On the positivity of
weak supersolutions of
non-uniformly elliptic equations

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The classical maximum principle for homogeneous, second order,
uniformly elliptic equations implies that non-negative, classical
supersolutions are either positive or vanish identically in the
interior of their domain of definition. This paper is concerned
with an extension of this result to weak supersolutions of non-
uniformly elliptic equations subject to only mild coefficient
restrictions.

This note is concerned with the positivity of weak non-negative
supersolutions of second order elliptic equations subject to only mild
coefficient restrictions. Specifically we consider operators of the form

(1) $$Lu = -D_i\left(a^{ij}D_j u + b^i u\right) + c^i D_i u + du ,$$

with coefficients a^{ij}, b^i, c^i, d $(i, j = 1, \ldots, n)$, measurable
functions on a domain Ω in euclidean n space $\mathbf{R}^n$. We call L
elliptic in Ω if the principal coefficient matrix $A = \left[a^{ij}\right]$ is positive
almost everywhere in Ω . Letting λ and Λ denote the minimum and
maximum eigenvalues of A_s , the symmetric part of A , we further assume
that λ^{-1} and Λ are locally integrable in Ω , together with the
function μ defined by

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(2) $\mu = \alpha_{ij}\left(b^i b^j + c^i c^j\right) + |d|$,

where $A^{-1} = \left[\alpha_{ij}\right]$ is the inverse of A . Finally we assume that there
exists a positive constant K such that

(3) $\left|a^{ij}(x)\xi_i\eta_j\right|^2 \leq K^2\left(a^{ij}(x)\xi_i\xi_j\right)\left(a^{ij}(x)\eta_i\eta_j\right)$

for all $x \in \Omega$, $\xi, \eta \in \mathbb{R}^n$. If A is symmetric, inequality (3) with
$K = 1$ is an immediate consequence of the Cauchy Schwarz inequality. Note
that in the expressions (1), (2), and (3), and in what follows, we are
employing the standard summation convention that repeated indices indicate
summation from 1 to n .

        We now have the following theorem.

        THEOREM. *Let u be a non-negative function in the space $H(A, \mu, \Omega)$*
satisfying the inequality, $Lu \geq 0$, weakly in Ω . Then either u is
positive almost everywhere in Ω or u vanishes almost everywhere in Ω .

        The spaces $H(A, \mu, \Omega)$ are defined in the papers [3], [4]. Under the
stricter hypotheses, $\lambda^{-1} \in L^s(\Omega)$, $\Lambda, \mu \in L^t(\Omega)$, $\frac{1}{s} + \frac{1}{t} < \frac{2}{n}$, Theorem 1
is a consequence of the weak Harnack inequality established in [3]. In
this case we can conclude that either u vanishes almost everywhere in Ω
or else

                          $\underset{\Omega'}{\text{ess inf }} u > 0$

for each domain Ω' strictly contained in Ω . The special case of the
above theorem where A is symmetric, $b^i = c^i = 0$ and $d \geq 0$ was proved
in [1]. The proof given below which depends on an appropriate choice of
test function, is completely different to that in [1].

        Proof. The weak inequality, $Lu \geq 0$, is equivalent to the integral
inequality

(4) $L(u, \varphi) \equiv \int_\Omega \left(a^{ij}D_i\varphi D_j u + b^i u D_i\varphi + c^i \varphi D_i u + du\varphi\right)$

                          ≥ 0 for all $\varphi \geq 0$, $\varphi \in H^0(A, \mu, \Omega)$; (see [3]).

Let us suppose that u vanishes on a subset E of Ω of positive measure and let B be any ball, strictly contained in Ω, such that $B \cap E$ has positive measure. We then choose as a test function in (4),

$$\varphi = \frac{\zeta^2}{u+\varepsilon} \,,$$

where $\zeta \in C_0^1(\Omega)$, $\zeta = 1$ on B, and $\varepsilon > 0$. Since

$$D_i\varphi = \frac{2\zeta D_i\zeta}{u+\varepsilon} - \frac{\zeta^2 D_i u}{(u+\varepsilon)^2} \,,$$

it follows from the local integrability of Λ and μ that $\varphi \in H^0(\Lambda, \mu, \Omega)$ and is accordingly a valid test function in (4). We therefore obtain, on substitution,

$$\int_\Omega \frac{\zeta^2 a^{ij} D_i u D_j u}{(u+\varepsilon)^2} \leq \int_\Omega \left\{ \frac{1}{u+\varepsilon} \left[2\zeta a^{ij} D_i \zeta D_j u + 2\zeta b^i u D_i \zeta + \zeta^2 c_i D_i u + \zeta^2 du \right] - \frac{\zeta^2 b_i u D_i u}{(u+\varepsilon)^2} \right\} .$$

so that, writing

$$w = \log\left(1 + \frac{u}{\varepsilon}\right) \,,$$

we have, by (3),

$$\int_\Omega \zeta^2 a^{ij} D_i w D_j w \leq \int_\Omega K\left\{ 2\zeta \left(a^{ij} D_i \zeta D_j \zeta\right)^{\frac{1}{2}} + \zeta^2 |a^{ij}\left(b^i b^j + c^i c^j\right)|^{\frac{1}{2}} \right\} \left(a^{ij} D_i w D_j w\right)^{\frac{1}{2}}$$

$$+ \int_\Omega \left\{ 2K\zeta \left(a^{ij} D_i \zeta D_j \zeta\right)^{\frac{1}{2}} \left(\alpha_{ij} b^i b^j\right)^{\frac{1}{2}} + d\zeta^2 \right\} .$$

Consequently, by the Schwarz inequality,

$$\int_B |Dw| \leq \left(\int_B \lambda^{-1}\right)^{\frac{1}{2}} \left(\int_B \lambda |Dw|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\int_B \lambda^{-1}\right)^{\frac{1}{2}} \left(\int_\Omega \zeta^2 a^{ij} D_i w D_j w\right)^{\frac{1}{2}}$$

$$\leq C\left\{\int_B \lambda^{-1} \int_\Omega \left(\Lambda |D\zeta|^2 + \mu\zeta^2\right)\right\}^{\frac{1}{2}} \,,$$

where C is a constant depending only on K. Since w vanishes in B

on a set of positive measure, we have, by a variant of the Poincaré inequality (see [2, Theorem 6.4]), that

$$\int_B \log\left(1 + \frac{u}{\varepsilon}\right)$$

is bounded independently of ε. Hence by letting ε tend to zero, we must have $u = 0$ almost everywhere in B. It then follows that $u = 0$ almost everywhere in Ω.   □

By replacing u by $-u$, we see that a non-positive subsolution of the equation $Lu = 0$ must either vanish or be negative almost everywhere in Ω. Furthermore if the inequality

$$d - D_i b^i \geq 0$$

holds weakly in Ω, then a non-constant subsolution (supersolution) can only attain a positive maximum (negative minimum) in Ω on a set of measure zero. The above theorem can thus be viewed as a "weak strong maximum principle". Results of this nature can be applied to the study of eigenfunctions of the operator L (see [1]).

References

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