Probability in the Engineering and Informational Sciences (2024), **38:**1 65–81
doi:10.1017/S026996482200050X

CAMBRIDGE
UNIVERSITY PRESS

RESEARCH ARTICLE

Valuation of vulnerable European options with market liquidity risk

Yihao Pan, Dan Tang and Xingchun Wang

School of International Trade and Economics, University of International Business and Economics, Beijing 100029, China.
E-mail: dantang@uibe.edu.cn.

Keywords: Counterparty default risk, Liquidity discount factor, Market liquidity risk, Vulnerable options

JEL Classification: G13

Abstract

In this paper, we investigate the pricing of vulnerable European options in a market where the underlying stocks are not perfectly liquid. A liquidity discount factor is used to model the effect of liquidity risk in the market, and the default risk of the option issuer is incorporated into the model using a reduced-form model, where the default intensity process is correlated with the liquidity risk. We obtain a semiclosed-form pricing formula of vulnerable options through the inverse Fourier transform. Finally, we illustrate the effects of default risk and liquidity risk on option prices numerically.

1. Introduction

Option pricing has been a focus of mathematical research in finance since the publication of the Black–Scholes formula [4]. It is often considered a kind of financial derivative with the underlying stock transacting in a perfectly liquid market. However, it is widely acknowledged that not all securities are perfectly liquid. Many prior studies have provided evidence that investors ask for illiquidity premiums due to liquidity risk (see, e.g., [1,2,26]). Brunetti and Caldarera [6] built a theoretical model that studied the effects of aggregate liquidity/illiquidity on asset return volatility and correlations. Cetin *et al.* [7] considered option pricing in an extended Black–Scholes economy in which the underlying asset was not perfectly liquid. In Feng *et al.* [10], the specification of Brunetti and Caldarera [6] was extended to develop a new option pricing model with stochastic market liquidity. Leippold and Schärer [19] extended the discrete-time constant liquidity model of Madan [22], and their model successfully replicated the term and skew structures of bid-ask spreads observed in option markets. Referring to Feng *et al.* [10], Pasricha *et al.* [25] considered all the possible correlations among the process of stock price, the mean-reversion process of liquidity risk and the process of the liquidity discount factor.

In recent decades, concerns about financial derivatives subject to default risk in the over-the-counter (OTC) markets have grown rapidly since the mid-2007, when the financial crisis erupted. Because there is no trading mechanism to guarantee the promised payment, the option holders are vulnerable to default risk, and these options with default risk are called vulnerable options. Johnson and Stulz [15] was the first to study vulnerable options by incorporating default risk into the option pricing model. In Hull and White [13], both the probability of default and the size of the proportional recovery from the default were set to be random and the authors showed that model parameters could be informed by data on bonds issued by the counterparty. Jarrow and Turnbull [14] considered two types of credit risks induced from the underlying assets and the writers of these derivatives. Klein [16] then moved forward to price vulnerable options with correlated default risk in the classical Black–Scholes model and extended the focus on stochastic default barriers in Klein and Inglis [17]. Additionally, many other studies have modified the classical Black–Scholes model and worked under stochastic volatility models [18,30,32]. The pricing of vulnerable options was also investigated in the jump-diffusion model and/or stochastic interest rate environment [20,21,24,28]. Wang *et al.* [31] presented a model with risky collateral under the assumption that holders of vulnerable options could recover a proportion of the option value using the collateral account when default occurred. Additionally, because most of the assets of financial institutions are financial assets, there exists a certain relationship between liquidity risk and default risk (see, e.g., [29]). He and Xiong [11] found that deterioration of debt market liquidity leads to not only an increase in the liquidity premium of corporate bonds but also credit risk. Brogaard *et al.* [5] focused on U.S. firms and found a negative effect of stock liquidity on default risk, suggesting that the informational efficiency of stock prices and corporate governance should be improved to enhance market liquidity and, thus, to control the level of default risk. Nadarajah *et al.* [23] further confirmed this negative effect between market liquidity and default risk on a larger scale by using a sample of 46,949 firm-year observations for 4,043 nonfinancial firms across 46 countries during the 2004–2018 period.

Motivated by the empirical results mentioned above, in this paper, we mainly focus on the pricing of vulnerable European options with market liquidity risk. To model the dynamics of the imperfectly liquid underlying asset, we extend the specification in Brunetti and Caldarera [6] and Feng *et al.* [10]. More specifically, in our model, the stock price, the market liquidity and the liquidity discount factor are all correlated with each other and this assumption is closer to reality. Furthermore, we take the dynamic relationship between liquidity and credit risk into consideration. Default risk is described by a reduced-form model where the default intensity process is affected by the market liquidity measure. Then by utilizing the characteristic function approach and the Feynman–Kac theorem, we obtain a semi-closed form for the prices of vulnerable European options with market liquidity risk. Finally, numerical examples are presented to illustrate the effects of both liquidity risk and default risk on option prices.

The remainder of this paper is organized as follows. In Section 2, the theoretical framework is introduced. In Section 3, after providing the construction of a suitable martingale measure, we price vanilla European options in the first subsection, and value vulnerable European options in the second subsection. Section 4 is devoted to numerical analysis. Finally, we draw the conclusion of the paper in Section 5. The detailed proofs are presented in the Appendix.

2. Model settings

Consider a model with a finite time horizon $T > 0$, and the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ models the uncertainty in the economy, where P is the physical probability measure. Suppose that there are two types of assets in the market: stocks and money market accounts. Following Brunetti and Caldarera [6], Feng *et al.* [10] and Wang [29], we assume that the stocks are not perfectly liquid, and account for liquidity risk using the liquidity discount factor. The imperfectly liquid stock price is the price that makes the demand for the stock clear its supply, and the liquidity discount factor is introduced in the demand function. Specifically, the demand for the stock depends on three factors: the stock-specific information I_t, the liquidity discount factor γ_t and the stock price S_t. The demand function, $D(S_t, \gamma_t, I_t)$, is given by

$$D(S_t, \gamma_t, I_t) = g\left(\frac{I_t^{\nu}}{\gamma_t S_t}\right), \tag{2.1}$$

where $g(\cdot)$ is a smooth, strictly increasing function, and ν is a positive constant.

Brunetti and Caldarera [6] proposed the following form of the liquidity discount factor γ_t,

$$\gamma_t = \exp\left(-\beta\left(\int_0^t L_s\, \mathrm{d}s + \int_0^t L_s\, \mathrm{d}W_s^{\gamma}\right)\right), \quad t \geq 0, \tag{2.2}$$

where $\{W_t^{\gamma}\}_{t\geq 0}$ is a standard Brownian motion under P, L_t is a market liquidity measure and β is a nonnegative constant representing the sensitivity of the stock to market illiquidity.

https://doi.org/10.1017/S026996482200050X Published online by Cambridge University Press

Suppose the supply for the stock is fixed and equals $\bar{S}$, and then the market clearing condition yields the expression of the imperfectly liquid stock price S_t as follows,

$$S_t = \frac{1}{\gamma_t}\left(\frac{I_t^{\gamma}}{g^{-1}(\bar{S})}\right), \quad t \geq 0. \tag{2.3}$$

Specially, when the liquidity discount factor $\gamma_t \equiv 1$, the stock price (2.3) degenerates to

$$S_t^L = \frac{I_t^{\gamma}}{g^{-1}(\bar{S})}, \quad t \geq 0. \tag{2.4}$$

Obviously, the price of the underlying stock affected by market liquidity can be formulated by

$$S_t = \frac{1}{\gamma_t} S_t^L, \quad t \geq 0. \tag{2.5}$$

As we know, liquidity risk is a financial risk that for a certain period of time a given financial asset, security or commodity cannot be traded quickly enough in the market without impacting the market price. Liquidity risk usually arises from situations in which a party interested in trading an asset cannot do it because nobody in the market wants to trade for that asset. Hence, risk-averse investors naturally require higher expected return as compensation for liquidity risk. In order to understand the effect of liquidity risk more clearly, we now consider a special case of (2.5) when L_t is deterministic and the liquidity discount factor is independent of S_t^L. In this special case, from (2.2) and (2.5), we have the following conditional expectation,

$$\begin{aligned} E^P[S_t \mid S_t^L] &= E^P\left[\exp\left(\beta\left(\int_0^t L_s \,\mathrm{d}s + \int_0^t L_s \,\mathrm{d}W_s^{\gamma}\right)\right) S_t^L \mid S_t^L\right] \\ &= S_t^L \exp\left(\int_0^t \left(\beta L_s + \frac{1}{2}\beta^2 L_s^2\right) \mathrm{d}s\right). \end{aligned} \tag{2.6}$$

When βL_t is small, the integral in (2.6) will have the same sign as its first term, and hence, $\exp(\int_0^t (\beta L_s + \frac{1}{2}\beta^2 L_s^2)\,\mathrm{d}s)$ can be interpreted as a convenience yield caused by the illiquidity. Therefore, the value of L_t can be interpreted as the level of market liquidity at time t, and $L_t = 0$ means that the market liquidity is at the perfect level. Additionally, $L_t > 0$ corresponds to shortages, while $L_t < 0$ corresponds to gluts (see, e.g., [6]).

