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On coinduced corepresentations

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We prove a Frobenius reciprocity type theorem for coinduced comodules.

We shall assume familiarity with the notation and results of the first two chapters of Sweedler [1]. Throughout this paper, C and D will denote coalgebras over a field k, and $f : C \to D$ will denote a coalgebra surjection. W will be a right D-comodule with structure map ψ_W and V will be a right C-comodule with structure map ψ_V. ε_C, ε_D, Δ_C, Δ_D will denote the counits and comultiplications of C and D.

Trushin [2, page 176], has defined the coinduced comodule $W \otimes^D C$ to be the kernel of the map

$$\xi = \left|(1 \otimes f \otimes 1)\left(1 \otimes \Delta_C\right) - \left(\psi_W \otimes 1\right)\right| : M \otimes_k C \to M \otimes_k D \otimes_k C ,$$

with structure map $\psi_{W \otimes^D C} = \left(1 \otimes \Delta_C\right)\big|_{\ker\xi}$.

We shall use comod-C and comod-D to denote the categories of right D- and C-comodules respectively, and F to denote the forgetful functor F : comod-$C \to$ comod-D.

THEOREM. *For every* $W \in$ comod-D *and every* $V \in$ comod-C, *there is an isomorphism*

$$\rho_{VW} : \text{comod-}C\left(V, W \otimes^D C\right) \to \text{comod-}D(FV, W)$$

which is natural in V *and* W.

Proof. If $\phi : V \to W \otimes^D C$ is a morphism of C-comodules, then we set

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$\rho_{VW}(\phi) = \mu \circ (1 \otimes \varepsilon_C) \circ F\phi$, where $\mu : W \otimes k \to W$ is the scalar multiplication map. We also define

$$\lambda_{VW} : \text{comod-}D(FV, W) \to \text{comod-}C(V, W \otimes^D C)$$

by

$$\lambda_{VW}(\chi) = (\chi \otimes 1) \circ \psi_V \quad \text{for} \quad \chi \in \text{comod-}D(FV, W) .$$

We have to check that

(1) $\rho_{VW}(\phi)$ is a D-comodule morphism,

(2) $\operatorname{im} \lambda_{VW}(\chi) \subseteq \ker \xi = W \otimes^D C$,

(3) $\lambda_{VW}(\chi)$ is a D-comodule morphism,

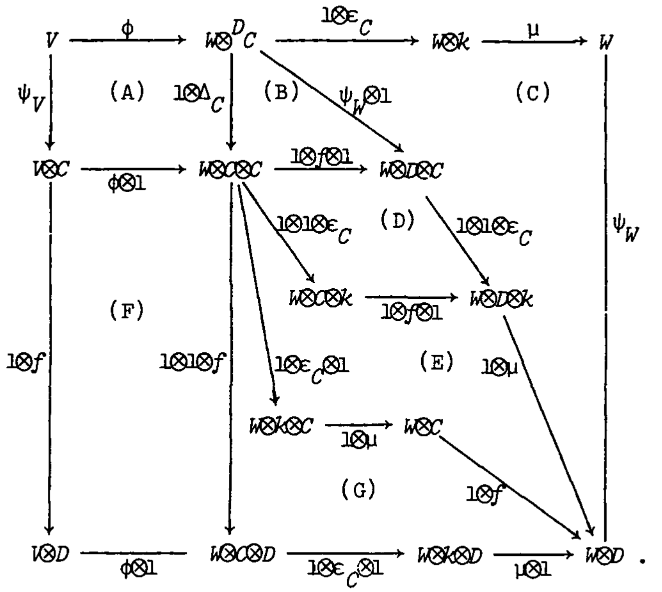
(4) ρ_{VW} and λ_{VW} are inverse to each other, and

(5) ρ_{VW} is natural in V and W .

(1) We must show that

$$\psi_W \circ (\mu \circ (1 \otimes \varepsilon_C) \circ \phi) = [(\mu \otimes 1)(1 \otimes \varepsilon_C \otimes 1)(\phi \otimes 1)] \circ [(1 \otimes f) \circ \psi_V] ;$$

that is, that the outside rectangle of the following diagram commutes. This will follow if we show that subdiagrams (A) to (G) commute.

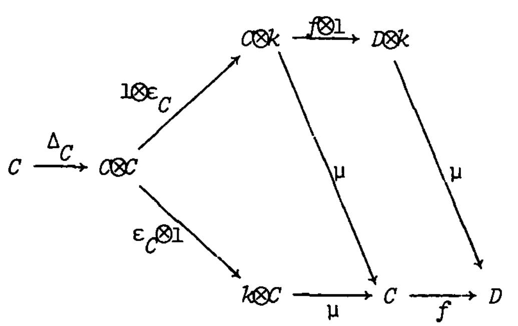


(A) commutes, since ϕ is a C-morphism.

(B) commutes, by definition of $W \otimes^D C$.

Easy calculations show that (C), (D), (F), and (G) commute.

(E) may not commute in general, but



commutes, by definition of coalgebra, so (E) commutes in this context.

