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# EXTENSIONS OF McCOY'S THEOREM

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**Abstract.** McCoy proved that for a right ideal $A$ of $S = R[x_1, \dots, x_k]$ over a ring $R$, if $r_S(A) \neq 0$ then $r_R(A) \neq 0$. We extend the result to the Ore extensions, the skew monoid rings and the skew power series rings over non-commutative rings and so on.

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Over a commutative ring $R$, McCoy [**4**, Theorem 2] obtained the following in 1942: $f(x)$ is a zero divisor in $R[x]$ if and only if $f(x)c = 0$ for some non-zero $c \in R$, where $R[x]$ is the polynomial ring with indeterminate $x$ over $R$. But Weiner [**9**] showed that this theorem fails in non-commutative rings.

Based on these results, Nielsen [**6**] called a ring $R$ *right McCoy* when the equation $f(x)g(x) = 0$ implies $f(x)c = 0$ for some non-zero $c \in R$, where $f(x)$, $g(x)$ are non-zero polynomials in $R[x]$. Left McCoy rings are defined similarly. If a ring is both left and right McCoy then the ring is called a *McCoy* ring. Nielsen [**6**, Theorem 2] proved that if a ring $R$ is reversible (i.e. for $a, b \in R$, $ab = 0$ implies $ba = 0$) then $R$ is McCoy.

As stated above, McCoy's theorem fails in non-commutative rings. However McCoy [**5**] proved the following result.

THEOREM †. Let $R$ be a ring and $A$ a right ideal of $S = R[x_1, \dots, x_k]$. If $r_S(A) \neq 0$ then $r_R(A) \neq 0$.

In 2002, Hirano [**3**, Theorem 2.2] proved independently that if for $f(x) \in R[x]$, $r_{R[x]}(f(x)R[x]) \neq 0$ then $r_R(f(x)R[x]) \neq 0$.

On the other hand, McCoy's theorem fails in the formal power series ring $R[[x]]$ over a commutative ring $R$ by [**1**, Example 3] in general. However, Gilmer [**2**] provided several conditions that are sufficient in order that the analogue of McCoy's theorem should be valid in a commutative $R[[x]]$. Such conditions include the reducedness

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and the von Neumann regularity of the total quotient ring, etc. Moreover, Fields [**1**, Theorem 5] proved that if $R$ is a commutative Noetherian ring in which $Q_1 \cap Q_2 \cap \cdots \cap Q_n = 0$ is a shortest primary representation of 0, then $f(x)g(x) = 0$ implies $f(x)c = 0$ for some non-zero $c \in R$.

We extend, in this paper, Theorem † to the Ore extensions of several types, the skew monoid rings and the skew power series rings over non-commutative rings, and so on.

Throughout this paper, $R$ denotes associative ring with identity. We denote the right annihilator of $A$ in $R$ by $r_R(A)$, where $A$ is a subset of an extension of $R$. We assume that $\sigma$ is an automorphism of $R$ and $\delta$ is a $\sigma$-derivation of $R$. Recall that the *Ore extension* $R[x; \sigma, \delta]$ of a ring $R$ is the ring obtained by giving the polynomial ring over $R$ with the new multiplication $xr = \sigma(r)x + \delta(r)$ for any $r \in R$.

THEOREM 1. *Let $R$ be a ring and and $A$ a right ideal of $S = R[x; \sigma, \delta]$. If $r_S(A) \neq 0$ then $r_R(A) \neq 0$.*

*Proof.* Let $g(x) = b_0 + b_1x + \cdots + b_nx^n$ be a non-zero element in $r_S(A)$ with minimal degree. Then $Ag(x) = 0$ and so $f(x)Sg(x) = 0$ for any $f(x) = a_0 + a_1x + \cdots + a_mx^m \in A$. Note that for any $r \in R$,

$$rx^i = x^i\sigma^{-i}(r) - \left(\sum_{s+t=i-1} \sigma^s\delta\sigma^t(\sigma^{-i}(r))\right)x^{i-1}$$
$$- \cdots - \left(\sum_{s+t=i-1} \delta^s\sigma\delta^t(\sigma^{-i}(r))\right)x - \delta^i(\sigma^{-i}(r)).$$