In what follows, we focus on the dynamics of the market liquidity measure L_t and the liquidity discount factor γ_t. Using S&P 500 index data, Feng *et al.* [10] found that market liquidity tends to fluctuate around the mean, which means that L_t has the mean-reverting property, so we model it as

$$\mathrm{d}L_t = \kappa_L(\theta_L - L_t)\,\mathrm{d}t + \sigma_L \,\mathrm{d}W_t^L, \quad t \geq 0, \tag{2.7}$$

where $\{W_t^L\}_{t\geq 0}$ is also a standard Brownian motion under P; κ_L is the mean-reversion speed of market liquidity; θ_L is the mean level and σ_L is the volatility. According to Itô's lemma, the liquidity discount factor γ_t in (2.2) can be written in the following form:

$$\frac{\mathrm{d}\gamma_t}{\gamma_t} = \left(-\beta L_t + \frac{1}{2}\beta^2 L_t^2\right)\mathrm{d}t - \beta L_t \,\mathrm{d}W_t^{\gamma}, \quad t \geq 0, \ \gamma_0 = 1. \tag{2.8}$$

Next, we turn to the dynamics of S_t^L, and then using (2.5), we can obtain the time-t price S_t of the imperfectly liquid stock. Note that under the assumptions of the fixed supply for the stock and the form of the specific demand function, Brunetti and Caldarera [6] proved that S_t^L is a geometric Brownian motion which is consistent with the classical Black–Scholes model. Here, we also adopt the classical

https://doi.org/10.1017/S026996482200050X Published online by Cambridge University Press

B-S model,

$$\frac{\mathrm{d}S_t^L}{S_t^L} = \mu_S\,\mathrm{d}t + \sigma_S\,\mathrm{d}W_t^S, \quad t \geq 0, \tag{2.9}$$

where μ_S, σ_S are positive constants and $\{W_t^S\}_{t\geq 0}$ is a standard Brownian motion under P.

In this paper, we work in a more general framework by assuming that W_t^S, W_t^γ and W_t^L are correlated with each other. Moreover, their correlation structure is listed below:

$$\begin{aligned} \langle \mathrm{d}W_t^S, \mathrm{d}W_t^\gamma \rangle &= \rho_1\,\mathrm{d}t, \\ \langle \mathrm{d}W_t^S, \mathrm{d}W_t^L \rangle &= \rho_2\,\mathrm{d}t, \\ \langle \mathrm{d}W_t^L, \mathrm{d}W_t^\gamma \rangle &= \rho_3\,\mathrm{d}t. \end{aligned} \tag{2.10}$$

Now from (2.5), (2.8) and (2.9), Itô's lemma implies that the time-t price S_t of the imperfectly liquid stock is given by the following process:

$$\frac{\mathrm{d}S_t}{S_t} = \left(\mu_S + \beta L_t + \frac{1}{2}\beta^2 L_t^2 + \rho_1\sigma_S\beta L_t\right)\mathrm{d}t + \sigma_S\,\mathrm{d}W_t^S + \beta L_t\,\mathrm{d}W_t^\gamma. \tag{2.11}$$

From the above equation, it is easy to see that market liquidity measure L_t affects the return and the volatility of the stock simultaneously under the physical probability measure P.

3. Options pricing

This section presents the procedures to derive the prices of vanilla European options and vulnerable European options when the underlying assets are not perfectly liquid.

3.1. Pricing of vanilla European options with liquidity risk

To price options, we need to determine an equivalent martingale measure. Bingham and Kiesel [3] illustrated that all possible martingale measures could be characterized by their Girsanov identities. Here, we select a suitable equivalent martingale measure using the following Radon–Nikodym derivative:

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}P}\right|_{\mathcal{F}_t} = \exp\left\{-\int_0^t \lambda_s^S\,\mathrm{d}W_s^S - \int_0^t \lambda_s^\gamma\,\mathrm{d}W_s^\gamma - \frac{1}{2}\int_0^t (\lambda_s^S)^2\,\mathrm{d}s - \frac{1}{2}\int_0^t (\lambda_s^\gamma)^2\,\mathrm{d}s - \rho_1\int_0^t \lambda_s^S\lambda_s^\gamma\,\mathrm{d}s\right\}, \tag{3.1}$$

where λ_t^S and λ_t^γ satisfy

$$\lambda_t^S(\rho_1\beta L_t + \sigma_S) + \lambda_t^\gamma(\beta L_t + \rho_1\sigma_S) = \mu_S + \beta L_t + \tfrac{1}{2}\beta^2 L_t^2 + \rho_1\sigma_S\beta L_t - r, \tag{3.2}$$

with r being a constant risk-free interest rate. Using Girsanov's theorem, the three-dimensional process $\{W_t^Q = (W_t^{Q,L},\ W_t^{Q,\gamma},\ W_t^{Q,S});\ 0 \leq t < \infty\}$ defined by

$$\begin{aligned} \mathrm{d}W_t^{Q,L} &= \mathrm{d}W_t^L + \rho_2\lambda_t^S\,\mathrm{d}t + \rho_3\lambda_t^\gamma\,\mathrm{d}t, \\ \mathrm{d}W_t^{Q,\gamma} &= \mathrm{d}W_t^\gamma + \lambda_t^\gamma\,\mathrm{d}t + \rho_1\lambda_t^S\,\mathrm{d}t, \\ \mathrm{d}W_t^{Q,S} &= \mathrm{d}W_t^S + \lambda_t^S\,\mathrm{d}t + \rho_2\lambda_t^\gamma\,\mathrm{d}t, \end{aligned} \tag{3.3}$$

is a standard Brownian motion under Q, and it has the same correlation structure as that under the physical probability measure P. Furthermore, the stock price dynamics under Q can be written as

$$\frac{\mathrm{d}S_t}{S_t} = r\,\mathrm{d}t + \beta L_t\,\mathrm{d}W_t^{Q,\gamma} + \sigma_S\,\mathrm{d}W_t^{Q,S}, \tag{3.4}$$

https://doi.org/10.1017/S026996482200050X Published online by Cambridge University Press

with

$$\mathrm{d}L_t = (\kappa_L\theta_L - \kappa_L L_t - \sigma_L\rho_2\lambda_t^S - \sigma_L\rho_3\lambda_t^\gamma)\,\mathrm{d}t + \sigma_L\,\mathrm{d}W_t^{Q,L}. \tag{3.5}$$

According to the expression of $\mathrm{d}W_t^{Q,L}$ in (3.3), the market liquidity risk premium is $\rho_2\lambda_t^S + \rho_3\lambda_t^\gamma$. Following the idea of Heston [12] as the means of achieving tractability, we assume that the liquidity risk premium is proportional to the level of market liquidity, that is,

$$\rho_2\lambda_t^S + \rho_3\lambda_t^\gamma = \frac{\xi L_t}{\sigma_L}, \tag{3.6}$$

where ξ is a constant. In other words, the measure change adjusts the drift of the market liquidity measure L_t by the term $\xi L_t/\sigma_L$ (also see [10]). Thus, we can rewrite the dynamics of the market liquidity measure under Q,

$$\mathrm{d}L_t = \kappa(\theta - L_t)\,\mathrm{d}t + \sigma_L\,\mathrm{d}W_t^{Q,L}, \tag{3.7}$$

where $\kappa := \kappa_L + \xi$ and $\theta := \kappa_L\theta_L/(\kappa_L + \xi)$.

To facilitate our analysis of the stock price, under the equivalent martingale measure Q, we intend to rewrite Brownian motion $W_t^Q = (W_t^{Q,L}, W_t^{Q,\gamma}, W_t^{Q,S})$ as a linear transformation of a three-dimensional standard Brownian motion. First, denote the correlation matrix of W_t^Q by Λ, which is given by

$$\Lambda = \begin{pmatrix} 1 & \rho_3 & \rho_2 \\ \rho_3 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{pmatrix}.$$

Applying the Cholesky decomposition, we can decompose Λ into the product of a lower triangular matrix A and its conjugate transpose, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \rho_3 & \sqrt{1-\rho_3^2} & 0 \\ \rho_2 & \zeta & \sqrt{1-\rho_2^2-\zeta^2} \end{pmatrix},$$

and $\zeta = (\rho_1 - \rho_2\rho_3)/\sqrt{1-\rho_3^2}$. Then, one obtains that

$$\begin{aligned} W_t^{Q,L} &= W_{1,t}^Q, \\ W_t^{Q,\gamma} &= \rho_3 W_{1,t}^Q + \sqrt{1-\rho_3^2}\,W_{2,t}^Q, \\ W_t^{Q,S} &= \rho_2 W_{1,t}^Q + \zeta W_{2,t}^Q + \sqrt{1-\rho_2^2-\zeta^2}\,W_{3,t}^Q, \end{aligned} \tag{3.8}$$

and $W_{1,t}^Q$, $W_{2,t}^Q$ and $W_{3,t}^Q$ are independent standard Brownian motions under Q. Therefore, the stock price can be rewritten as:

$$\begin{aligned} \frac{\mathrm{d}S_t}{S_t} &= r\,\mathrm{d}t + (\rho_2\sigma_S + \rho_3\beta L_t)\,\mathrm{d}W_{1,t}^Q \\ &\quad + (\zeta\sigma_S + \sqrt{1-\rho_3^2}\beta L_t)\,\mathrm{d}W_{2,t}^Q + \sqrt{1-\rho_2^2-\zeta^2}\sigma_S\,\mathrm{d}W_{3,t}^Q, \end{aligned} \tag{3.9}$$

with the market liquidity measure L_t given below,

$$\mathrm{d}L_t = \kappa(\theta - L_t)\,\mathrm{d}t + \sigma_L\,\mathrm{d}W_{1,t}^Q. \tag{3.10}$$

