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HOMEOMORPHISMS WITHOUT THE PSEUDO-ORBIT TRACING PROPERTY

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§ 0. Introduction

Recently, A. Morimoto [5] proved that every isometry of a compact Riemannian manifold of positive dimension has not the pseudo-orbit tracing property, and that if a homeomorphism of a compact metric space has the pseudo-orbit tracing property then $E_\varphi = O_\varphi$ (see § 1 for definition). The purpose of this paper is to show that every distal homeomorphism of a compact connected metric space has not the pseudo-orbit tracing property.

The author benefited from reading the papers by A. Morimoto [5, 6].

§ 1. Definitions

Let $\varphi: X \to X$ be a (self-) homeomorphism of a compact metric space X with distance function d. A sequence of points $\{x_i\}_{i \in (a,b)}$ $(-\infty \leq a < b \leqslant \infty)$ is called a *δ-pseudo-orbit* of φ if $d(\varphi(x_i), x_{i+1}) < \delta$ for $i \in (a, b-1)$. A sequence $\{x_i\}$ is called to be *ε-traced* by $x \in X$ if $d(\varphi^i(x), x_i) < \varepsilon$ holds for $i \in (a, b)$. We say (X, φ) to have the *pseudo-orbit tracing property* (abbrev. P.O.T.P.) if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit of φ can be ε-traced by some point $x \in X$. The system (X, φ) is said to be *minimal* if a φ-invariant closed set K is necessarily $K = \emptyset$ or $K = X$. Let A be a subset of the integer group Z. Then A is *syndetic* if there is a finite subset K of Z with $Z = K + A$. Let $x \in X$. Then x is an *almost periodic point* if $\{n \in Z: \varphi^n(x) \in U\}$ is a syndetic set for all neighborhoods U of x. Let (X, φ) be *distal*, that is, if $\inf_{n \in Z} d(\varphi^n(x), \varphi^n(y)) = 0$ then $x = y$. Then every $x \in X$ is an almost periodic point and the converse is true (p. 36 of [2]). It is clear that every equi-continuous homeomorphism has this property and is hence distal. But the converse does not hold. To check this for example, let T^2 be a 2-dimensional torus

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and define a homeomorphism $\varphi\colon T^2 \to T^2$ by $\varphi(x_1, x_2) = (\alpha + x_1, nx_1 + x_2)$ $((x_1, x_2) \in T^2)$ where $\alpha \in T^1$ and $0 \neq n \in Z$. Then it will be easily checked that φ is distal but not equi-continuous. A point $x \in X$ is said to be *non-wandering* (with respect to φ) if for every neighborhood U of x, there is an $n > 0$ with $U \cap \varphi^n(U) \neq \emptyset$. The set of all nonwandering points is called the *nonwandering set* and denoted by $\Omega(\varphi)$. Since X is compact, we get $\Omega(\varphi) \neq \emptyset$. If in particular (X, φ) is distal, then it is easily proved that $\Omega(\varphi) = X$ since every $x \in X$ is almost periodic. We know (cf. p. 132 of [7]) that there is always a Borel probability measure μ on X which is preserved by φ and φ^{-1}, and (cf. p. 135 of [7]) that if (X, σ) is minimal then $\mu(U) > 0$ for all non-empty open set U.

The set 2^X of all closed non-empty subsets of X will be a compact metric space by the distance function $\bar{d}$ defined by

$$\bar{d}(A, B) = \mathrm{Max}\,\{\mathrm{Max}_{b \in B}\, d(A, b),\ \mathrm{Max}_{a \in A}\, d(a, B)\} \qquad (A, B \in 2^X)$$

where $d(A, b) = \inf_{a \in A} d(a, b)$ (cf. p. 45 of [4]). We denote by $\mathrm{Orb}^{\delta}(\varphi)$ the set of all δ-pseudo-orbits of φ and by $\overline{\mathrm{Orb}^{\delta}(\varphi)}$ the set of all $A \in 2^X$, for which there is $\{x_i\} \in \mathrm{Orb}^{\delta}(\varphi)$ such that $A = \mathrm{Cl}\,\{x_i\colon i \in Z\}$, Cl denoting the closure. Let E_{φ} denote the set of all $A \in 2^X$ such that for every $\varepsilon > 0$ there is $A_{\varepsilon} \in \overline{\mathrm{Orb}^{\varepsilon}(\varphi)}$ with $\bar{d}(A, A_{\varepsilon}) < \varepsilon$. An element A of E_{φ} is called an *extended orbit* of φ. On the other hand, we define $O_{\varphi} = \mathrm{Cl}\,\{O_{\varphi}(x)\colon i \in Z\}$ where $O_{\varphi}(x) = \mathrm{Cl}\,\{\varphi^i(x)\colon i \in Z\}$. We can easily see that E_{φ} is closed in 2^X and that $O_{\varphi} \subset E_{\varphi}$ holds.

