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EXPANSION FORMULAE FOR GENERAL TRIPLE HYPERGEOMETRIC SERIES

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Abstract

The main object of present paper is to obtain a finite summation of Srivastava's general triple hypergeometric series in terms of Kampé de Fériet's double hypergeometric series. A number of finite sums of Kampé de Fériet's double hypergeometric polynomials in terms of different kinds of single hypergeometric polynomials of higher order, are obtained. Some known results of Manocha and Sharma [9], [10], Munot [11], Pathan [12], Qureshi [15], Qureshi and Pathan [16] and Srivastava [26] are deduced as special cases. A result of Pathan [13, page 316 (1.2)] is also corrected here.

1. Introduction

A unification of Lauricella's fourteen triple hypergeometric functions $F_1, \ldots, F_{14}$ [7, pages 113–114], the extended F_k function of Sharma [20, page 613 (2)] and three additional functions H_A, H_B and H_C of Srivastava [22, pages 99–100, see also 23 and 25] was introduced by Srivastava [24, page 428] in the form of a general triple hypergeometric series $F^{(3)}$, defined as

$$F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\ (d)::(e);(e');(e''):(f);(f');(f'');\end{matrix}\quad x, y, z\right]$$
$$= \sum_{m,n,p=0}^{\infty} \frac{[a]_{m+n+p}[b]_{m+n}[b']_{n+p}[b'']_{p+m}}{[d]_{m+n+p}[e]_{m+n}[e']_{n+p}[e'']_{p+m}}$$
$$\cdot \frac{[c]_m[c']_n[c'']_p}{[f]_m[f']_n[f'']_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{1.1}$$

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where (a) is the abbreviation for the array of A parameters $a_1, a_2, \ldots, a_A$. It is to be noted here that there are A of the (a) parameters, B of the (b) parameters and so on. Thus $[a]_m$ is interpreted as

$$[a]_m = \prod_{i=1}^{A} (a_i)_m = \prod_{i=1}^{A} \frac{\Gamma(a_i + m)}{\Gamma(a_i)} \tag{1.2}$$

with similar interpretations for $[b]$, $[c]$, et cetera. It will be assumed throughout the paper that the absence of parameters shown by horizontal dashes mean that there exists no parameter, and in that case, from (1.2), the conventional value of an empty product will be unity, i.e.,

$$\prod_{i=1}^{0} (a_i)_m = 1.$$

Also, numerator parameters like (a), (b), (b'), (b'') et cetera may be zero or negative integers, but the denominator parameters like (d), (e) et cetera are not allowed to be zero or negative integers.

The region of convergence of the above triple power series (1.1) is given in the recent literature [3, page 156; see also 4, page 40].

The main object of the present paper is to obtain the following finite sum of Srivastava's general triple hypergeometric series in terms of Kampé de Fériet's double hypergeometric series [1, page 150]

$$\sum_{n=0}^{m} \frac{(-m)_n}{n!} F^{(3)} \left[\begin{matrix} (a') :: (a); (g'); - : -n, (d); -m+n, (g); (c); \\ (b') :: (b); (h'); - : (e); (h); (f); \end{matrix} \; x, y, z \right]$$

$$= \frac{x^m [a]_m [a']_m [d]_m}{[b]_m [b']_m [e]_m}$$

$$\cdot F \left[\begin{matrix} (g'): (a'+m), (c); -m, (1-e-m), (g); \\ (h'): (b'+m), (f); (1-d-m), (h); \end{matrix} \; z, (-1)^{D-E} \frac{y}{x} \right] \tag{1.3}$$

where each of the parameters $(d) = \{d_1, d_2, \ldots, d_D\}$ need not be an integer. Also the variable x should not be zero. The notation used here for Kampé de Fériet's double series is due to Burchnall and Chaundy [2, page 112].

Some finite sums for hypergeometric polynomials of Krall and Frink, Rainville, Gegenbauer, generalized Rice, Shively's Pseudo-Laguerre, Bateman, Generalized Laguerre and Jacobi are deduced as special cases of (1.3). A number of known results are also obtained from our main result in Section 3.