Obviously, $\{S_t, t \geq 0\}$ is a martingale after discounted by the risk-free cash account under Q.

https://doi.org/10.1017/S026996482200050X Published online by Cambridge University Press

Now, we are ready to derive the price of vanilla European options. At time $t = 0$, the price of a European call option with maturity T and strike price K is given by

$$\begin{aligned} C_0 &= e^{-rT} E^Q [\max(S_T - K, 0)] \\ &= e^{-rT} E^Q [(S_T - K) I_{\{S_T > K\}}] \\ &= e^{-rT} E^Q [S_T I_{\{\ln S_T > \ln K\}}] - K e^{-rT} Q(\ln S_T > \ln K), \end{aligned} \tag{3.11}$$

where $I_{\{\cdot\}}$ is the indicator function. To calculate C_0, we use the characteristic function of $\ln(S_T)$ defined by $f_1(\phi) := E^Q[e^{\phi \ln S_T}]$ for any complex number ϕ. Using the Fourier inversion formula and following [27], we can obtain the explicit expressions of the two components in (3.11) as follows:

$$\begin{aligned} I_1 &:= E^Q [S_T I_{\{\ln S_T > \ln K\}}] \\ &= \frac{1}{2} f_1(1) + \frac{1}{\pi} \int_0^\infty \mathrm{Re} \left[\frac{K^{-i\phi} f_1(i\phi + 1)}{i\phi} \right] \mathrm{d}\phi, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} I_2 &:= Q(\ln S_T > \ln K) \\ &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re} \left[\frac{K^{-i\phi} f_1(i\phi)}{i\phi} \right] \mathrm{d}\phi. \end{aligned} \tag{3.13}$$

Additionally, the explicit expression of the characteristic function $f_1(\phi) := E^Q[e^{\phi \ln S_T}]$ will be given as a special case in the following subsection.

3.2. Pricing of vulnerable European options with liquidity risk

In this subsection, we incorporate default risk of option issuers into the pricing model. Here, we describe default risk in a reduced-form model, and work under the equivalent martingale measure Q directly for valuation purposes.[1] Assume that the underlying asset price is driven by (3.9), and let τ be the default time modeled by the first jump time of a doubly stochastic Poisson process with the following intensity process:

$$\lambda_t = \eta_0 + \eta_1 L_t + \eta_2 L_t^2 + X_t. \tag{3.14}$$

And X_t is captured by a mean-reverting square root process,

$$\mathrm{d}X_t = (\gamma_X - \alpha_X X_t)\,\mathrm{d}t + \sigma_X \sqrt{X_t}\,\mathrm{d}B_t^Q, \tag{3.15}$$

with $X_0 > 0$ and B_t^Q being a standard Brownian motion under Q, independent of $W_{1,t}^Q$, $W_{2,t}^Q$ and $W_{3,t}^Q$. To ensure that λ_t is nonnegative, we need to pose the assumptions on the parameters that $\eta_2 > 0$ and $4\eta_0\eta_2 \geq \eta_1^2$. There are two remarks on the assumption of the intensity process. First, the default intensity (3.14) consists of two parts: market liquidity L_t and idiosyncratic risk X_t, and market liquidity is a common factor to default intensity processes and all stocks in the market. Second, we propose a general form of the intensity process, which could also allow us to achieve tractability. Specially, with $\eta_1 = 0$, the model could capture negative effects between market liquidity ($|L_t|$) and default risk (λ_t). We refer interested readers to Brogaard *et al.* [5] and Nadarajah *et al.* [23] for this negative effect.

Now we are ready to price vulnerable European options. Let $\bar{\alpha}$ be the recovery rate, and then the vulnerable option price is given by

$$D_0 = e^{-rT} E^Q [I_{\{\tau > T\}} (S_T - K)^+] + \bar{\alpha} e^{-rT} E^Q [I_{\{0 < \tau \leq T\}} (S_T - K)^+]. \tag{3.16}$$

[1] Alternatively, we can first assume the intensity process under physical measure P and then follow Duffie [9] and Wang *et al.* [31] to find one martingale measure by assuming a risk premium for uncertainty associated with the timing of the default (see [9] or Section 2.1 in [31]).

In the proposed pricing model, we can derive the semi-closed form of the vulnerable option price D_0. To this end, we define the Fourier transform of $(\ln(S_T), \int_0^T \lambda_s \, ds)$, denoted by $f(\phi, \psi)$,

$$f(\phi, \psi) = E^Q[e^{\phi \ln S_T + \psi \int_0^T \lambda_s \, ds}], \tag{3.17}$$

where ϕ and ψ are complex numbers. Because $\{X_t, t \geq 0\}$ is independent of the other processes, we obtain that

$$\begin{aligned} f(\phi, \psi) &= E^Q[e^{\phi \ln S_T + \psi \int_0^T (\eta_0 + \eta_1 L_s + \eta_2 L_s^2) \, ds}] \times E^Q[e^{\psi \int_0^T X_s \, ds}] \\ &:= f_{SL}(\phi, \psi) \times f_X(\psi), \end{aligned} \tag{3.18}$$

where $f_{SL}(\phi, \psi) = E^Q[e^{\phi \ln S_T + \psi \int_0^T (\eta_0 + \eta_1 L_s + \eta_2 L_s^2) \, ds}]$ and $f_X(\psi) = E^Q[e^{\psi \int_0^T X_s \, ds}]$. In addition, the closed-form expressions of $f_X(\psi)$ and $f_{SL}(\phi, \psi)$ are shown in the following proposition.

Proposition 3.1. *Let $\mu_1(\psi) = \sqrt{\alpha_X^2 - 2\psi\sigma_X^2}$ and $\mu_2(\psi) = (\alpha_X + \mu_1(\psi))/(\alpha_X - \mu_1(\psi))$. The closed-form expressions of $f_X(\psi)$ and $f_{SL}(\phi, \psi)$ are given by*

$$\begin{aligned} f_X(\psi) = \exp\Bigg\{ &\frac{(\alpha_X + \mu_1(\psi))(1 - e^{\mu_1(\psi)T})}{\sigma_X^2 (1 - \mu_2(\psi) e^{\mu_1(\psi)T})} X_0 \\ &+ \frac{\mu_1(\psi)}{\sigma_X^2} \left((\alpha_X + \mu_1(\psi))T - 2 \ln\left(\frac{1 - \mu_2(\psi) e^{\mu_1(\psi)T}}{1 - \mu_2(\psi)} \right) \right) \Bigg\}, \end{aligned} \tag{3.19}$$

and

$$f_{SL}(\phi, \psi) = \exp\{\phi Y_0 + \tfrac{1}{2} A_1(0, T) L_0^2 + A_2(0, T) L_0 + A_3(0, T)\}, \tag{3.20}$$

where

$$Y_0 = \ln S_0 + \left(r - \frac{1}{2}\sigma_S^2 - \frac{\tilde{\kappa}\tilde{\theta}\rho_2\sigma_S}{\sigma_L} - \frac{\sigma_L \rho_3 \beta}{2} + \frac{1}{2}\phi\sigma_S^2(1 - \rho_2^2) + \frac{\psi\eta_0}{\phi} \right) T - \frac{\rho_3\beta}{2\sigma_L} L_0^2 - \frac{\rho_2\sigma_S}{\sigma_L} L_0,$$

and $A_1(0, T)$, $A_2(0, T)$ and $A_3(0, T)$ are given by (A.10)–(A.12) in the Appendix.

Proof. See the Appendix. □

Note that when $\psi = 0$, $f(\phi, 0)$ is the characteristic function of $\ln(S_T)$, that is, $f(\phi, 0) = f_1(\phi) := E^Q[e^{\phi \ln S_T}]$. Using the Fourier inversion formula and the characteristic functions, we can derive the price of vulnerable options and the results are given in the following proposition.

Proposition 3.2. *Under the risk-neutral martingale measure Q, the time-t price of vulnerable European call options with liquidity risk can be calculated as follows:*

$$D_0 = (1 - \bar{\alpha}) e^{-rT} (I_3 - K I_4) + \bar{\alpha} C_0, \tag{3.21}$$

where C_0 is given in (3.11), and

$$I_3 = \frac{1}{2} f(1, -1) + \frac{1}{\pi} \int_0^\infty Re\left[\frac{K^{-i\phi} f(i\phi + 1, -1)}{i\phi} \right] d\phi, \tag{3.22}$$

$$I_4 = \frac{1}{2} f(0, -1) + \frac{1}{\pi} \int_0^\infty Re\left[\frac{K^{-i\phi} f(i\phi, -1)}{i\phi} \right] d\phi. \tag{3.23}$$

https://doi.org/10.1017/S026996482200050X Published online by Cambridge University Press

Proof. It can be easily seen that D_0 in (3.16) can be rewritten as

$$D_0 = (1 - \bar{\alpha})e^{-rT}(I_3 - KI_4) + \bar{\alpha}C_0,$$

where C_0 is the price of vanilla European options with liquidity risk given in (3.11), and

$$I_3 := E^Q[S_T I_{\{\tau>T,\, S_T \geq K\}}],$$
$$I_4 := E^Q[I_{\{\tau>T,\, S_T \geq K\}}].$$

Employing the inverse Fourier transform, we can obtain the following expressions of I_3 and I_4:

$$I_3 = \frac{1}{2}f(1,-1) + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\phi}f(i\phi+1,-1)}{i\phi}\right]\mathrm{d}\phi,$$
$$I_4 = \frac{1}{2}f(0,-1) + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\phi}f(i\phi,-1)}{i\phi}\right]\mathrm{d}\phi.$$

This completes the proof of Proposition 3.2. □

4. Numerical analysis

In this section, we illustrate the effect of market liquidity risk on the prices of (vulnerable) European options. To show the results explicitly, we mainly illustrate prices of call options in three situations: the proposed framework with both liquidity risk and default risk, the case without liquidity risk (i.e., $\beta = 0$), and the case without default risk (i.e., $\bar{\alpha} = 1$).