§ 2. Results

Throughout this section, X will be a compact metric space with distance function d and φ will be a self-homeomorphism of X.

THEOREM. *Assume that X is connected. If (X, φ) is distal, then (X, φ) has not P.O.T.P.*

LEMMA 1. *If (X, φ) has P.O.T.P., for every $\varepsilon > 0$ and every $x_0 \in \Omega(\varphi)$ there is a point $y \in X$ and an integer $k = k(x_0, \varepsilon) > 0$ such that $O_{\varphi^k}(y) \subset U_{\varepsilon}(x_0)$.*

Proof. Since $x_0 \in \Omega(\varphi)$, for $\delta > 0$ with $\delta < \varepsilon$ there are a point $x \in X$ and an integer $k > 0$ such that x and $\varphi^k(x)$ belong to $U_{\delta/2}(x_0)$. Now, set $x_{nk+i} = \varphi^i(x)$ for $n \in Z$ and $0 \leq i < k$. Obviously, $\{x_i\}_{i \in Z} = \{\cdots, x, \varphi(x), \cdots, \varphi^{k-1}(x), \cdots\} \in \mathrm{Orb}^{\delta}(\varphi)$. Hence we can find a point $y \in X$ such that $d(\varphi^i(y),$

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$x_i) < \varepsilon$ for $i \in Z$. In particular, $d(\varphi^{nk}(y), x_{nk}) < \varepsilon$ and hence $d(\varphi^{nk}(y), x) < \varepsilon$ for $n \in Z$. Therefore we have $O_{\varphi^k}(y) \subset U_\varepsilon(x_0)$.

COROLLARY 1. *Assume that* X *is connected and not one point. If* (X, φ) *is minimal, then* (X, φ) *has not P.O.T.P.*

Proof. Let $\varepsilon = \text{diameter}(X)/3$ and assume that (X, φ) has P.O.T.P. By Lemma 1 we have that for some $x_0 \in X$ there are $y \in X$ and $k > 0$ with $O_{\varphi^k}(y) \subset U_\varepsilon(x_0)$. Since X is connected, $O_{\varphi^k}(y) = O_\varphi(y) = X$ and so diameter $(X) \leqslant 2\varepsilon$. This is a contradiction.

COROLLARY 2. *If* (X, φ) *is minimal, then* $E_\varphi = O_\varphi$.

Proof. It is proved by A. Morimoto that every $A \in E_\varphi$ is φ-invariant ($\varphi(A) = A$). In fact, for every $\varepsilon > 0$ there is $\varepsilon > \varepsilon_1 > 0$ such that $d(\varphi(x), \varphi(y)) < \varepsilon$ when $d(x, y) < \varepsilon_1$. By definition we can find $\{x_i\} \in \text{Orb}^{\varepsilon_1}(\varphi)$ with $\bar{d}(A, \text{Cl}\{x_i\}) < \varepsilon_1$. Set $y_i = \varphi(x_i)$ for $i \in Z$, then $d(y_i, x_{i+1}) < \varepsilon_1$ and so $\bar{d}(\text{Cl}\{x_i\}, \text{Cl}\{y_i\}) < \varepsilon_1$. It is clear that $d(\varphi(y_i), y_{i+1}) < \varepsilon$ for $i \in Z$. Hence, $\{y_i\} \in \text{Orb}^\varepsilon(\varphi)$. Let $A' = \text{Cl}\{x_i\}$. Then $\bar{d}(A', \varphi(A')) < \varepsilon_1$ and since $\bar{d}(A, A') < \varepsilon_1$ we get $\bar{d}(\varphi(A), \varphi(A')) < \varepsilon$. Therefore

$$\bar{d}(\varphi(A), A) < \bar{d}(\varphi(A), \varphi(A')) + \bar{d}(\varphi(A'), A') + \bar{d}(A', A) < 3\varepsilon$$

and so $\bar{d}(\varphi(A), A) = 0$; i.e. $\varphi(A) = A$. Therefore we get $E_\varphi = \{X\} = O_\varphi$.