2. Proof of the main result (1.3)

Consider the series

$$S = \sum_{n=0}^{m} \frac{(-m)}{n!} F^{(3)}\left[\begin{matrix}(a')::(a);(g');-:-n,(d);-m+n,(g);(c); \\ (b')::(b);(h');-:(e);(h);(f);\end{matrix}\ x,y,z\right]$$

$$= \sum_{n=0}^{m} \frac{(-m)_n}{n!} \sum_{p=0}^{\infty} \frac{[a']_p[g']_p[c]_p}{[b']_p[h']_p[f]_p} \frac{z^p}{p!}$$

$$\sum_{r=0}^{n} \sum_{s=0}^{m-n} \frac{[a]_{r+s}[a'+p]_{r+s}[d]_r[g]_s[g'+p]_s}{[b]_{r+s}[b'+p]_{r+s}[e]_r[h]_s[h'+p]_s}(-n)_r(n-m)_s\frac{x^r}{r!}\frac{y^s}{s!}. \qquad (2.1)$$

Now using the finite triple series identity of Srivastava [26, page 95]

$$\sum_{n=0}^{m} \sum_{r=0}^{n} \sum_{s=0}^{m-n} A(m,n,r,s) = \sum_{s=0}^{m} \sum_{r=0}^{m-s} \sum_{n=0}^{m-r-s} A(m,n+r,r,s) \qquad (2.2)$$

and many Pochhammer's identities like

$$\left.\begin{aligned} [a]_m &= (-1)^{mA}[1-a-m]_m \\ [b]_m &= [b]_s[b+s]_{m-s} \\ [d]_{m-s} &= [d]_m\frac{(-1)^{sD}}{[1-d-m]_s} \\ (-n-s)_s &= (-1)^s\frac{(n+s)!}{n!} \end{aligned}\right\}, \qquad (2.3)$$

we have

$$S = m! \sum_{p=0}^{\infty} \frac{[a']_p[g']_p[c]_p}{[b']_p[h']_p[f]_p} \frac{z^p}{p!} \sum_{s=0}^{m} \frac{[a'+p]_s[a]_s[g'+p]_s[g]_s}{[b]_s[b'+p]_s[h]_s[h'+p]_s} \frac{(-y)^s}{s!}$$

$$\cdot \sum_{r=0}^{m-s} \frac{[a'+p+s]_r[a+s]_r[d]_r x^r}{[b+s]_r[b'+p+s]_r[e]_r r!(m-r-s)!} \sum_{n=0}^{m-r-s} (-1)^n \binom{m-r-s}{n}. \qquad (2.4)$$

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Now using the combinatorial identity

$$\sum_{n=0}^{k}(-1)^n\binom{k}{n}=\begin{cases}1, & \text{if } k=0,\\ 0, & \text{if } k=1,2,3,\ldots,\end{cases} \tag{2.5}$$

in (2.4), we find that the value of the r-summation reduces to that of its last term, all other terms vanishing.

So for $r=(m-s)$, we have

$$\begin{aligned} S={}&\frac{[a]_m[a']_m[d]_m x^m}{[b]_m[b']_m[e]_m}\\ &\sum_{p=0}^{\infty}\sum_{s=0}^{m}\frac{[g']_{p+s}[a'+m]_p[c]_p(-m)_s[1-e-m]_s[g]_s}{[h']_{p+s}[b'+m]_p[f]_p[1-d-m]_s[h]_s}\\ &\cdot\frac{(-1)^{s(D-E)}}{s!}\left(\frac{y}{x}\right)^s\frac{z^p}{p!}\end{aligned} \tag{2.6}$$

which reduces to our main result (1.3)