Following Pasricha *et al.* [25], we use the following parameter values: $\sigma_S = 0.2$, $\sigma_L = 0.9$, $\rho_1 = 0.25$, $\rho_2 = 0.35$, $\rho_3 = 0$, $\tilde{\kappa} = 0.3$, $\tilde{\theta} = 0.2$ and $\beta = 0.5$. Theoretically, the less liquidity the financial market holds, the more likely option issuers are to default. As a result, the assumption that the parameters in (3.14) are positive is reasonable. We find that when setting $\eta_0 = 0.02$, $\eta_1 = 0.02$ and $\eta_2 = 0.02$, the default probability in 2 years is approximately 7.34%. Additionally, $\bar{\alpha} = 0.6$ means that 60% of the loss can be recovered after default, which is a conservative value. Without loss of generality, we set the interest rate $r = 0.01$, $S_0 = 100$, $L_0 = 0.3$, and the option is at the money ($S_0 = K = 100$) with a maturity of 2.0 years.

First, we focus on the effect of stochastic liquidity risk on the prices of both vanilla and vulnerable call options. From Figure 1, we can intuitively find that call option prices decrease with higher strike prices. To go further, it can also be found that by comparing the distance between the two lines, both the liquidity risk and default risk have small effects on in-the-money options. In contrast, with the augmentation of strike prices, the impact of liquidity risk always remains pronounced, while the impact of default risk becomes relatively more negligible, and these observations are consistent with Wang [29]. The large price discrepancies suggest that when pricing options, option issuers should take market liquidity into consideration. In addition, option buyers should be clear about the credit status of each issuer with the help of credit rating agencies or other possible channels and only accept reasonable prices that include default risk premiums.

Figure 2 displays the relationship between call option prices and the values of the sensitivity level of the stock to market illiquidity. Undoubtedly, option prices in the pricing model without liquidity risk are not affected by the changes of β, hence, we obtain a constant number in this case. Regarding the other two cases, call option prices rise when the stock becomes more sensitive to market illiquidity. Comparing the distance between two lines, the effect of default risk on the prices becomes relatively evident with a larger β. It should be noted that the values of β affect the total volatility of the underlying stock.

Figure 3 illustrates call option prices with different volatilities of the underlying stock. A higher volatility corresponds to a higher option price. We can observe two different trends from this graph:

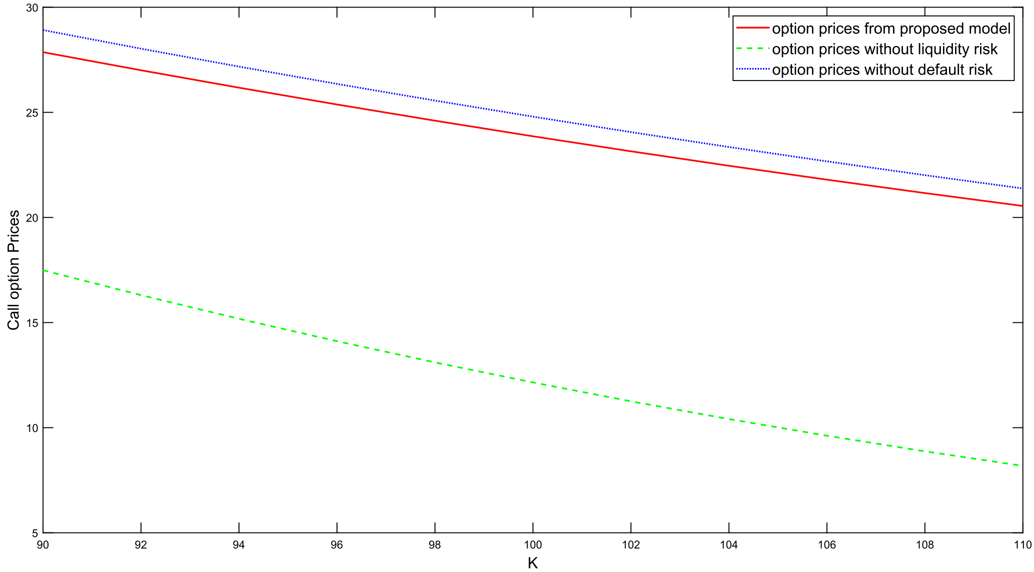

Figure 1. *Call option prices against strike prices. The solid, dashed and dotted lines correspond to prices in the proposed framework, prices without liquidity risk ($\beta = 0$) and prices without default risk ($\bar{\alpha} = 1$), respectively.*

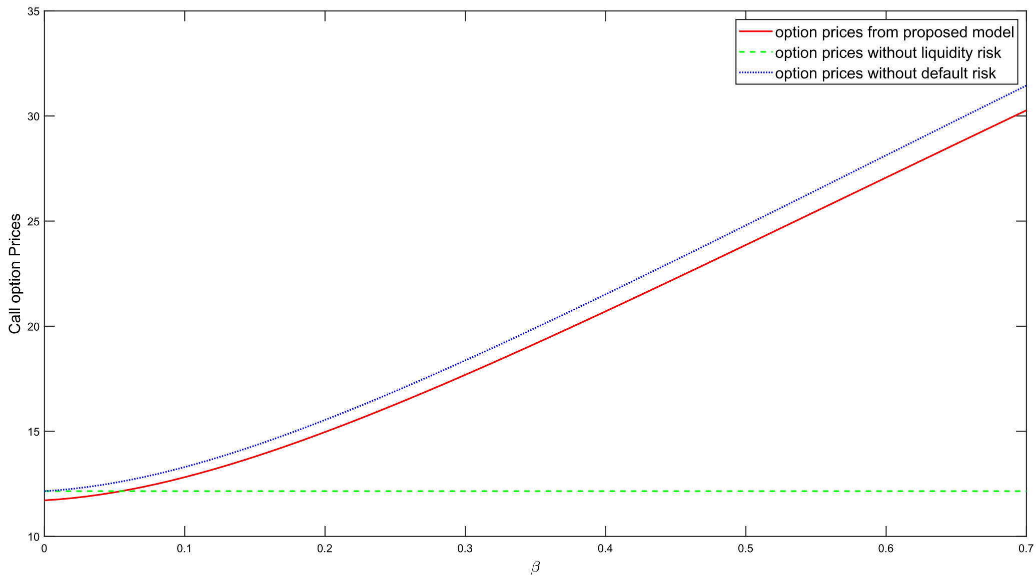

Figure 2. *Call option prices against the values of the sensitivity level of the stock to the market illiquidity. The solid, dashed and dotted lines correspond to prices in the proposed framework, prices without liquidity risk ($\beta = 0$) and prices without default risk ($\bar{\alpha} = 1$), respectively.*

with an increasing volatility of the underlying stock, the dotted line and the solid line tend to be farther away from each other, while the solid line and the dashed line get closer to each other, showing the strengthening effect of default risk and the weakening impact of liquidity risk. This is because the instantaneous total variance of the underlying stock is $\sigma_S^2 + \beta^2 L_t^2 + 2\rho_1\sigma_S\beta L_t$, and the effect of liquidity risk is not so significant with a bigger σ_S. We need to mention that in the real world, the probability of

https://doi.org/10.1017/S026996482200050X Published online by Cambridge University Press

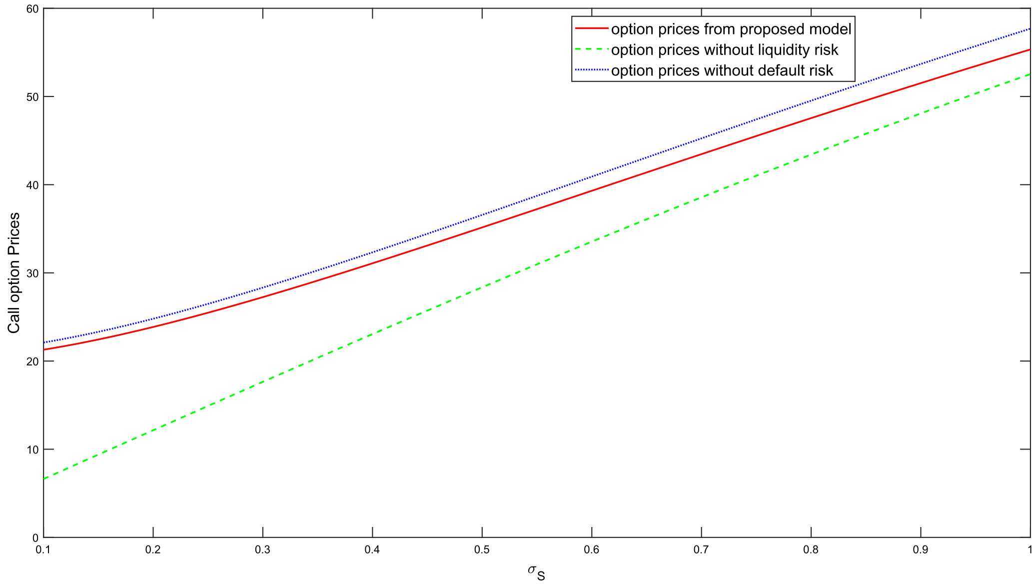

Figure 3. *Call option prices against volatilities of the underlying stock. The solid, dashed and dotted lines correspond to prices in the proposed framework, prices without liquidity risk ($\beta = 0$) and prices without default risk ($\bar{\alpha} = 1$), respectively.*