(2) We must show that

$$(1 \otimes f \otimes 1)(1 \otimes \Delta_C)(\chi \otimes 1)\psi_V = (\psi_W \otimes 1)(\chi \otimes 1)\psi_V .$$

Now since χ is a D-morphism,

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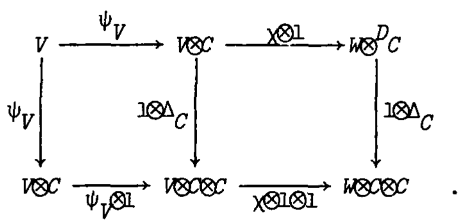
$$(\psi_W \otimes 1)(\chi \otimes 1)\psi_V = (\chi \otimes 1 \otimes 1)(1 \otimes f \otimes 1)(\psi_V \otimes 1)\psi_V$$
$$= (1 \otimes f \otimes 1)(\chi \otimes 1 \otimes 1)(1 \otimes \Delta_C)\psi_V ,$$

since ψ_V is a C-comodule structure map; so

$$(\psi_W \otimes 1)(\chi \otimes 1)\psi_V = (1 \otimes f \otimes 1)(1 \otimes \Delta_C)(\chi \otimes 1)\psi_V ,$$

as required.

(3) Obviously the diagram below commutes, so $\lambda_{VW}(\chi)$ is a C-comodule morphism:



(4) If $\phi \in$ comod-$C(V, W \otimes^D C)$, then

$$\lambda_{VW}(\rho_{VW}(\phi)) = (\mu \otimes 1)(1 \otimes \varepsilon_C \otimes 1)(\phi \otimes 1)\psi_V$$
$$= (\mu \otimes 1)(1 \otimes \varepsilon_C \otimes 1)(1 \otimes \Delta_C)\phi ,$$

since ϕ is a C-comodule morphism,

$$= (\mu \otimes 1)(1 \otimes \varepsilon_D \otimes 1)(1 \otimes f \otimes 1)(1 \otimes \Delta_C)\phi , \text{ since } \varepsilon_D f = \varepsilon_C ,$$
$$= (\mu \otimes 1)(1 \otimes \varepsilon_D \otimes 1)(\psi_W \otimes 1)\phi , \text{ since } \operatorname{im}\phi \subseteq \ker \xi ,$$
$$= \phi , \text{ by a comodule axiom.}$$

If $\chi \in$ comod-$D(FV, W)$, then

$$\rho_{VW}(\lambda_{VW}(\chi)) = \mu(1 \otimes \varepsilon_C)(\chi \otimes 1)\psi_V$$
$$= \mu(1 \otimes \varepsilon_D)(1 \otimes f)(\chi \otimes 1)\psi_V \text{ since } \varepsilon_D f = \varepsilon_C ,$$
$$= \mu(1 \otimes \varepsilon_D)(\chi \otimes 1)(1 \otimes f)\psi_V$$
$$= \mu(1 \otimes \varepsilon_D)\psi_W \chi \text{ since } \chi \text{ is a } D\text{-morphism,}$$
$$= \chi \text{ by a comodule axiom.}$$

(5) Naturality. We must show that the following diagram commutes:

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$$\begin{array}{ccc}
\text{comod-}C\left(V_2, W_2 \otimes^D C\right) & \xrightarrow{\rho_{V_2W_2}} & \text{comod-}D\left(FV_2, W_2\right) \\
\Big\downarrow {\scriptstyle \text{comod-}C(\eta, \theta\otimes 1)} & & \Big\downarrow {\scriptstyle \text{comod-}D(F\eta, \theta)} \\
\text{comod-}C\left(V_1, W_1 \otimes^D C\right) & \xrightarrow[\rho_{V_1W_1}]{} & \text{comod-}D\left(FV_1, W_1\right)
\end{array}$$

where $\eta \in \text{comod-}C\left(V_1, V_2\right)$ and $\theta \in \text{comod-}D\left(W_2, W_1\right)$.

If $\phi \in \text{comod-}C\left[V_2, W_2 \otimes^D C\right]$, then

$$\begin{aligned}
\rho_{V_1W_1}\left(\text{comod-}C(\eta, \theta \otimes 1)(\phi)\right) &= \mu\left(1 \otimes \varepsilon_C\right) \circ F[(\theta \otimes 1) \circ \theta \circ \eta] \\
&= \theta \circ \mu\left(1 \times \varepsilon_C\right) \circ F\phi \circ F\eta \\
&= \theta \circ \left(\mu\left(1 \times \varepsilon_C\right) \circ F\phi\right) \circ F\eta \\
&= \text{comod-}D(F\eta, \theta)\left(\rho_{V_2W_2}(\phi)\right) . \qquad \square
\end{aligned}$$

References

[1] Moss E. Sweedler, *Hopf algebras* (Benjamin, New York, 1969).

[2] David Trushin, "A theorem on induced corepresentations and applications to finite group theory", *J. Algebra* 42 (1976), 173-183.

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