)\mathcal{R}(s) - \bar{b}L(s)\mathcal{R}(s))^2 ds + \lambda_\varepsilon e^{-r_0 T}(X(T) - X_\varepsilon^* \times L(T))^2\right\}. \tag{11}$$

The problem can be solved similarly to Problem (5) via dynamic programming. The resulting optimal risk allocation rule preserves linearity as Equation (6) with an explicit form given in the online appendix.

With the dynamic asset allocation, the actual benefit can be perfectly maintained at the target level ($\bar{b} \times L(t)$), which seems that a TB design is not necessary. However, the stable benefit payments are guaranteed with a cost in the sense that a highly volatile portfolio will be observed. Figure 11 displays the 95% fan-chart for the optimal equity share.

The optimal strategy requires the weight in equity to fluctuate between −100,000% and 100,000%. Thus, the optimal equity share obtained from Problem (11) is highly impractical. This also suggests that a risk sharing design is necessary for the market risk mitigation as long as the trustee does not have absolute control over its portfolio mix.

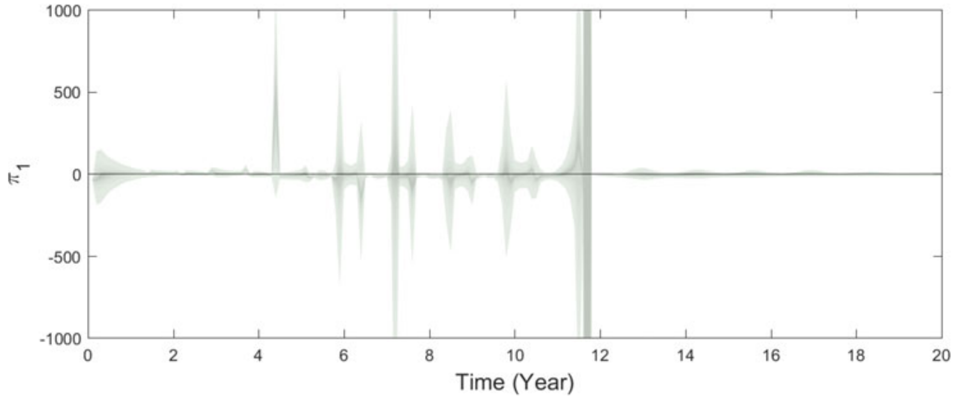

Figure 11. The optimal equity share π during the regular market, when including the equity share as a control variable.

5. Conclusion

This paper constructs an optimal TB pension plan that balances between retirees' income security and pension sustainability. We obtain an explicit solution using a regime-switching model, and display the trade-off faced by a pension sponsor between the transparency of the risk sharing mechanism and the flexibility of the asset allocation. Through our analysis, we provide the following insights to both practitioners and academia. First, the optimal TB design should be pro-cyclical for the general circumstance. However, it should set up additional counter-cyclical policies for extremely underfunded plans to reduce the retirees' already significant income losses. Second, we provide a natural interpretation of the pension's discontinuity penalty term, which is equivalent to the risk sharing level for a regime-stable risk sharing design. Last, contradicting to the practice of the DB plan where a 60/40 investment rule is often applied, we would recommend that the equity share for a TB plan should be more aggressive.

The analysis of this paper rests on a simple market model and a stylized pension plan. These may be replaced by more complicated assumptions. For example, to overcome the limitations of a frictionless market model, we may incorporate investment constraints into our optimization problem. To resolve the issue of unobservable switches between regimes, a hidden Markov model might be more appropriate. The sustainability advantage of the TB designs might be better demonstrated by including negative jumps in the equity price process or stochastic interest rates.

In addition, important topics that we cannot address in the scope of this paper include the market-consistent liability valuation of the TB plan, the relationship between intergenerational risk-sharing, demographic developments, intra-generational fairness, etc.

Supplementary material. The supplementary material for this article can be found at https://doi.org/10.1017/S1474747222000099

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