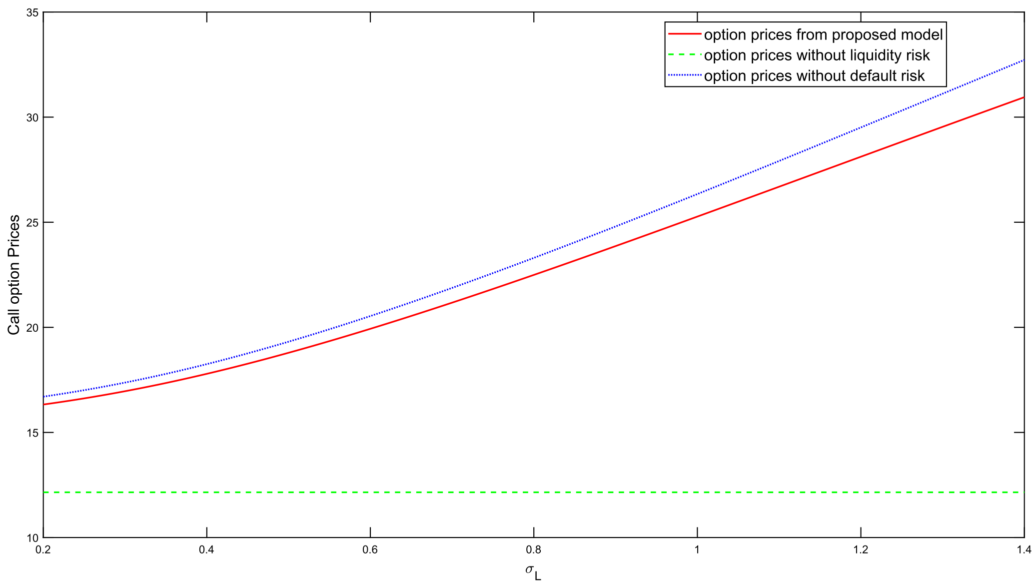

Figure 4. *Call option prices against volatilities of the stock market liquidity. The solid, dashed and dotted lines correspond to prices in the proposed framework, prices without liquidity risk ($\beta = 0$) and prices without default risk ($\bar{\alpha} = 1$), respectively.*

stock prices fluctuating fiercely in a short period is relatively low. As a result, we shall focus more on call option prices with smaller σ_S, and pay more attention to the influence of the stochastic liquidity risk.

Figure 4 depicts call option prices with different volatilities of market liquidity risk itself. Evidently, we obtain a horizontal line without liquidity risk. In regard to the dotted line and the solid one, a larger

https://doi.org/10.1017/S026996482200050X Published online by Cambridge University Press

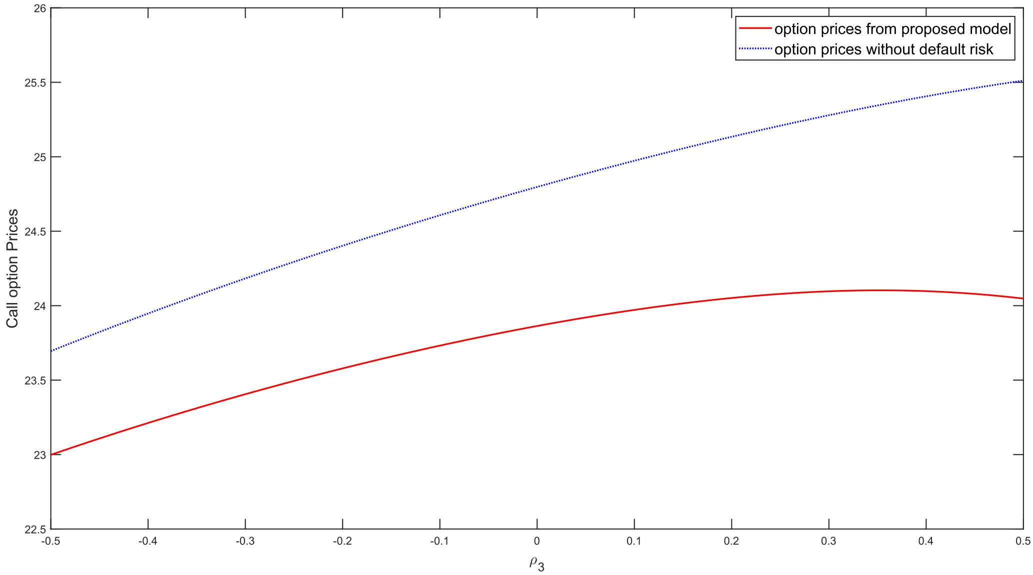

Figure 5. *Call option prices against ρ_3. The solid and dotted lines correspond to prices in the proposed framework and prices without default risk ($\bar{\alpha} = 1$), respectively.*

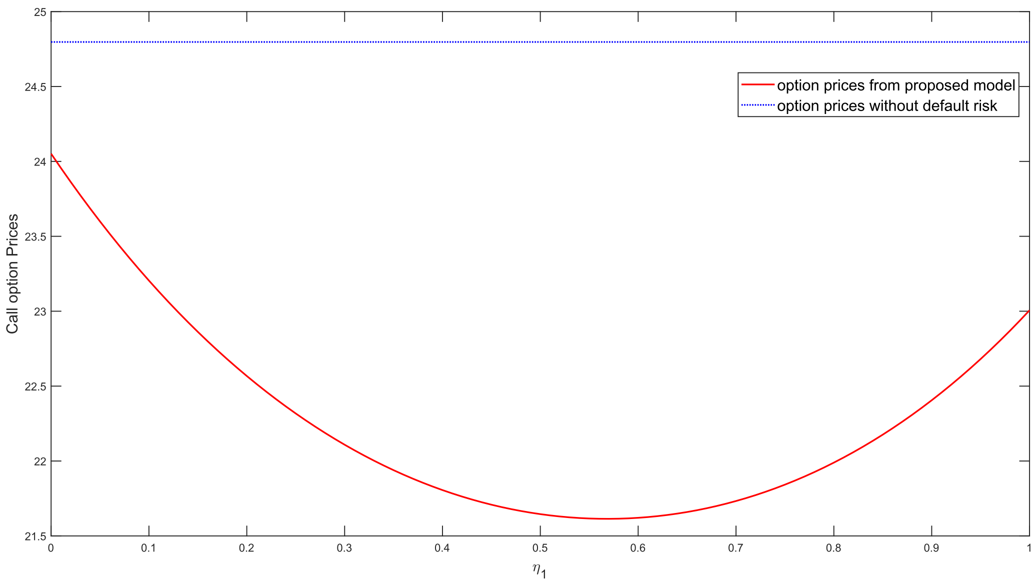

Figure 6. *Call option prices against η_1. The solid and dotted lines correspond to prices in the proposed framework and prices without default risk ($\bar{\alpha} = 1$), respectively.*

σ_L induces a correspondingly higher option price. Observing the differences between the solid and dashed lines, we find that the effect of liquidity risk is enhanced as the values of σ_L rise. Based on the bid-ask spread, market liquidity can be captured even though it is an invisible variable (see, e.g., [1,8]). Therefore, investors could easily estimate the value of σ_L and eventually trade options around a reasonable price using the model developed in this paper.