LEMMA 2. *If* (X, φ) *has P.O.T.P., for every integer* $k > 0$, (X, φ^k) *has also P.O.T.P.*

Proof. For every $\varepsilon > 0$ there is $\delta > 0$ such that $\{x_i\} \in \text{Orb}^\delta(\varphi)$ is ε-traced by a point in X. Take $\{y_i\} \in \text{Orb}^\delta(\varphi)$ and put $x_{nk+i} = \varphi^i(y_n)$ for $n \in Z$ and $0 \leqslant i \leqslant k-1$. Obviously, $\{x_i\} \in \text{Orb}^\delta(\varphi)$. Hence there is $y \in X$ with $d(\varphi^i(y), x_i) < \varepsilon$ for $i \in Z$. In particular, $d(\varphi^k)^n(y), y_n) = d(\varphi^{nk}(y), x_{nk}) < \varepsilon$ for $n \in Z$. This completes the proof of Lemma 2.

LEMMA 3. *Let* (X, φ) *be distal. Then for every* $x \in X$, $(O_\varphi(x), \varphi)$ *is minimal.*

Proof. Since every $x \in X$ is almost periodic under φ, for a neighborhood U of x there is a finite set $K = \{n_1, \cdots, n_k\}$ of Z such that $Z = A + K$ where $A = \{n \in Z: \varphi^n(x) \in U\}$. Hence $O_\varphi(x) = \text{Cl}\{\varphi^n(x): n \in A\} \cup \text{Cl}\{\varphi^{n+n_1}(x): n \in A\} \cup \cdots \cup \text{Cl}\{\varphi^{n+n_k}(x): n \in A\}$. Let $y \in O_\varphi(x)$. Then $O_\varphi(y) \cap U \neq \emptyset$. This implies that $x \in O_\varphi(y)$. Hence $O_\varphi(x) = O_\varphi(y)$.

Remark 1. If (X, φ) is distal and topologically transitive, then it is clearly minimal (by Lemma 3).

We shall now give a proof of the theorem.

Assuming that (X, φ) has P.O.T.P., we shall draw a contradiction. To do this, let $\varepsilon = \text{diameter}(X)/9$. Then there is $\delta > 0$ with $\delta < \varepsilon$ such that every $\{z_i\} \in \text{Orb}^\delta(\varphi)$ is ε-traced by a point of X. Lemma 1 insures us that for $y_0 \in \Omega(\varphi)$ there are $y \in X$ and $k > 0$ with $O_{\varphi^k}(y) \subset U_\varepsilon(y_0)$. Put $\psi = \varphi^k$. Then (X, ψ) has P.O.T.P. (by Lemma 2) and is distal. Since X is connected and compact, we can take a sequence of points $\{p_i\}_{i=1}^N$ in X such that $p_1 = y$, $d(p_i, p_{i+1}) < \delta/2$ for $1 \leqslant i \leqslant N-1$ and such that $\bigcup_{i=1}^N U_\delta(p_i) = X$. Since (X, ψ) is distal, every point of X is almost periodic. Hence for $1 \leqslant i \leqslant N$ there is an integer $c(i) > 0$ such that $d(p_i, \psi^{c(i)}(p_i)) < \delta/2$. Let us put

$$
\begin{aligned}
&x_i = \psi^{-i}(p_1) && (i < 0)\\
&x_i = \psi^i(p_1) && (0 \leqslant i \leqslant c(1)-1)\\
&x_{c(1)+i} = \psi^i(p_2) && (0 \leqslant i \leqslant c(2)-1)\\
&\cdots\\
&x_{c(1)+\cdots+c(N-1)+i} = \psi^i(p_N) && (0 \leqslant i \leqslant c(N)-1)\\
&x_{c(1)+\cdots+c(N)+i} = \psi^i(p_{N-1}) && (0 \leqslant i \leqslant c(N-1)-1)\\
&\cdots\\
&x_{c(1)+2c(2)+\cdots+2c(N-1)+c(N)+i} = \psi^i(p_1) && (i \geqslant 0)\,.
\end{aligned}
$$

Obviously, $\{x_i\}_{i\in Z} \in \text{Orb}^\delta(\psi)$ and $\bar{d}(\text{Cl}\{x_i\}, X) < \delta$. By assumption, there is $z \in X$ with $d(\psi^i(z), x_i) < \varepsilon$ $(i \in Z)$ so that $\bar{d}(O_\psi(z), X) < \delta + \varepsilon < 2\varepsilon$, and in particular

$$
\begin{aligned}
&d(\psi^i(z), \psi^i(p_1)) = d(\psi^i(z), \psi^i(y)) < \varepsilon && (i < 0)\,,\\
&d(\psi^{i+c}(z), \psi^i(p_1)) = d(\psi^{c+i}(z), \psi^i(y)) < \varepsilon && (i \geqslant 0)
\end{aligned}
$$

