J. Austral. Math. Soc. (*Series A*) 34 (1983), 55–59

INVARIANT MEASURES FOR PIECEWISE LINEAR FRACTIONAL MAPS

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(Received 28 July 1981)

Communicated by A. J. van der Poorten

Abstract

Let T: $[0, 1] \to [0, 1]$ be a map which is given piecewise as a linear fractional map such that $T0 = T1 = 0$ and $T'0 < 1$. Then T is ergodic and admits an invariant measure which can be calculated explicitly.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 28 D 05, 10 K 10.

1. Introduction

Recent years have seen some interest in the study of maps T: $[0, 1] \to [0, 1]$ with the following property: There is a number α, $0 < \alpha < 1$ such that

(i) $T0 = 0$, $T\alpha = 1$, $T1 = 0$

(ii) T is increasing on $[0, \alpha]$

(iii) T is decreasing on $[\alpha, 1]$.

In this paper we consider the case where T is given piecewise as a linear fractional map. One sees easily that in this case T is given by

$$Tx = T_0 x = \frac{\alpha x}{\alpha p + (\alpha - p)x} \quad \text{for } 0 \leqslant x \leqslant \alpha,$$

$$Tx = T_1 x = \frac{q(1-\alpha) - q(1-\alpha)x}{q - q\alpha - \alpha + (1 - q + q\alpha)x} \quad \text{for } \alpha \leqslant x \leqslant 1$$

where $p > 0$ and $q > 0$ are real numbers. If $p > 1$ the point $x = 0$ is an attractive fixed point. Therefore we restrict our attention to $0 < p \leqslant 1$. We will show that T

is ergodic with respect to Lebesgue measure and an invariant measure is given explicitly by the following

THEOREM. *Let*

$$S_0 x = \frac{\alpha - p + \alpha x}{\alpha p},$$

$$S_1 x = \frac{1 - q + q\alpha - q(1-\alpha)x}{q - q\alpha - \alpha + q(1-\alpha)x}.$$

Let β, γ, δ *be defined by the equations* $S_0\delta = \delta$, $S_1\gamma = \delta$, $S_0\beta = \gamma$ *then* $S_1\beta = \gamma$ *and a density of an invariant measure with respect to* T *is given as follows*:

Case (a): *If* $\gamma \neq \delta$ *then*

$$f(x) = \left| \frac{1}{x + 1/\gamma} - \frac{1}{x + 1/\delta} \right|.$$

Case (b): *If* $\beta = \gamma = \delta \neq 0$ *then*

$$f(x) = \frac{1}{(x + 1/\beta)^2}.$$

Case (c): *If* $\beta = \gamma = \delta = 0$ *then*

$$f(x) \equiv 1.$$

REMARK. If $\delta = 0$ then the formula for f reduces to

$$f(x) = \frac{1}{x + 1/\gamma}.$$

If $\delta = \infty$ then f is given by

$$f(x) = \left| \frac{1}{x + 1/\gamma} - \frac{1}{x} \right|.$$

In this case the measure defined by f is σ-finite.

2. Proof of the theorem

The equation $S_0\delta = \delta$ gives

$$\delta = \frac{\alpha - p}{\alpha(p-1)}.$$

This shows that $\delta = 0$ corresponds to $\alpha = p$, that is, T_0 is a straight line. The special case $\delta = \infty$ corresponds to $p = 1$. In this case $x = 0$ is a fixed point of slope 1 for T_0.

https://doi.org/10.1017/S1446788700019753 Published online by Cambridge University Press

Solving next $S_1\gamma = \delta$ we obtain

$$\gamma = \frac{\alpha(1+pq) - qp}{pq(1-\alpha)}.$$

The solution of $S_0\beta = \gamma$ and $S_1\beta = \gamma$ gives the same value

$$\beta = \frac{\alpha^2(1+q+qp) + \alpha(-2qp-q) + pq}{q\alpha(1-\alpha)}.$$

We assume first that $\gamma < \delta$. We denote by V_0: $[0,1] \to [0,\alpha]$, V_1: $[0,1] \to [\alpha,1]$ the inverse branches of T. The map S: $[\gamma,\delta] \to [\gamma,\delta]$ defined as $Sx = S_0x$ on $[\beta,\delta]$ and $Sx = S_1x$ on $[\gamma,\beta]$ has the inverse branches U_0: $[\gamma,\delta] \to [\beta,\delta]$, U_1: $[\gamma,\delta] \to [\gamma,\beta]$.

