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# THE ABSOLUTE GALOIS GROUP OF A RATIONAL FUNCTION FIELD IN CHARACTERISTIC ZERO IS A SEMI-DIRECT PRODUCT

BY

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ABSTRACT. Let $K$ be a field of characteristic 0 and $t$ an indeterminate. It is shown that the absolute Galois group of $K(t)$ is the semi-direct product of a free profinite group with the absolute Galois group of $K$.

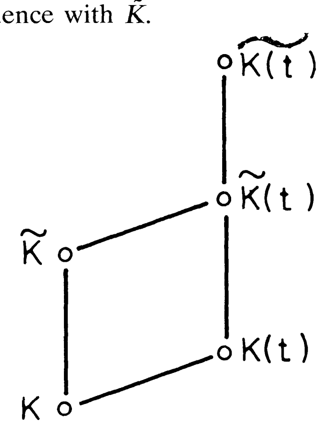
NOTATION. If $K$ is a field, let $\tilde{K}$ denote its algebraic closure, let $\mathcal{G}(K) = \mathrm{Gal}(\tilde{K} \mid K)$ be the profinite group of automorphisms of $\tilde{K}$ which fix $K$; $G(K)$ is called the absolute Galois group of $K$.

Let $K((t^{1/\infty})) = \bigcup_{m=1}^{\infty} K((t^{1/m}))$ be the field of Puiseux series over $K$.

If $K$ is any field and $t$ an indeterminate, we have the exact sequence of profinite groups

$$(*) \qquad \mathcal{G}(\tilde{K}(t)) \overset{i}{\rightarrowtail} \mathcal{G}(K(t)) \overset{\pi}{\twoheadrightarrow} \mathcal{G}(K)$$

where $i$ is the natural inclusion and $\pi$ is the restriction. If $K$ has characteristic 0, Douady proved [1] that $\mathcal{G}(\tilde{K}(t))$ is the free profinite group on a set which is in one-to-one correspondence with $\tilde{K}$.




We shall prove here the following for $K$ of characteristic 0:

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THEOREM. *The exact sequence* $(*)$ *splits, that is there is a continuous homomorphism* $s: \mathcal{G}(K) \to \mathcal{G}(K(t))$ *such that* $\pi \circ s = \text{identity}$.

**Proof.** We identify $\mathcal{G}(K)$ with $\text{Gal}(\tilde{K}(t) \mid K(t))$. Let $\sigma \in \mathcal{G}(K)$. We first extend $\sigma$ to an automorphism $\hat{\sigma}$ of the field $\tilde{K}((t^{1/\infty}))$ of Puiseux series, by putting

$$\hat{\sigma}(a_{-n}t^{-n/m} + a_{-n+1}t^{-n+1/m} + \cdots + a_0 + a_1t^{1/m} + \cdots)$$
$$= \sigma(a_{-n})t^{-n/m} + \sigma(a_{-n+1})t^{-n+1/m} + \cdots + \sigma(a_0) + \sigma(a_1)t^{1/m} + \cdots.$$

Obviously $\sigma \mapsto \hat{\sigma}$ defines an embedding of the group $\mathcal{G}(K)$ into the group $\text{Aut}(\tilde{K}((t^{1/\infty})) \mid K(t))$.

Since $\tilde{K}((t^{1/\infty}))$ is algebraically closed (see for example, Walker [6]), we may consider $\widetilde{K(t)}$ as embedded into $\tilde{K}((t^{1/\infty}))$. Let $s(\sigma)$ be the restriction of $\hat{\sigma}$ to $\widetilde{K(t)}$, so $s(\sigma) \in \mathcal{G}(K(t))$ and this defines the mapping $s: \mathcal{G}(K) \to \mathcal{G}(K(t))$. Clearly $s$ is a group-homomorphism and $\pi \circ s = \text{identity}$.

