Canad. Math. Bull. Vol. **32** (2), 1989

ON INVARIANT RADICALS

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ABSTRACT. It is shown that the concepts of normal radical as introduced by Jaegermann [2] and of invariant radical as introduced by Anderson, Divinsky and Sulinski [1] are equivalent.

The concept of an invariant radical was introduced by Anderson, Divinsky and Sulinski [1] in relation to 2-graded rings. Given such a ring (R_0, R_1) an ideal A_0 of R_0 is said to be special if $R_1A_0R_1 \subseteq R_0$; A_0 is said to be invariant in R_0 if it is special in every such 2-graded ring. The radical β is said to be invariant if $\beta(R_0)$ is invariant in R_0 for all rings R_0. More generally one may consider G-graded rings where G is any additive group. Such a ring S is a direct sum, as an additive group, of subgroups S_α, $\alpha \in G$, and satisfies the multiplicative conditions $S_\alpha S_\beta \subseteq S_{\alpha+\beta}$. If S is a G-graded ring an ideal A_0 of S_0 is said to be G-special if $S_{-\alpha}A_0S_\alpha \subseteq A_0$ for all $\alpha \in G$; A_0 is said to be G-invariant if it is G-special in every such G-graded ring. The radical β is said to be G-invariant if $\beta(R_0)$ is G-invariant in R_0 for every ring R_0. The radical β is said to be fully invariant if it is G-invariant for all groups G. In this notation invariant, as defined in [1], means $\mathbf{Z}(2)$-invariant. Our purpose in this note is to show that the concepts of invariance and full invariance coincide for radicals and that they coincide with the concept of normal radical as defined by Jaegermann [2].

In [1] it has been shown that for radicals the concept of invariance and the concept of having the exchange property coincide. A radical β is defined to have the exchange property if for every ring A and every idempotent e of A one has $A_{01}\beta(A_{11})A_{10} \subseteq \beta(A_{00})$, where in the Peirce decomposition one has $A_{11} = eAe$, $A_{01} = (1-e)Ae$, $A_{10} = eA(1-e)$ and $A_{00} = (1-e)A(1-e)$. It is not assumed that A is a ring with 1; multiplication by $1-e$ is defined by the usual embedding of A into its Dorroh extension $(1, A)$ or just formally by $(1-e)a = a - ea$, $a(1-e) = a - ae$. We recall that a radical β is normal if for every Morita context ring $\left[\begin{smallmatrix} R & V \\ W & S \end{smallmatrix}\right]$, where R, S are rings, V is an R-S bimodule and W an S-R bimodule, one has $V\beta(S)W \subseteq \beta(R)$. We observe that given a ring A with idempotent e one has a Morita context ring $\left[\begin{smallmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{smallmatrix}\right]$ isomorphic to A. Indeed this is just another way of expressing the Peirce decomposition.

THEOREM. *The following conditions on a radical β are equivalent: (i) β is normal, (ii) β is fully invariant, (iii) β is invariant, (iv) β has the exchange property.*

Received by the editors May 25, 1988.
AMS Subject Classification, 16A21, 16A03.

PROOF. (i) $\Rightarrow$ (ii). Let S be a G-graded ring. For each $\alpha \in G$ we have a Morita context ring $\begin{bmatrix} S_0 & S_{-\alpha} \\ S_\alpha & S_0 \end{bmatrix}$. Since β is normal we have $S_{-\alpha}\beta(S_0)S_\alpha \subseteq S_0$. Thus β is G-invariant. Since this holds for each group G, β is fully invariant. (ii) $\Rightarrow$ (iii) This result is immediate. (iii) $\Leftrightarrow$(iv) This is proved in [1, Theorem 5]. (iii) $\Rightarrow$ (i). Let $A = \begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be a Morita context ring. Then taking $A_0 = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix}$ and $A_1 = \begin{bmatrix} 0 & V \\ W & 0 \end{bmatrix}$ we have a 2-grading of A. From $A_1\beta(A_0)A_1 \subseteq \beta(A_0)$ and $\beta(A_0) = \begin{bmatrix} \beta(R) & 0 \\ 0 & \beta(S) \end{bmatrix}$ it follows that $V\beta(S)W \subseteq \beta(R)$. Therefore β is normal. $\square$

Direct proofs of the equivalence of (i) and (iv) could be given. That (i) $\Rightarrow$ (iv) follows from the Morita context ring associated with the Peirce decomposition. That (iv) $\Rightarrow$ (i) may be deduced by embedding $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ in $\begin{bmatrix} R & V \\ W & (1,S) \end{bmatrix}$ and taking $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

We now consider the question at the conclusion of [1]. As is shown in [5], using results from [2], normal radicals which contain all zero rings are precisely the N-radicals of [4]. These radicals are not only hereditary but also left hereditary and right hereditary. Other hereditary normal radicals do exist. They are classified in [3] as the intersections of hereditary A-radicals and of N-radicals. They also are left and right hereditary. So in this sense the condition that β contain all zero rings may be weakened. However there are A-radicals and hence normal radicals which are not hereditary. So it is not possible to prove the results in [1, Theorem 6] without some condition to replace the condition that all zero rings are in β.

REFERENCES

1. T. Anderson, N. Divinsky, A. Sulinski, *Simple grings and invariant radicals*, Colloq. Math. Soc. Janos Bolyai, vol. **38** (1985), pp. 59–92.

2. M. Jaegermann, *Morita contexts and radicals*, Bull. Acad. Polon. Sci., vol. **20** (1972), pp. 619–623.

3. M. Jaegermann and A. D. Sands, *On normal radicals, N-radicals and A-radicals*, J. Al., vol. **50** (1978), pp. 337–349.

4. A. D. Sands, *Radicals and Morita contexts*, J. Al., vol. **24** (1973), pp. 335–345.

5. ——— *On normal radicals*, J. London Math. Soc. (2), vol. **11** (1975), pp. 361–365.

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