We note that S is the dual algorithm to T in the sense of Schweiger [3] (see also Tanaka-Ito [6] and Nakada [2] for a similar approach to some continued fraction like algorithms). Next we define

$$f(x) = \int_\gamma^\delta \frac{dy}{(1+xy)^2}.$$

The essential property of the kernel $(1+xy)^{-2}$ now is

$$\frac{|V_i'x|}{(1+(V_ix)y)^2} = \frac{|U_i'y|}{(1+x(U_i'(y)))^2}, \qquad i = 0,1.$$

Therefore

$$\begin{aligned} f(V_0x)|V_0'x| + f(V_1x)|V_1'x| &= \int_\gamma^\delta \frac{|V_0'x|dy}{(1+(V_0x)y)^2} + \int_\gamma^\delta \frac{|V_1'x|dy}{(1+(V_1x)y)^2} \\ &= \int_\gamma^\delta \frac{|U_0'y|dy}{(1+x(U_0y))^2} + \int_\gamma^\delta \frac{|U_1'y|dy}{(1+x(U_1y))^2} \\ &= \int_\gamma^\delta \frac{dz}{(1+xz)^2} + \int_\gamma^\beta \frac{dz}{(1+xz)^2} = f(x). \end{aligned}$$

This shows that f is an invariant density. A similar discussion applies to $\delta < \gamma$.

Now let $\beta = \gamma = \delta$. Then one calculates that this is equivalent to

$$\alpha^2\left(1 + \frac{p-1}{p^2q}\right) - 2\alpha + 1 = 0 \quad \text{resp.} \quad \left(\frac{\alpha-1}{\alpha}\right)^2 = \frac{1-p}{p^2q}.$$

If $\beta \neq 0$ a heuristic argument shows that

$$f(x) = \lim_{n\to 0} \frac{1}{n}\left(\frac{1}{x+1/\beta+n} - \frac{1}{x+1/\beta}\right)$$

should give an invariant density. Now let

$$f(x) = \frac{1}{(x + 1/\beta)^2}.$$

Then a calculation gives

$$f(V_0 x)|V_0' x| + f(V_1 x)|V_1' x| = \frac{p}{(x + 1/\beta)^2} + \frac{1}{q(1 + \beta)^2 (x + 1/\beta)^2}.$$

But the condition $pq(1 + \beta)^2 + 1 = q(1 + \beta)^2$ can be seen as equivalent to the condition mentioned before if one inserts

$$\beta = \frac{\alpha + \alpha pq - pq}{pq(1 - \alpha)}.$$

If $\beta = 0$ then $\alpha = p$ and $q(1 - \alpha) = 1$. In this case T is piecewise linear. Therefore Lebesgue measure is invariant.

3. T is ergodic

We define

$$w(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \left|(V_{\varepsilon_1} \circ V_{\varepsilon_n} \circ \cdots \circ V_{\varepsilon_n})'\right|$$

for any sequence $\varepsilon_i = 0, 1$ and $1 \leqslant i \leqslant n$. Then

$$w(\varepsilon_1, \varepsilon_n, \ldots, \varepsilon_n; x) = \frac{A_n}{(B_n + C_n x)^2}.$$

The numbers A_n, B_n, C_n can be calculated by recursion relations. For our purpose it is sufficient to note that

$$\frac{C_{n+1}}{B_{n+1}} = U_i\left(\frac{C_n}{B_n}\right) \quad \text{if } \varepsilon_{n+1} = i.$$

Let us consider first the case $p < 1$. Then δ is an attractive fixed point for U_0. We note that $\delta > -1$. The function U_1 has a fixed point $\eta > -1$. One finds

$$\eta = \frac{-2q + 2q\alpha + \alpha + \sqrt{\alpha^2 + 4q(\alpha - 1)^2}}{2q(1 - \alpha)}.$$

Therefore

$$|U_1'(\eta)| = \frac{1}{q(1 + \eta)^2} < 1.$$

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Hence η is also attractive. Since the starting values for C_1/B_1 are given by

$$\frac{C_1}{B_1} = \frac{p - \alpha}{\alpha} > -1 \quad \text{if } \varepsilon_1 = 0,$$

$$\frac{C_1}{B_1} = \frac{1 - q + q\alpha}{q(1 - \alpha)} > -1 \quad \text{if } \varepsilon_1 = 1$$

the sequence C_n/B_n is bounded by a constant M, say. Therefore

$$\frac{w(\varepsilon_1, \ldots, \varepsilon_n; x)}{w(\varepsilon_1, \ldots, \varepsilon_n; y)} \leqslant \left(1 + \frac{C_n}{B_n}\right)^2 \leqslant (1 + M)^2.$$

Hence Rényi's condition (C) applies (see Schweiger [4], Fischer [1]) and T is ergodic. Actually, T is an exact endomorphism. If $p = 1$ then $\delta = \infty$ also is attractive. But since

$$\lim_{x \to \infty} U_1(x) = \frac{-q + q\alpha + \alpha}{q(1 - \alpha)} > -1$$

one sees again that the sequence C_n/B_n is bounded if the last digit satisfies $\varepsilon_n = 1$. Since $\lim_{n \to \infty} V_0^n x = 0$ the jump transformation can be applied (see Schweiger [5]). Therefore T is ergodic. It should be pointed out that we only have $|T'0| \geqslant 1$ but $|T'x| < 1$ may occur for some $x > 0$.

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