Figure 5 illustrates call option prices with respect to the correlation coefficients between the liquidity measure L_t and the liquidity discount factor γ_t. When ρ_3 is negative, call option prices increase with

https://doi.org/10.1017/S026996482200050X Published online by Cambridge University Press

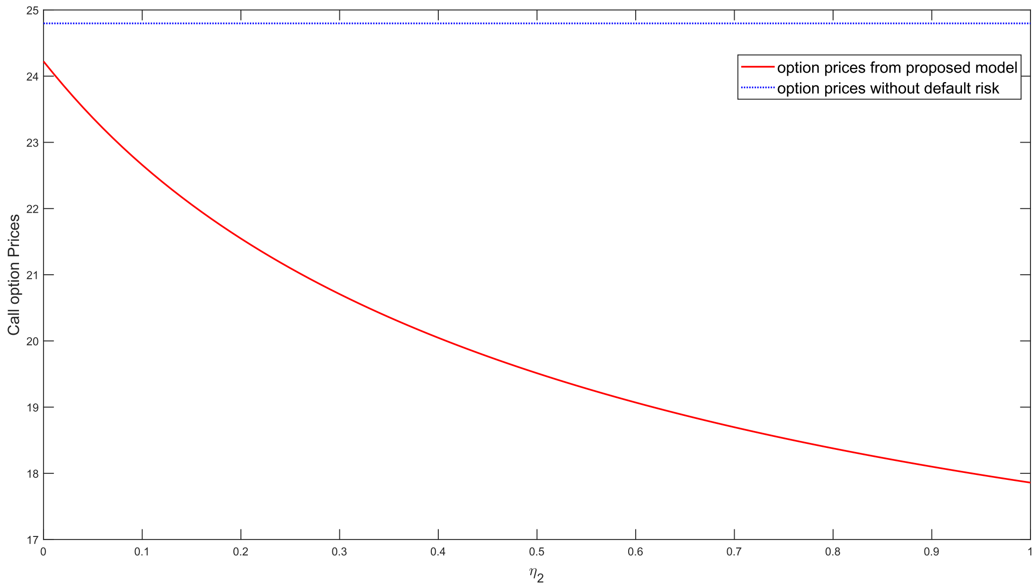

Figure 7. *Call option prices against η_2. The solid and dotted lines correspond to prices in the proposed framework and prices without default risk ($\bar{\alpha} = 1$), respectively.*

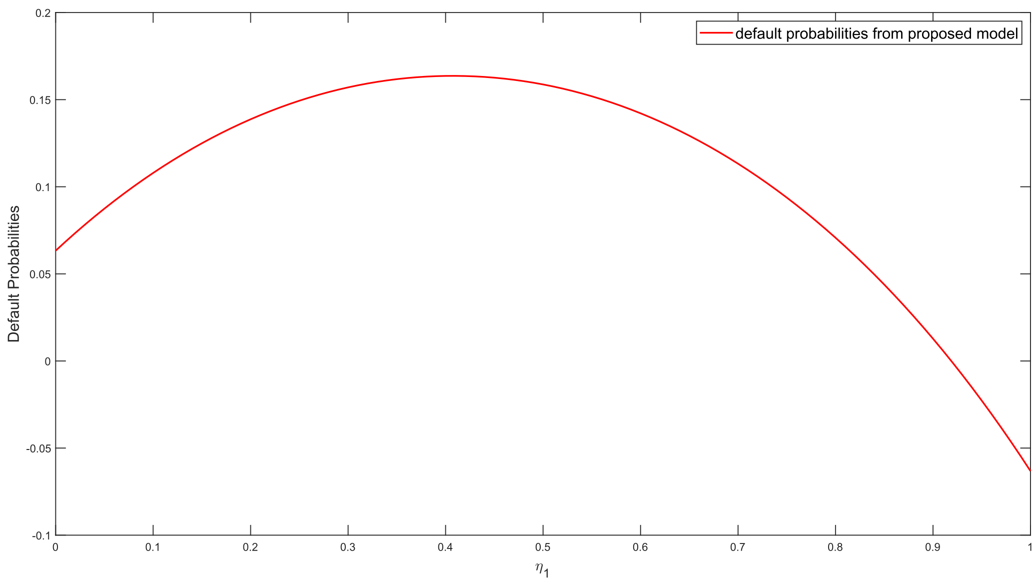

Figure 8. *Default probabilities against η_1. The solid line corresponds to default probabilities in the proposed framework.*

the correlation coefficients approaching to zero. In contrast, in the case of $\rho_3 > 0$, the marginal rise of call option prices decreases and even becomes negative when ρ_3 is large enough. In addition, the effect of default risk on option prices is much more pronounced when ρ_3 takes a larger value.

Figure 6 shows call option prices against η_1, displaying a U-shaped curve in the proposed model. η_1 is the coefficient of the first-order term in the intensity process λ_t. Call option prices first decrease and then increase as η_1 increases. Comparing the distance between two lines, we can easily find that the effect of default risk is evident when η_1 is small and the effect reaches the maximum under a certain

https://doi.org/10.1017/S026996482200050X Published online by Cambridge University Press

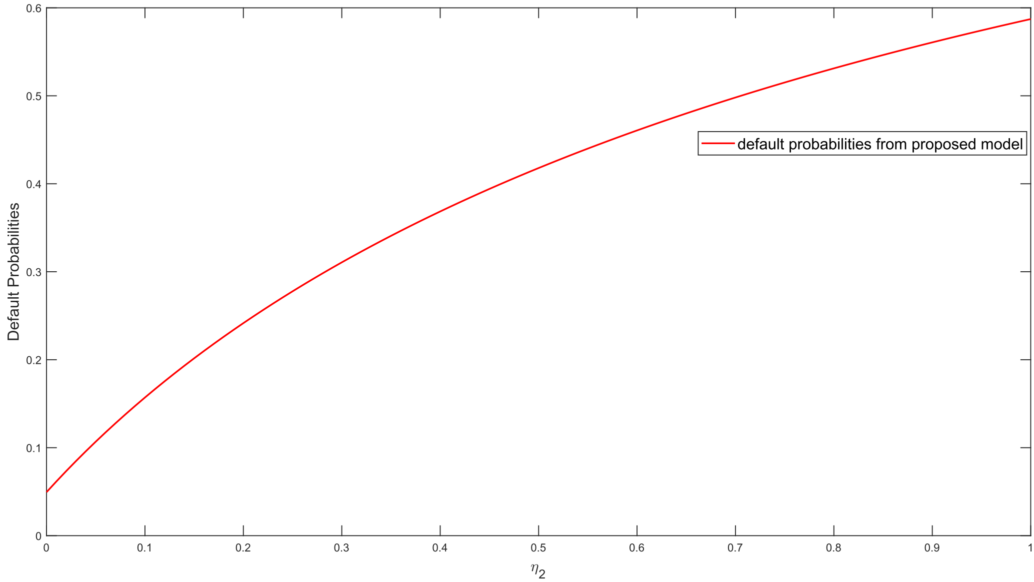

Figure 9. *Default probabilities against η_2. The solid line corresponds to default probabilities in the proposed framework.*

η_1. This is because the corresponding default probabilities first increase and then decrease as shown in Figure 8. Figure 7 shows call option prices with respect to η_2. η_2 is the coefficient of the second-order term in the intensity process λ_t. Since L_t^2 is always positive, default probabilities increase and then vulnerable call option prices drop with an increase of η_2, which can also be verified in Figure 9.

5. Conclusion

In this paper, we contribute to the literature on vulnerable European options by taking the possibility of default risk caused by the counterparty and market liquidity risk into consideration. A general correlation structure among the underlying asset, the liquidity discount factor and the default intensity process is specified. Utilizing the characteristic function and the Feynman–Kac theorem, we obtain the semi-closed form pricing formulae of vulnerable European options with market liquidity risk. Finally, numerical experiments are performed to illustrate the effects of liquidity risk and default risk on the prices of vulnerable European options.

Acknowledgments. The authors would like to thank anonymous referees and editors for providing a number of valuable comments that led to several important improvements. All errors are our own responsibilities.

Competing interests. The authors declare no conflict of interest.

References

[1] Amihud, Y. (2002). Illiquidity and stock returns: Cross-section and time-series effects. *Journal of Financial Markets* 5(1): 31–56.
[2] Amihud, Y. & Mendelson, H. (1991). Liquidity, asset prices and financial policy. *Financial Analysts Journal* 47(6): 56–66.
[3] Bingham, N.H. & Kiesel, R. (2013). *Risk-neutral valuation: Pricing and hedging of financial derivatives*. London: Springer Science & Business Media.
[4] Black, F. & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* 81(3): 637–654.
[5] Brogaard, J., Li, D., & Xia, Y. (2017). Stock liquidity and default risk. *Journal of Financial Economics* 124(3): 486–502.
[6] Brunetti, C. & Caldarera, A. (2004). Asset prices and asset correlations in illiquid markets. *Available at SSRN 625184*.
[7] Cetin, U., Jarrow, R., Protter, P., & Warachka, M. (2006). Pricing options in an extended Black Scholes economy with illiquidity: Theory and empirical evidence. *Review of Financial Studies* 19(2): 493–529.