where $c = c(1) + c(N) + 2\sum_{i=2}^{N-1} c(i)$. This implies that

$$
\begin{aligned}
&\psi^i(z) \in U_\varepsilon(\psi^i(y)) \subset U_\varepsilon(O_\psi(y)) && (i < 0)\,,\\
&\psi^{c+i}(z) \in U_\varepsilon(\psi^i(y)) \subset U_\varepsilon(O_\psi(y)) && (i \geqslant 0)
\end{aligned}
$$

where $U_\varepsilon(O_\psi(y)) = \bigcup_{h\in O_\psi(y)} U_\varepsilon(h)$. Put $O_\psi^-(z) = \text{Cl}\{\psi^i(z): i < 0\}$ and $O_\psi^+(z) = \text{Cl}\{\psi^i(z): i \geqslant 0\}$. Then we have that $O_\psi^-(z) \subset U_\varepsilon(O_\psi(y))$ and $\psi^c O_\psi^+(z) \subset U_\varepsilon(O_\psi(y))$. Since $O_\psi^-(z) \cup O_\psi^+(z) = O_\psi(z)$, by Baire's theorem either $O_\psi^-(z)$ or $O_\psi^+(z)$ has non-empty interior in the set $O_\psi(z)$.

Let μ be a ψ-invariant Borel probability measure of $O_\psi(z)$. Since

$(O_\psi(z), \psi)$ is minimal by Lemma 3, every non-empty open set in $O_\psi(z)$ has μ-positive measure. When the interior of $O_\psi^-(z)$ in $O_\psi(z)$ is non-empty, it is easy to see that $O_\psi^-(z) = \psi O_\psi^-(z)$ and so $O_\psi^-(z) = O_\psi(z)$. Indeed, assume $\psi^{-1}O_\psi^-(z) \subsetneqq O_\psi^-(z)$. Then $V = \bigcap_{k\geqslant 0}\psi^{-k}O_\psi^-(z)$ does not contain the interior of $O_\psi^-(z)$ in $O_\psi(z)$. Hence $\mu(O_\psi^-(z)\backslash V) > 0$. Since $O_\psi^-(z) = \bigcup_{k\geqslant 0}\psi^{-k}\{O_\psi^-(z)\backslash \psi^{-1}O_\psi^-(z)\} \cup V$, we get $\mu(O_\psi^-(z)\backslash\psi^{-1}O_\psi^-(z)) > 0$, thus contradicting $\mu(O_\psi^-(z)) \leqslant 1$. If the interior of $O_\psi^+(z)$ in $O_\psi(z)$ is non-empty; i.e. $\mu(O_\psi^+(z)) > 0$, then it follows that $O_\psi^+(z) = O_\psi(z)$. Obviously $O_\psi(z) = \psi^c O_\psi^+(z)$. In any case we get $O_\psi(z) \subset U_\varepsilon(O_\psi(y))$ so that $O_\psi(z) \subset U_\varepsilon(O_\psi(y)) \subset U_{2\varepsilon}(y_0)$ (because $O_\psi(y) \subset U_\varepsilon(y_0)$). Since $2\varepsilon > \bar{d}(O_\psi(z), X) = \max_{x\in X} d(O_\psi(z), x)$, we have $X = U_{2\varepsilon}(O_\psi(z))$ from which $X = U_{4\varepsilon}(y_0)$; i.e. diameter $(X) \leqslant 8\varepsilon$. This is a contradiction.

Remark 2. We know (Application 2 of [1]) that every (group) automorphism σ of a zero-dimensional compact metric group X has P.O.T.P. If (X, σ) has zero topological entropy (the existence of such automorphisms is known), then we can prove (cf. Lemma 14 of [1]) that X contains a sequence $X = X_0 \supset X_1 \supset \cdots$ of completely σ-invariant normal subgroups such that $\cap X_n$ is trivial and for every $n \geqslant 0$, X_n/X_{n+1} is a finite group. Hence for $x, y \in X$ $(x \neq y)$ there is $n > 0$ such that $xy^{-1} \notin X_n$. Since $\sigma^j(X_n) = X_n$ for all $j \in \boldsymbol{Z}$, we get easily $\sigma^j(xy^{-1}) \notin X_n$ $(j \in \boldsymbol{Z})$, which implies that $d(\sigma^j(x), \sigma^j(y)) \geqslant d(\sigma^j(x)X_n, \sigma^j(y)X_n) > 0$ (the distance function d is a translation invariant metric of X). Since X_n/X_{n+1} is a finite group, we get $\inf_j d(\sigma^j(x), \sigma^j(y)) > 0$; i.e. (X, σ) is distal. Therefore every zero-dimensional automorphism with zero topological entropy is distal and has P.O.T.P. This shows that the assumption of connectedness in the theorem can not drop out.

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