Then we can rewrite $f(x) = c_0 + xc_1 + \cdots + x^mc_m$. Thus we have the following:

$$(c_0 + xc_1 + \cdots + x^mc_m)R(b_0 + b_1x + \cdots + b_nx^n) = 0. \qquad (*)$$

We will show that $f(x)b_j = 0$ for any $0 \leq j \leq n$. If $n = 0$, then we are done. Suppose that $n \geq 1$. From equation $(*)$, we have $c_mb_n = 0$. Then $f(x)R(c_mg(x)) \subseteq f(x)Rg(x) = 0$ and so equation $(*)$ becomes

$$(c_0 + xc_1 + \cdots + x^mc_m)R(c_mb_0 + c_mb_1x + \cdots + c_mb_{n-1}x^{n-1}) = 0.$$

By the choice of $g(x)$, we have $c_mb_0 + c_mb_1x + \cdots + c_mb_{n-1}x^{n-1} = 0$ and so $c_mb_j = 0$ for any $0 \leq j \leq n$. Assume that $c_ib_j = 0$, where $i = t+1, \ldots, m$ and $0 \leq j \leq n$ and that for each $0 \leq i \leq t$, $c_ib_j \neq 0$ for some $j$. Then equation $(*)$ becomes

$$0 = f(x)Rg(x) = (c_0 + xc_1 + \cdots + x^tc_t)R(b_0 + b_1x + \cdots + b_nx^n).$$

Thus we also have $c_tb_n = 0$. Then $f(x)R(c_tg(x)) \subseteq f(x)Rg(x) = 0$ and so $f(x)R(c_tb_0 + c_tb_1x + \cdots + c_tb_{n-1}x^{n-1}) = 0$. By the choice of $g(x)$, we have $c_tb_0 + c_tb_1x + \cdots + c_tb_{n-1}x^{n-1} = 0$ and so $c_tb_j = 0$ for any $0 \leq j \leq n$, which is a contradiction. Consequently $n$ must be zero. Hence $f(x)b_0 = 0$ and therefore $Ab_0 = 0$ with $b_0 \neq 0$. □

COROLLARY 2. *For a ring $R$, let $T$ be $R[x; \sigma]$, $R[x, x^{-1}; \sigma]$ or $R[x; \delta]$ and $A$ a right ideal of $T$. If $r_T(A) \neq 0$ then $r_R(A) \neq 0$.*

Recall that a monoid $G$ is called a *unique product monoid* (simply, *u.p.-monoid*) if any two non-empty finite subsets $A, B \subseteq G$ there exists $c \in G$ uniquely presented in the form $ab$ where $a \in A$ and $b \in B$. The class of u.p.-monoids is quite large and important (see [**7**] and [**8**] for details). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups.

Let $R$ be a ring and $G$ a u.p.-monoid. Assume that $G$ acts on $R$ by means of a homomorphism into the automorphism group of $R$. We denote by $\sigma_g(r)$ the image of $r \in R$ under $g \in G$. The skew monoid ring $R * G$ is a ring which as a left $R$-module is free with basis $G$ and multiplication defined by the rule $gr = \sigma_g(r)g$.

THEOREM 3. *Let $R$ be a ring, $G$ a u.p.-monoid and $A$ a right ideal of $R * G$. If $r_{R*G}(A) \neq 0$ then $r_R(A) \neq 0$.*

*Proof.* Let $\beta = b_0h_0 + b_1h_1 + \cdots + b_nh_n$ be a non-zero element in $r_{R*G}(A)$ with minimal non-zero terms, where $b_j \in R$ and $h_j \in G$. Then $A\beta = 0$ and so $\alpha(R * G)\beta = 0$ for any $\alpha = a_0g_0 + a_1g_1 + \cdots + a_mg_m \in A$ with $a_i \in R$ and $g_i \in G$. Thus we have the following:

$$(a_0g_0 + a_1g_1 + \cdots + a_mg_m)R(b_0h_0 + b_1h_1 + \cdots + b_nh_n) = 0. \qquad (**)$$