3. Cases of reducibility

When $y=0$ and $A=B=G'=H'=0$ in (1.3), we have

$$\begin{aligned}&\sum_{n=0}^{m}\frac{(-m)_n}{n!}F\left[\begin{matrix}(a')\colon -n,(d);(c);\\ (b')\colon\ (e);(f);\end{matrix}\ x,z\right]\\ &=\frac{[a']_m[d]_m x^m}{[b']_m[e]_m}\,{}_{A'+C}F_{B'+F}\left[\begin{matrix}(a'+m),(c);\\ (b'+m),(f);\end{matrix}\ z\right].\end{aligned} \tag{3.1}$$

When $z=0$ and $A'=B'=G'=H'=0$ in (1.3), we have

$$\begin{aligned}&\sum_{n=0}^{m}\frac{(-m)_n}{n!}F\left[\begin{matrix}(a)\colon -n,(d);-m+n,(g);\\ (b)\colon (e);(h);\end{matrix}\ x,y\right]\\ &=\frac{[a]_m[d]_m x^m}{[b]_m[e]_m}\,{}_{1+E+G}F_{D+H}\left[\begin{matrix}-m,(1-e-m),(g);\\ (1-d-m),(h);\end{matrix}\ (-1)^{D-E}\frac{y}{x}\right].\end{aligned} \tag{3.2}$$

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Setting $A = E = H = 1$ and $B = D = G = 0$ in (3.2), and making suitable adjustment of parameters, using the definition of Jacobi's polynomials [18, page 255 (8)], we have a known result of Munot [11, page 691 (2.1); see also 26, page 95 (7.13)]

$$\sum_{n=0}^{m} \frac{(-b-m)_n(1+b)_{m-n}}{n!(m-n)!} F_2[d; -n, n-m; 1+a, 1+b; x, y] = \frac{(d)_m(-1)^m}{(1+a)_m}(x+y)^m P_m^{(a,b)}\left(\frac{y-x}{y+x}\right), \tag{3.3}$$

where F_2 is Appell's polynomial of second kind [5, page 224 (7)].

In fact, the result (3.3) was obtained by Munot [11] using operational calculus techniques, while Srivastava [26, page 95] derived the same result using method of series manipulation.

On taking $A = B = 0$ in (3.2), we have a known result of Manocha and Sharma [10, page 233 (16)]

$$\sum_{n=0}^{m} \frac{(-m)_n}{n!} {}_{D+1}F_E\left[\begin{matrix} -n, (d); \\ (e); \end{matrix} \; x\right] {}_{G+1}F_H\left[\begin{matrix} -m+n, (g); \\ (h); \end{matrix} \; y\right] = \frac{[d]_m x^m}{[e]_m} {}_{1+E+G}F_{D+H}\left[\begin{matrix} -m, (1-e-m), (g); \\ (1-d-m), (h); \end{matrix} \; (-1)^{D-E}\frac{y}{x}\right], \tag{3.4}$$

which was obtained by the method of fractional derivative.

Taking $D = E = G = H = 1$ in (3.4), we get a known result of Qureshi and Pathan [16, page 180 equation (3.5)].

Setting $A = A' = B' = C = E = H = 1$ and $B = D = F = G = G' = H' = 0$ in (1.3), adjusting parameters and variables suitably and using the definition of Jacobi's polynomials [18, page 255 (8)], we get a known result of Pathan [12, page 59 (2.3)]

$$\sum_{n=0}^{m} \frac{(-m)_n}{n!} F^{(3)}\left[\begin{matrix} d :: a; -; -: -n; -m+n; c; \\ b :: -; -; -: 1+e; 1+h; -; \end{matrix} \; x, y, z\right] = \frac{m!(-)^m (a)_m (d)_m}{(1+h)_m (b)_m (1+e)_m}(y+x)^m {}_2F_1\left[\begin{matrix} d+m, c; \\ b+m; \end{matrix} \; z\right] P_m^{(e,h)}\left(\frac{y-x}{y+x}\right), \tag{3.5}$$

which was obtained by the method of operational calculus.