[8] Corwin, S.A. & Schultz, P. (2012). A simple way to estimate bid-ask spreads from daily high and low prices. *Journal of Finance* 67(2): 719–760.
[9] Duffie, D. (2005). Credit risk modeling with affine processes. *Journal of Banking and Finance* 29: 2751–2802.
[10] Feng, S., Hung, M., & Wang, Y. (2014). Option pricing with stochastic liquidity risk: Theory and evidence. *Journal of Financial Markets* 18: 77–95.
[11] He, Z. & Xiong, W. (2012). Rollover risk and credit risk. *Journal of Finance* 67(2): 391–430.
[12] Heston, S.L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* 6(2): 327–343.
[13] Hull, J. & White, A. (1995). The impact of default risk on the prices of options and other derivative securities. *Journal of Banking and Finance* 19(2): 299–322.
[14] Jarrow, R. & Turnbull, S. (1995). Pricing derivatives on financial securities subject to credit risk. *Journal of Finance* 50(1): 53–85.
[15] Johnson, H. & Stulz, R. (1987). The pricing of options with default risk. *Journal of Finance* 42(2): 267–280.
[16] Klein, P. (1996). Pricing Black-Scholes options with correlated credit risk. *Journal of Banking and Finance* 20(7): 1211–1229.
[17] Klein, P. & Inglis, M. (2001). Pricing vulnerable European options when the option payoff can increase the risk of financial distress. *Journal of Banking and Finance* 25(5): 993–1012.
[18] Lee, M.-K., Yang, S.-J., & Kim, J.-H. (2016). A closed form solution for vulnerable options with Heston's stochastic volatility. *Chaos, Solitons & Fractals* 86: 23–27.
[19] Leippold, M. & Schärer, S. (2017). Discrete-time option pricing with stochastic liquidity. *Journal of Banking and Finance* 75: 1–16.
[20] Ma, Y., Shrestha, K., & Xu, W. (2017). Pricing vulnerable options with jump clustering. *Journal of Futures Markets* 37(12): 1155–1178.
[21] Ma, C., Yue, S., Wu, H., & Ma, Y. (2020). Pricing vulnerable options with stochastic volatility and stochastic interest rate. *Computational Economics* 56(2): 391–429.
[22] Madan, D. (2010). Conserving capital by adjusting deltas for gamma in the presence of skewness. *Journal of Risk and Financial Management* 3(1): 1–25.
[23] Nadarajah, S., Duong, H.N., Ali, S., Liu, B., & Huang, A. (2021). Stock liquidity and default risk around the world. *Journal of Financial Markets* 55: 100597.
[24] Niu, H. & Wang, D. (2016). Pricing vulnerable options with correlated jump-diffusion processes depending on various states of the economy. *Quantitative Finance* 16(7): 1129–1145.
[25] Pasricha, P., Zhu, S.-P., & He, X.-J. (2022). A closed-form pricing formula for European options with market liquidity risk. *Expert Systems with Applications* 189: 116128.
[26] Pástor, L. & Stambaugh, R. (2003). Liquidity risk and expected stock returns. *Journal of Political Economy* 111(3): 642–685.
[27] Shephard, N. (1991). From characteristic function to distribution function: A simple framework for the theory. *Econometric Theory* 7(4): 519–529.
[28] Tian, L., Wang, G., Wang, X., & Wang, Y. (2014). Pricing vulnerable options with correlated credit risk under jump-diffusion processes. *Journal of Futures Markets* 34(10): 957–979.
[29] Wang, X. (2021). Pricing vulnerable options with jump risk and liquidity risk. *Review of Derivatives Research* 24(3): 243–260.
[30] Wang, G., Wang, X., & Zhou, K. (2017). Pricing vulnerable options with stochastic volatility. *Physica A: Statistical Mechanics and its Applications* 485: 91–103.
[31] Wang, G., Wang, X., & Shao, X. (2020). The valuation of vulnerable European options with risky collateral. *European Journal of Finance* 26(13): 1315–1331.
[32] Yang, S.-J., Lee, M.-K., & Kim, J.-H. (2014). Pricing vulnerable options under a stochastic volatility model. *Applied Mathematics Letters* 34: 7–12.

Appendix

Here, we derive the expressions of $f_X(\psi)$ and $f_{SL}(\phi,\psi)$. First, using the dynamics of X_t in (3.15), we can derive the expression of $f_X(\psi)$ easily,

$$\begin{aligned} f_X(\psi) = E^Q[e^{\psi\int_0^T X_s \mathrm{d}s}] = \exp\Bigg\{ & \frac{(\alpha_X + \mu_1(\psi))(1 - e^{\mu_1(\psi)T})}{\sigma_X^2(1 - \mu_2(\psi)e^{\mu_1(\psi)T})} X_0 \\ & + \frac{\mu_1(\psi)}{\sigma_X^2}\left((\alpha_X + \mu_1(\psi))T - 2\ln\left(\frac{1 - \mu_2(\psi)e^{\mu_1(\psi)T}}{1 - \mu_2(\psi)}\right)\right)\Bigg\}, \end{aligned} \tag{A.1}$$

https://doi.org/10.1017/S026996482200050X Published online by Cambridge University Press

where $\mu_1(\psi) = \sqrt{\alpha_X^2 - 2\psi\sigma_X^2}$ and $\mu_2(\psi) = (\alpha_X + \mu_1(\psi))/(\alpha_X - \mu_1(\psi))$.

Next, we turn to the calculation of $f_{SL}(\phi,\psi)$. Note that

$$\begin{aligned}\ln S_T &= \ln S_0 + \left(r - \frac{1}{2}\sigma_S^2\right)T + \int_0^T (\rho_2\sigma_S + \rho_3\beta L_s)\,\mathrm{d}W_{1,s}^Q \\ &\quad + \int_0^T (\zeta\sigma_S + \beta\sqrt{1-\rho_3^2}L_s)\,\mathrm{d}W_{2,s}^Q \\ &\quad + \sigma_S\sqrt{1-\rho_2^2-\zeta^2}W_{3,T}^Q - \frac{1}{2}\beta^2\int_0^T L_s^2\,\mathrm{d}s - \rho_1\beta\sigma_S\int_0^T L_s\,\mathrm{d}s. \end{aligned} \tag{A.2}$$

From the dynamics of L_t in (2.7), we apply Itô's lemma to $L_t^2 + (2\rho_2\sigma_S/\rho_3\beta)L_t$, and obtain the following result,

$$\begin{aligned}\mathrm{d}\left(L_t^2 + \frac{2\rho_2\sigma_S}{\rho_3\beta}L_t\right) &= 2\left(L_t + \frac{\rho_2\sigma_S}{\rho_3\beta}\right)\mathrm{d}L_t + \sigma_L^2\,\mathrm{d}t \\ &= \left(2\tilde{\kappa}(\tilde{\theta} - L_t)\left(L_t + \frac{\rho_2\sigma_S}{\rho_3\beta}\right) + \sigma_L^2\right)\mathrm{d}t + 2\sigma_L\left(L_t + \frac{\rho_2\sigma_S}{\rho_3\beta}\right)\mathrm{d}W_{1,t}^Q, \end{aligned} \tag{A.3}$$

which in turn implies that

$$\begin{aligned}\int_0^T \left(L_s + \frac{\rho_2\sigma_S}{\rho_3\beta}\right)\mathrm{d}W_{1,s}^Q &= -\frac{1}{2\sigma_L}L_0^2 - \frac{\rho_2\sigma_S}{\sigma_L\rho_3\beta}L_0 - \left(\frac{\tilde{\kappa}\tilde{\theta}\rho_2\sigma_S}{\sigma_L\rho_3\beta} + \frac{\sigma_L}{2}\right)T \\ &\quad + \frac{1}{2\sigma_L}L_T^2 + \frac{\rho_2\sigma_S}{\sigma_L\rho_3\beta}L_T - \frac{\tilde{\kappa}}{\sigma_L}\left(\tilde{\theta} - \frac{\rho_2\sigma_S}{\rho_3\beta}\right)\int_0^T L_s\,\mathrm{d}s + \frac{\tilde{\kappa}}{\sigma_L}\int_0^T L_s^2\,\mathrm{d}s. \end{aligned} \tag{A.4}$$

Therefore, one can get that

$$\begin{aligned} f_{SL}(\phi,\psi) &= E^Q[e^{\phi\ln S_T + \psi\int_0^T(\eta_0+\eta_1L_s+\eta_2L_s^2)\,\mathrm{d}s}] \\ &= \exp\left\{\phi\left(\ln S_0 + \left(r - \frac{1}{2}\sigma_S^2\right)T\right)\right\}E^Q\left[\exp\left\{\phi\left(\int_0^T(\rho_2\sigma_S + \rho_3\beta L_s)\,\mathrm{d}W_{1,s}^Q\right.\right.\right. \\ &\quad + \int_0^T(\zeta\sigma_S + \beta\sqrt{1-\rho_3^2}L_s)\mathrm{d}W_{2,s}^Q + \sigma_S\sqrt{1-\rho_2^2-\zeta^2}W_{3,T}^Q \\ &\quad \left.\left.\left. -\frac{1}{2}\beta^2\int_0^T L_s^2\,\mathrm{d}s - \rho_1\beta\sigma_S\int_0^T L_s\,\mathrm{d}s\right) + \psi\int_0^T(\eta_0 + \eta_1L_s + \eta_2L_s^2)\,\mathrm{d}s\right\}\right] \\ &= \exp\left\{\phi\left(\ln S_0 + \left(r - \frac{1}{2}\sigma_S^2 - \frac{\tilde{\kappa}\tilde{\theta}\rho_2\sigma_S}{\sigma_L} - \frac{\sigma_L\rho_3\beta}{2}\right)T - \frac{\rho_3\beta}{2\sigma_L}L_0^2 - \frac{\rho_2\sigma_S}{\sigma_L}L_0\right) + \psi\eta_0T\right\} \\ &\quad \times E^Q\left[\exp\left\{\phi\left(\frac{\rho_3\beta}{2\sigma_L}L_T^2 + \frac{\rho_2\sigma_S}{\sigma_L}L_T - \frac{\tilde{\kappa}}{\sigma_L}(\tilde{\theta}\rho_3\beta - \rho_2\sigma_S)\int_0^T L_s\,\mathrm{d}s + \frac{\tilde{\kappa}\rho_3\beta}{\sigma_L}\int_0^T L_s^2\,\mathrm{d}s\right)\right.\right. \\ &\quad + \frac{1}{2}\phi^2\zeta^2\sigma_S^2T + \phi^2\zeta\sigma_S\beta\sqrt{1-\rho_3^2}\int_0^T L_s\mathrm{d}s \\ &\quad + \frac{1}{2}\phi^2\beta^2(1-\rho_3^2)\int_0^T L_s^2\mathrm{d}s + \frac{1}{2}\phi^2\sigma_S^2(1-\rho_2^2-\zeta^2)T \\ &\quad \left.\left. -\frac{1}{2}\phi\beta^2\int_0^T L_s^2\,\mathrm{d}t - \phi\rho_1\beta\sigma_S\int_0^T L_s\,\mathrm{d}s + \psi\eta_1\int_0^T L_s\mathrm{d}s + \psi\eta_2\int_0^T L_s^2\,\mathrm{d}s\right\}\right] \end{aligned}$$