We will show that $a_iR\sigma_{g_i}(b_j) = 0$ for any $0 \leq i \leq m$ and $0 \leq j \leq n$. If $n = 0$, then

$$\begin{aligned} 0 &= (a_0g_0 + a_1g_1 + \cdots + a_mg_m)r(b_0h_0) \\ &= a_0\sigma_{g_0}(rb_0)g_0h_0 + a_1\sigma_{g_1}(rb_0)g_1h_0 + \cdots + a_m\sigma_{g_m}(rb_0)g_mh_0. \end{aligned}$$

By [**7**, Lemma 1, p.119], $g_ih_0 \neq g_jh_0$ if $i \neq j$. Thus $a_iR\sigma_{g_i}(b_0) = 0$. Suppose that $n \geq 1$. Since $G$ is a u.p.-monoid, there exist $g_p, h_q$ such that $g_ph_q$ is uniquely presented by considering two subsets $A = \{g_0, g_1, \ldots, g_m\}$ and $B = \{h_0, h_1, \ldots, h_n\}$ of $G$. After reordering if necessary, we may assume that $p = m$ and $q = n$. Then from equation $(**)$, we have $a_mR\sigma_{g_m}(b_n) = 0$. Since $\sigma_{g_m}$ is an automorphism of $R$, $\sigma_{g_m}^{-1}(a_m)Rb_n = 0$. Now for any $s \in R$, $\alpha R(\sigma_{g_m}^{-1}(a_m)s\beta) \subseteq \alpha R\beta = 0$ and so $\alpha R(\sigma_{g_m}^{-1}(a_m)s\beta) = 0$, where $\sigma_{g_m}^{-1}(a_m)s\beta = \sigma_{g_m}^{-1}(a_m)sb_0h_0 + \sigma_{g_m}^{-1}(a_m)sb_1h_1 + \cdots + \sigma_{g_m}^{-1}(a_m)sb_{n-1}h_{n-1}$. By the choice of $\beta$, $\sigma_{g_m}^{-1}(a_m)s\beta = 0$, and hence $a_mR\sigma_{g_m}(b_j) = 0$ for any $0 \leq j \leq n$. After reordering if necessary, assume that $a_iR\sigma_{g_i}(b_j) = 0$, where $i = t + 1, \ldots, m$ and $0 \leq j \leq n$ and that for each $0 \leq i \leq t$, $a_iR\sigma_{g_i}(b_j) \neq 0$ for some $j$. Then from equation $(**)$, we have $\alpha R\beta = (a_0g_0 + a_1g_1 + \cdots + a_tg_t)R(b_0h_0 + b_1h_1 + \cdots + b_nh_n) = 0$. Since $G$ is an u.p.-monoid, there exist $p, q$ with $0 \leq p \leq t$ and $0 \leq q \leq n$ such that $g_ph_q$ is uniquely presented by considering two subsets $A = \{g_0, g_1, \ldots, g_t\}$ and $B = \{h_0, h_1, \ldots, h_n\}$ of $G$. After reordering if necessary, we may assume that $p = t$ and $q = n$. Then $a_tR\sigma_{g_t}(b_n) = 0$ and so $\sigma_{g_t}^{-1}(a_t)Rb_n = 0$. Hence

$$\begin{aligned} 0 &= \alpha R(\sigma_{g_t}^{-1}(a_t)s\beta) \\ &= \alpha R(\sigma_{g_t}^{-1}(a_t)sb_0h_0 + \sigma_{g_t}^{-1}(a_t)sb_1h_1 + \cdots + \sigma_{g_t}^{-1}(a_t)sb_{n-1}h_{n-1}). \end{aligned}$$

By choice of $\beta$, we have $\sigma_{g_t}^{-1}(a_t)sb_0h_0 + \sigma_{g_t}^{-1}(a_t)sb_1h_1 + \cdots + \sigma_{g_t}^{-1}(a_t)sb_{n-1}h_{n-1} = 0$ and hence $a_tR\sigma_{g_t}(b_j) = 0$ for any $0 \leq j \leq n$, which is a contradiction. Consequently $n$ must be zero. Hence we have $\alpha b_0 = 0$, and therefore $Ab_0 = 0$ with $b_0 \neq 0$. □

By [**1**, Example 3], McCoy's theorem fails in the formal power series ring $R[[x]]$ over a commutative ring $R$. However, Gilmer [**2**] proved that a commutative ring satisfies

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McCoy's theorem for the formal power series ring case, when it is reduced (i.e. a ring with no non-zero nilpotent elements).