Again, setting $A' = B' = C = A = E = H = 1$, $B = G' = H' = D = F = 0$, changing n, m into m, n in (1.3) and making suitable adjustment of parameters and variables, we obtain

$$\sum_{m=0}^{n} \frac{(-n)_m}{m!} F^{(3)}\left[\begin{matrix} a :: b; -; -: -m; m-n; c-b; \\ \\ c :: -; -; -: 1+e; 1+h; -; \end{matrix} \frac{x}{1+z}, \frac{-x}{1+z}, \frac{z}{1+z}\right]$$
$$= \frac{(1+e+h)_{2n}(a)_n(b)_n}{(1+e)_n(1+h)_n(1+e+h)_n(c)_n}\left(\frac{x}{1+z}\right)^n {}_2F_1\left[\begin{matrix} a+n, c-b; \\ \\ c+n; \end{matrix} \frac{z}{1+z}\right], \tag{3.6}$$

which is the correct form of the following result of Pathan [13, page 316 (1.2)]

$$\sum_{m=0}^{n} \frac{(-h-n)_m(1+h)_{n-m}}{m!(n-m)!}$$
$$\cdot F^{(3)}\left[\begin{matrix} a :: b; -; -: -m; m-n; c-b; \\ \\ c :: -; -; -: 1+e; 1+h; -; \end{matrix} \frac{x}{1+z}, \frac{-x}{1+z}, \frac{z}{1+z}\right]$$
$$= \frac{(1+e+h)_{2n}(a)_n(b)_n}{(1+e)_n(1+h)_n(1+e+h)_n(c)_n}\left(\frac{x}{1+z}\right)^n$$
$$\cdot {}_2F_1\left[\begin{matrix} a+n, c-b; \\ \\ c+n; \end{matrix} \frac{z}{1+z}\right]. \tag{3.7}$$

Also, in (3.6) taking $z = 0$ and $z = x$ in conjunction with the use of a transformation of Pathan [14, page 372 (1.3)] we may obtain two other correct forms of the results of Pathan [13, page 317 (3.2, 3.4)].

When $A = E = H = 0$ and $B = D = G = 1$ in (3.2), we have a known result of Qureshi [15, page 48 (3.3.13)].

$$\sum_{n=0}^{m} \frac{(-m)_n}{n!} F_3[-n, -m+n; d, g; b; x, y]$$
$$= \frac{(d)_m x^m}{(b)_m} {}_2F_1\left[\begin{matrix} -m, g; \\ \\ 1-d-m; \end{matrix} \frac{-y}{x}\right], \tag{3.8}$$

where F_3 is Appell's polynomial of third kind [5, page 224 (8)].

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When $A = B = D = G = 0$, $E = H = 1$ in (3.2), using the definition of generalized Laguerre's polynomials [18, page 200 (1)] and Jacobi's polynomials [18, page 255 (8)], we get a known result of Manocha and Sharma [9, page 475 (33)]

$$\begin{aligned}\sum_{n=0}^{m} \frac{(-h-m)_n}{(1+e)_n} L_n^{(e)}(x) L_{m-n}^{(h)}(y) \\ = \frac{(-)^m}{(1+e)_m}(x+y)^m P_m^{(e,h)}\left(\frac{y-x}{y+x}\right).\end{aligned} \tag{3.9}$$

The result (3.9) was also obtained by Qureshi and Pathan [16, page 181 equation (3.8)] using the process of confluence [8, page 48 (§3.5)].

Setting $A = E = H = D = 0$, $B = G = 1$ in (3.2), making suitable adjustment of parameters and variables, using definitions of Humbert function Θ_1 [5, page 226 (25)] and thereafter using Krall and Frink's simple Bessel polynomials $y_m(z)$ and generalized Bessel polynomials $y_m(a, b, z)$ [6; see also 18, pages 293–294 (1.2)] and Rainville's polynomials $\phi_m(c, z)$ [18, page 294 (3)], we get three more relations in the form

$$\begin{aligned}\sum_{n=0}^{m} \frac{(-m)_n}{n!} \Theta_1\left[-m+n, -n; 1+m; b; \frac{-z}{2}, x\right] \\ = \frac{x^m}{(b)_m} y_m\left(\frac{z}{x}\right),\end{aligned} \tag{3.10}$$

