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ON THE DERIVED CUBOID OF AN EULERIAN TRIPLE

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One of the interesting mathematical problems is whether the system of four Diophantine equations

$$x^2+y^2=l^2, \qquad x^2+z^2=m^2, \qquad y^2+z^2=n^2, \tag{1}$$

$$x^2+y^2+z^2=w^2 \tag{2}$$

has a solution in x, y, z, l, m, n, w. To this day the problem has not been shown to be impossible, nor has it been solved.

Throughout this paper all symbols denote natural numbers. It is known that the Eulerian triple

$$x=a(4b^2-c^2), \qquad y=b(4a^2-c^2), \qquad z=4abc, \qquad (a^2+b^2=c^2)$$

which satisfies equation (1), cannot also satisfy equation (2) (SPOHN [1]). It is easy to see that if the x, y, z satisfies equation (1), then xy, zx, yz have the same property. It was shown by SPOHN [2] that the derived Eulerian triple

$$a(4b^2-c^2)b(4a^2-c^2), \qquad a(4b^2-c^2)4abc, \qquad b(4a^2-c^2)4abc$$

does not satisfy equation (2) except possibly when one of the generators of the equation $a^2+b^2=c^2$ is divisible by 705180. In the present paper we show the derived Eulerian triple does not satisfy equation (2). This is accomplished by the following theorem;

THEOREM. *Let a, b, c satisfy $a^2+b^2=c^2$ with $(a,b,c)=1$. Then the number $k=(a(4b^2-c^2)b(4a^2-c^2))^2+(4abca(4b^2-c^2))^2+(4abcb(4a^2-c^2))^2$ is never a perfect square.*

Proof. Let us assume that k is a perfect square and then we will reach a contradiction.

Since $a^2+b^2=c^2$, we have

$$\begin{aligned}\frac{k}{a^2b^2}&=25a^8+4a^6b^2+214a^4b^4+4a^2b^6+25b^8\\&=(4ab)^4-6(4ab)^2c^4+25c^8\\&=((4ab)^2-3c^4)^2+(4c^4)^2=h^2.\end{aligned}$$

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We observe that since $2\nmid c$ and $(a, b, c)=1$, we have $((4ab)^2-3c^4, 4c^4)=1$; then there exists e, f such that

$$(4ab)^2-3c^4=e^2-f^2, \qquad 4c^4=2ef, \qquad h=e^2+f^2. \qquad (e, f)=1.$$

Since c is odd, then $-3=e^2-f^2 \bmod 4$, and hence e is odd. Then

$$e=c_1^4, \qquad f=2c_2^4, \qquad (c=c_1c_2)$$

and

$$(4ab)^2=e^2+3c^4-f^2=(c_1^4+4c_2^4)(c_1^4-c_2^4).$$

Since $(c_1, c_2)=1$, we have $(c_1^4+4c_2^4, c_1^4-c_2^4)\mid 5$. By a well known result, the number $c_1^4-c_2^4$ is not a perfect square, except in trivial cases which are not acceptable here. So it follows that

$$(3) \qquad c_1^4+4c_2^4=5d_1^2, \qquad c_1^4-c_2^4=5d_2^2, \qquad (4ab=5d_1d_2)$$

or

$$d_1^2=d_2^2+c_2^4, \qquad c_1^4=d_1^2+4d_2^2.$$

We observe that since $(d_1, d_2)=1$ and $2\mid d_2$, then the solution of equation (3) is given by

$$d_1=s^2+t^2, \qquad c_2^2=s^2-t^2, \qquad d_2=2st,$$

and we obtain

$$c_1^4=(s^2+t^2)^2+4(2st)^2=s^4+18s^2t^2+t^4,$$

which is impossible (Pocklington [3]). The proof for the case $(4ab)^2-3c^4<0$ proceeds similarly by interchanging c_1 and c_2. This completes the proof of our theorem.

References

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3. H. C. Pocklington, *Some Diophantine impossibilities*, Proc. Cambridge Phil. Soc., **17** (1914), 110–118.

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