We here show that Theorem † holds for the skew power series rings and the skew Laurent power series rings over semi-prime rings, noting that Theorem † does not hold for the formal power series ring case in general.

LEMMA 4. *Let $R$ be a semi-prime ring. Then for $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x;\sigma]]$, $f(x)R[[x;\sigma]]g(x) = 0$ if and only if $a_i R\sigma^{i+t}(b_j) = 0$ for all $t, i, j \geq 0$.*

*Proof.* It is enough to show the necessity. Suppose that $f(x)R[[x;\sigma]]g(x) = 0$, equivalently, $f(x)x^t rg(x) = 0$ for any $r \in R$ and integer $t \geq 0$. So we have the following:

$$a_0\sigma^t(rb_0) = 0, \tag{0}$$

$$a_0\sigma^t(rb_1) + a_1\sigma^{t+1}(rb_0) = 0, \tag{1}$$

$$\cdots$$

$$a_0\sigma^t(rb_n) + a_1\sigma^{t+1}(rb_{n-1}) + \cdots + a_n\sigma^{t+n}(rb_0) = 0. \tag{$n$}$$

From equation (0), $a_0\sigma^t(rb_0) = 0$. In equation (1), we replace $r$ by $rb_0s$ for any $s \in R$. Then $0 = a_0\sigma^t(rb_0sb_1) + a_1\sigma^{1+t}(rb_0sb_0) = a_1\sigma^{1+t}(rb_0sb_0)$. Thus $a_1R\sigma^{1+t}(b_0)R\sigma^{1+t}(b_0) = 0$. Since $R$ is semi-prime, $a_1R\sigma^{1+t}(rb_0) = 0$ and so $a_1\sigma^{1+t}(rb_0) = 0$ for all $r \in R$. From equation (1), $a_0\sigma^t(rb_1) = 0$ for all $r \in R$. Now suppose that $a_i\sigma^{i+t}(rb_j) = 0$ for all $t \geq 0$ and $0 \leq i + j \leq n - 1$. In equation ($n$), we first replace $r$ by $rb_0s$. Then $a_n\sigma^{n+t}(rb_0sb_0) = 0$ and so $a_n\sigma^{n+t}(rb_0) = 0$ by the same method as above. So we have

$$a_0\sigma^t(rb_n) + a_1\sigma^{1+t}(rb_{n-1}) + \cdots + a_{n-1}\sigma^{n-1+t}(rb_1) = 0. \tag{$n'$}$$

Next, we replace $r$ by $rb_1s$ for any $s \in R$ in equation ($n'$). Then $a_{n-1}\sigma^{n-1+t}(rb_1) = 0$ using $R$ is semi-prime. Continuing this process, we have $a_i\sigma^{i+t}(rb_j) = 0$ for all $t \geq 0$ and $0 \leq i + j \leq n$. By induction, we have $a_i\sigma^{i+t}(rb_j) = 0$ and therefore $a_iR\sigma^{i+t}(b_j) = 0$ for all $k, i, j \geq 0$. □

We also have the same result as Lemma 4 for the skew Laurent power series ring $R[[x, x^{-1};\sigma]]$, using a slightly modified method. Now we have the following.

THEOREM 5. *Let $R$ be a semi-prime ring and $A$ a right ideal of $T = R[[x;\sigma]]$ or $T = R[[x, x^{-1};\sigma]]$. If $r_T(A) \neq 0$ then $r_R(A) \neq 0$.*

*Proof.* It is enough to show the skew power series ring case. Let $0 \neq g(x) = \sum_{j=0}^{\infty} b_j x^j \in r_T(A)$. Then $Ag(x) = 0$ and so $f(x)Tg(x) = 0$ for any $f(x) = \sum_{i=0}^{\infty} a_i x^i \in A$. By Lemma 4, we have $a_iR\sigma^{i+t}(b_j) = 0$ for any integers $t, i, j \geq 0$. Then $f(x)b_j = 0$ and therefore $Ac = 0$, where $c = b_j$ for any non-zero $b_j$. □

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