$$\begin{aligned}\sum_{n=0}^{m} \frac{(-m)_n}{n!} \Theta_1[-m+n, -n; a+m-1; b; z, -xc] \\ = \frac{(-xc)^m}{(b)_m} y_m\left(a, c, \frac{z}{x}\right),\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\sum_{n=0}^{m} \frac{(-m)_n}{n!} \Theta_1[-m+n, -n; c+m; b; y, x] \\ = \frac{m! x^m}{(b)_m (c)_m} \phi_m\left(c, \frac{y}{x}\right).\end{aligned} \tag{3.12}$$

In (3.2), putting $G = H = 2$, $E = D = 0$ and using the definition of generalized Rice polynomials $H_m^{(a,b)}[g, h, v]$ [27, page 73 (1.1); see also 19], we have

$$\begin{aligned}\sum_{n=0}^{m} \frac{(-m)_n}{n!} F\left[\begin{matrix}(a): -n; -m+n, 1+a+b+m, g; \\ (b); -; 1+a, h;\end{matrix}\; x, y\right] \\ = \frac{m! x^m [a]_m}{(1+a)_m [b]_m} H_m^{(a,b)}\left[g, h, \frac{y}{x}\right].\end{aligned} \tag{3.13}$$

Setting $D = E = 0$, $H = G = 1$ in (3.2), making suitable adjustment of parameters and variables and then using the definition of Gegenbauer's polynomials $C_m^g(x)$ [18, page 279 (15)], we have

$$\sum_{n=0}^{m} \frac{(-m)_n}{n!} F\left[\begin{matrix}(a)\colon -n; -m+n, 2g+m; \\ \\ (b); -; g+\frac{1}{2};\end{matrix} \quad x, y\right] = \frac{m! x^m [a]_m}{(2g)_m [b]_m} C_m^g\left(\frac{x-2y}{x}\right). \tag{3.14}$$

Similarly setting $G = E = D = 0$, $H = 1$ in (3.2) and using the definition of Shively's Pseudo-Laguerre polynomials $R_m(g, v)$ [21, page 54 (48)], we have

$$\sum_{n=0}^{m} \frac{(-m)_n}{n!} F\left[\begin{matrix}(a)\colon -n; -m+n; \\ (b)\colon -; m+h;\end{matrix} \quad x, y\right] = \frac{m! x^m (h)_m [a]_m}{(h)_{2m} [b]_m} R_m\left(h, \frac{y}{x}\right). \tag{3.15}$$

When $D = G = E = 0$, $H = 2$ in (3.2), adjusting parameters and variables suitably and then using the definition of Bateman's polynomials $J_m^{(g,h)}(v)$ [17, page 721], we obtain

$$\sum_{n=0}^{m} \frac{(-m)_n}{n!} F\left[\begin{matrix}(a)\colon -n; -m+n; \\ \\ (b)\colon -; g+1, h+\frac{g}{2}+1;\end{matrix} \quad x^2, y^2\right] = \frac{x^{2m+g} [a]_m m! \Gamma(g+1) \Gamma\left(h+\frac{g}{2}+1\right)}{y^g [b]_m \Gamma\left(h+\frac{g}{2}+m+1\right)} J_m^{(g,h)}\left(\frac{y}{x}\right). \tag{3.16}$$

When $D = E = G = H = 1$ in (3.2), adjusting parameters suitably and using the definition of Rainville's polynomials $\psi_m(h, g, x)$ [18, page 302 (ex. 3)], we have

$$\sum_{n=0}^{m} \frac{(-m)_n}{n!} F\left[\begin{matrix}(a)\colon -n, \frac{1}{2}+\frac{1}{2}g; -m+n, \frac{1}{2}-\frac{1}{2}g; \\ \\ (b)\colon h; h;\end{matrix} \quad x, y\right] = \frac{(-1)^m [a]_m}{[b]_m} x^m \psi_m\left(h, g, \frac{y}{x}\right). \tag{3.17}$$

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