Now we shall prove that $s$ is continuous. It is equivalent to show that if $\alpha(t) \in \tilde{K}((t^{1/\infty}))$ is algebraic over $K(t)$ there is a finite extension $L \mid K$, $L \subset \tilde{K}$, such that all coefficients of $\alpha$ are in $L$.

This follows from the next proposition:

PROPOSITION. *Let* $K$ *be a field of characteristic* 0 *and let* $L$ *range over the subfields of* $\tilde{K}$ *which are finite extensions of* $K$. *Then* $P = \bigcup_L L((t^{1/\infty}))$ *is an algebraic closure of* $K((t))$.

**Proof.** If $L = K(\gamma) \subset \tilde{K}$ then $L((t^{1/m})) = K((t))(\gamma, t^{1/m})$ hence $P$ is algebraic over $K((t))$. The fact that $P$ is an algebraically closed field may be inferred from a close reading of the constructive proof (in Walker [6]) that $\tilde{K}((t^{1/\infty}))$ is algebraically closed.

However, for the convenience of the reader we give an independent proof that $P$ is algebraically closed.

Let $v$ be the valuation of $P$ defined as follows: if $\alpha(t) \in P$, if $r \in \mathbb{Q}$ is the smallest exponent of the non-zero terms of the Puiseux series $\alpha(t)$, we define $v(\alpha(t)) = r$. The value group of $v$ is $\mathbb{Q}$ and the residue field is $\tilde{K}$. Each subfield $L((t^{1/m}))$ of $P$ is henselian with respect to $v$, so $P$ is also henselian. But, it is known that if a field is henselian with respect to a valuation with divisible value group and algebraically closed residue field of characteristic 0, then the field itself is algebraically closed; this concludes the proof. □

REMARKS. (1) The splitting morphism is uniquely defined by the $K(t)$-embedding of $\widetilde{K(t)}$ into $\tilde{K}((t^{1/\infty}))$.

(2) We would like to know more about the $s$-action of $\mathcal{G}(K)$ on the free profinite group $\mathcal{G}(\tilde{K}(t))$. In an attempt to determine this action we proceed as follows. After identifying each element $a$ of $\tilde{K}$ with the $\tilde{K}$-place having $t - a$ as uniformizing parameter, we consider any finite subset $S$ of $\tilde{K}$; since each such set is contained in a finite subset of $\tilde{K}$ which is invariant under the action of

$\mathcal{G}(K)$, we may assume without loss of generality that $S$ is invariant. Let $\tilde{K}(t)_S$ be the largest subfield of $\widetilde{K(t)}$ containing $\tilde{K}(t)$ and such that all points of $\tilde{K}\setminus S$ are unramified in $\tilde{K}(t)_S \mid \tilde{K}(t)$.

Let $\mathcal{F}_S = \mathrm{Gal}(\tilde{K}(t)_S \mid \tilde{K}(t))$. It is known that $\mathcal{F}_S$ is a free profinite group on a set with the same cardinality as $S$ and $\mathcal{G}(\tilde{K}(t))$ is the inverse limit of the groups $\mathcal{F}_S$ (see Ribes [5]). Since $S$ is invariant under the action of $\mathcal{G}(K)$ then $\tilde{K}(t)_S$ is a Galois extension not only of $\tilde{K}(t)$ but even of $K(t)$, and we have the exact sequence of profinite groups:

$$(**) \qquad \mathcal{F}_S \rightarrowtail \mathrm{Gal}(\tilde{K}(t)_S \mid K(t)) \twoheadrightarrow \mathcal{G}(K)$$

(with the morphisms of inclusion and restriction).

Once more, we have a splitting $s:\mathcal{G}(K)\to \mathrm{Gal}(\tilde{K}(t)_S \mid K(t))$, namely $s(\sigma)$ is the restriction of $\hat{\sigma}$ to $\tilde{K}(t)_S$.

In order to determine the $s$-action of $\mathcal{G}(K)$ on $\mathcal{F}_S$ it suffices to determine the action on a free generator set of $\mathcal{F}_S$. This has been done only in some special cases, cf. [1], [2], [3], [4].

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