https://doi.org/10.1017/S026996482200050X Published online by Cambridge University Press

$$
\begin{aligned}
&= \exp\left\{\phi\left(\ln S_0 + \left(r - \frac{1}{2}\sigma_S^2 - \frac{\tilde{\kappa}\tilde{\theta}\rho_2\sigma_S}{\sigma_L} - \frac{\sigma_L\rho_3\beta}{2} + \frac{1}{2}\phi\sigma_S^2(1-\rho_2^2) + \frac{\psi\eta_0}{\phi}\right)T\right.\right.\\
&\qquad\left.\left. - \frac{\rho_3\beta}{2\sigma_L}L_0^2 - \frac{\rho_2\sigma_S}{\sigma_L}L_0\right)\right\}\\
&\quad\times E^Q\left[\exp\left\{\frac{\phi\rho_3\beta}{2\sigma_L}L_T^2 + \frac{\phi\rho_2\sigma_S}{\sigma_L}L_T\right.\right.\\
&\quad + \left(\phi^2\zeta\sigma_S\beta\sqrt{1-\rho_3^2} - \frac{\phi\tilde{\kappa}}{\sigma_L}(\tilde{\theta}\rho_3\beta - \rho_2\sigma_S) - \phi\rho_1\beta\sigma_S + \psi\eta_1\right)\int_0^T L_s\,\mathrm{d}s\\
&\quad\left.\left. + \left(\frac{\phi\tilde{\kappa}\rho_3\beta}{\sigma_L} + \frac{1}{2}\phi^2\beta^2(1-\rho_3^2) - \frac{1}{2}\phi\beta^2 + \psi\eta_2\right)\int_0^T L_s^2\,\mathrm{d}s\right\}\right]. \qquad \text{(A.5)}
\end{aligned}
$$

To obtain the expression of the expectation in the above equation, we denote

$$
P(L,t,T) = E^Q\left[\exp\left\{-\omega_1\int_t^T L_s^2\,\mathrm{d}s - \omega_2\int_t^T L_s\,\mathrm{d}s + \omega_3 L_T^2 + \omega_4 L_T\right\}\middle|\mathcal{F}_t\right], \qquad \text{(A.6)}
$$

with terminal conditions $P(L,T,T) = e^{\omega_3 L_T^2 + \omega_4 L_T}$. Then, the expectation in (A.5) equals $P(L,0,T)$ with $\omega_1 = -\phi\tilde{\kappa}\rho_3\beta/\sigma_L - \frac{1}{2}\phi^2\beta^2(1-\rho_3^2) + \frac{1}{2}\phi\beta^2 - \psi\eta_2$, $\omega_2 = -\phi^2\zeta\sigma_S\beta\sqrt{1-\rho_3^2} + (\phi\tilde{\kappa}/\sigma_L)(\tilde{\theta}\rho_3\beta - \rho_2\sigma_S) + \phi\rho_1\beta\sigma_S - \psi\eta_1$, $\omega_3 = \phi\rho_3\beta/2\sigma_L$ and $\omega_4 = \phi\rho_2\sigma_S/\sigma_L$.

According to the Feynman–Kac theorem, for $0 \le t < T$, $P(L,t,T)$ satisfies the following partial differential equation:

$$
\frac{\partial P}{\partial t} + \frac{\partial P}{\partial L}\kappa(\theta - L_t) + \frac{1}{2}\frac{\partial^2 P}{\partial L^2}\sigma_L^2 - (\omega_1 L_t^2 + \omega_2 L_t)P = 0. \qquad \text{(A.7)}
$$

The solution of $P(L,t,T)$ has the following form:

$$
P(L,t,T) = \exp\{\tfrac{1}{2}A_1(t,T)L_t^2 + A_2(t,T)L_t + A_3(t,T)\}, \qquad \text{(A.8)}
$$

with terminal conditions $A_1(T,T) = 2\omega_3$, $A_2(T,T) = \omega_4$ and $A_3(T,T) = 0$. Specifically, the system of ordinary differential equations is given below:

$$
\begin{cases}
\dfrac{\mathrm{d}A_1}{\mathrm{d}t} + \sigma_L^2 A_1^2 - 2\tilde{\kappa}A_1 - 2\omega_1 = 0,\\
\dfrac{\mathrm{d}A_2}{\mathrm{d}t} - (\tilde{\kappa} - \sigma_L^2 A_1)A_2 + \tilde{\kappa}\tilde{\theta}A_1 - \omega_2 = 0,\\
\dfrac{\mathrm{d}A_3}{\mathrm{d}t} + \dfrac{1}{2}\sigma_L^2 A_2^2 + \tilde{\kappa}\tilde{\theta}A_2 + \dfrac{1}{2}\sigma_L^2 A_1 = 0.
\end{cases} \qquad \text{(A.9)}
$$

Additionally, we can obtain the solutions as follows:

$$
A_1(t,T) = \frac{1}{\sigma_L^2}\left(\tilde{\kappa} - \delta_1\frac{\sinh(\delta_1(T-t)) + \delta_2\cosh(\delta_1(T-t))}{\cosh(\delta_1(T-t) + \delta_2\sinh(\delta_1(T-t))}\right), \qquad \text{(A.10)}
$$

$$
\begin{aligned}
A_2(t,T) &= \frac{1}{\sigma_L^2\delta_1}\left(\frac{(\tilde{\kappa}\tilde{\theta} + \sigma_L^2\omega_4)\delta_1 - \delta_2\delta_3}{\cosh(\delta_1(T-t)) + \sinh(\delta_1(T-t))} - \tilde{\kappa}\tilde{\theta}\delta_1\right)\\
&\quad + \frac{\delta_3}{\sigma_L^2\delta_1}\left(\frac{\sinh(\delta_1(T-t)) + \delta_2\cosh(\delta_1(T-t))}{\cosh(\delta_1(T-t)) + \delta_2\sinh(\delta_1(T-t))}\right), \qquad \text{(A.11)}
\end{aligned}
$$

and

$$
\begin{aligned}
A_3(t,T) = {}& -\frac{1}{2}\ln(\cosh(\delta_1(T-t)) + \delta_2\sinh(\delta_1(T-t))) + \left(\frac{1}{2}\tilde{\kappa} + \frac{1}{2}\sigma_L^2\omega_4^2 + \tilde{\kappa}\tilde{\theta}\omega_4\right)(T-t) \\
& + \frac{(\tilde{\kappa}\tilde{\theta} + \sigma_L^2\omega_4)^2\delta_1^2 - \delta_3^2}{2\sigma_L^2\delta_1^3}\left(\frac{\sinh(\delta_1(T-t))}{\cosh(\delta_1(T-t)) + \delta_2\sinh(\delta_1(T-t))} - \delta_1(T-t)\right) \\
& + \frac{((\tilde{\kappa}\tilde{\theta} + \sigma_L^2\omega_4)\delta_1 - \delta_2\delta_3)\delta_3}{\sigma_L^2\delta_1^3}\left(\frac{\cosh(\delta_1(T-t)) - 1}{\cosh(\delta_1(T-t)) + \delta_2\sinh(\delta_1(T-t))}\right),
\end{aligned}
\tag{A.12}
$$

where $\delta_1 = \sqrt{2\sigma_L^2\omega_1 + \tilde{\kappa}^2}$, $\delta_2 = (1/\delta_1)(\tilde{\kappa} - 2\sigma_L^2\omega_3)$ and $\delta_3 = \tilde{\kappa}(\tilde{\kappa}\tilde{\theta} + \sigma_L^2\omega_4) - \sigma_L^2(\omega_2 + \tilde{\kappa}\omega_4)$.

Let $Y_0 = \ln S_0 + (r - \frac{1}{2}\sigma_S^2 - \tilde{\kappa}\tilde{\theta}\rho_2\sigma_S/\sigma_L - \sigma_L\rho_3\beta/2 + \frac{1}{2}\phi\sigma_S^2(1-\rho_2^2) + \psi\eta_0/\phi)T - (\rho_3\beta/2\sigma_L)L_0^2 - (\rho_2\sigma_S/\sigma_L)L_0$, and we obtain the following result,

$$
f_{SL}(\phi,\psi) = \exp\{\phi Y_0 + \tfrac{1}{2}A_1(0,T)L_0^2 + A_2(0,T)L_0 + A_3(0,T)\}.
$$

Cite this article: Pan Y, Tang D and Wang X (2024). Valuation of vulnerable European options with market liquidity risk. *Probability in the Engineering and Informational Sciences* **38**, 65–81. https://doi.org/10.1017/S026996482200050X