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COMPARING GRADED VERSIONS OF THE PRIME RADICAL

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ABSTRACT. Let G be a group with identity e, let λ be a normal supernilpotent radical in the category of associative rings and let λ_{ref} be the reflected radical in the category of G-graded rings. Then for A a G-graded ring, $\lambda_{\mathrm{ref}}(A)$ is the largest graded ideal of A whose intersection with A_e is $\lambda(A_e)$. For $\lambda = B$, the prime radical, we compare $B_{\mathrm{ref}}(A)$ to $B_G(A) = B(A)_G$, the largest graded ideal in $B(A)$.

0. **Introduction**. Given a radical λ in the category of associative rings and ring homomorphisms, one might seek a natural definition for a graded version of λ in the category of G-graded rings and grade-preserving homomorphisms, G a given group. One way of defining a graded version of λ would be to consider $\lambda(A)_G$, the largest graded ideal contained in $\lambda(A)$, A a G-graded ring. Another possibility would be to consider the largest graded ideal I of A such that $I \cap A_e = \lambda(A_e)$, A_e the identity graded component of A.

The first section of this note contains some necessary background material and definitions. In the second section we note that if λ is a normal radical, then $\lambda_{\mathrm{ref}}(A) \cap A_e = \lambda(A_e)$, where λ_{ref} is the reflected graded radical of [2]. If, as well, λ is supernilpotent (for example λ the Jacobson, Levitzki or prime radical), then $\lambda_{\mathrm{ref}}(A)$ is the largest graded ideal I of A such that $I \cap A_e = \lambda(A_e)$.

In the third section we study graded versions of the prime radical, namely $B(A)_G$, the largest graded ideal in $B(A)$, and $B_{\mathrm{ref}}(A)$, the reflected radical of [2] and the largest graded ideal of A whose intersection with A_e is $B(A_e)$ by the results of the preceding section. For G finite, $B_{\mathrm{ref}} = B_G$; therefore we focus our discussion on rings graded by an infinite group. An example shows that $B_{\mathrm{ref}}(A)$ may properly contain $B_G(A)$ (in fact $B_{\mathrm{ref}}(A) \cap A_e$ may properly contain $B(A)_G \cap A_e$) even if G is locally finite and A is strongly G-graded. We apply the main theorem of the second section to obtain an analogue to a theorem of Cohen and Montgomery for infinite G, show by examples that the implications in our theorem cannot be strengthened, and discuss conditions which ensure that $B_{\mathrm{ref}}(A) = B_G(A)$.

1. **Preliminaries.** Throughout, G will denote a group with identity e, and A a G-graded ring, not necessarily with identity. Unless otherwise stated, ideal means two-sided ideal.

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The G-grading on A is called nondegenerate if for any $g \in G$ and any $0 \neq a_g \in A_g$, $a_g A_{g^{-1}} \neq 0$ and $A_{g^{-1}} a_g \neq 0$. By [3, Lemma 2.8], if A has nondegenerate G-grading, then, for any nonzero left ideal L of A, $L_e \neq 0$, and for any nonzero right ideal R of A, $R_e \neq 0$.

If A does not have an identity, then, as in [2], let A^1 be the Dorroh extension of A and give A^1 a G-grading by $A^1_e = \{(a,n) : a \in A_e, n \in \mathbf{Z}\}$, $A^1_g = \{(b,0) : b \in A_g\}$ for $g \neq e$. Then A is a graded ideal in A^1 and $A^1/A \simeq A^1_e/A_e \simeq \mathbf{Z}$ with trivial G-grading.

In [1, Definition 2.1], $A \# G^*$, the generalized smash product of A and G, was defined to be the free left A-module $\oplus_{g\in G} A p_g$ with multiplication defined for elements ap_g and bp_h by $(ap_g)(bp_h) = ab_{gh^{-1}}p_h$, and extended to general elements of $A \# G^*$ by linearity. For λ a radical in the category of associative rings, the reflected radical λ_{ref} in the category of G-graded rings and grade preserving ring homomorphisms was defined in [2] to be $\lambda_{\mathrm{ref}}(A) = \{a \in A : ap_g \in \lambda(A \# G^*) \text{ for all } g \in G\}$. It is shown in [2] that $\lambda_{\mathrm{ref}}(A)$ is a graded ideal of A and that $\lambda_{\mathrm{ref}}(A) \# G^* = \lambda(A \# G^*)$.

Recall that a radical λ is called normal [7, Theorem 2] if the following hold.

i) For any idempotent $f = f^2 \in S$, $\lambda(fSf) = f\lambda(S)f$.

ii) If I is an ideal of a ring S such that $S/I \simeq \mathbf{Z}$, the ring of integers, then $\lambda(I) = \lambda(S) \cap I$.

Also a radical λ is called supernilpotent if $\lambda(S) = S$ for all nilpotent rings S.

LEMMA 1.1. *λ is normal iff (i) above holds along with*

(ii′). If I is an ideal of S such that S/I is a direct sum of copies of $\mathbf{Z}$, then $\lambda(I) = \lambda(S) \cap I$.

Also λ normal implies

(iii). If J is a graded ideal of a graded ring A such that $A/J \simeq \mathbf{Z}$, trivially graded, then $\lambda_{\mathrm{ref}}(J) = \lambda_{\mathrm{ref}}(A) \cap J$.

PROOF. Suppose λ is a normal radical, and S/I is a direct sum of copies of $\mathbf{Z}$. If $\lambda(\mathbf{Z}) \neq 0$, then by [6, Theorem 1.3], λ is supernilpotent, and therefore hereditary by [5, Theorem 2]. (Recall that a radical λ is called hereditary if $\lambda(I) = \lambda(S) \cap I$ for any ideal I of a ring S.) If $\lambda(\mathbf{Z}) = 0$, then $\lambda(S/I) = 0$ and $\lambda(S) \subseteq I$. Hence $\lambda(S) \subseteq \lambda(I)$, so that $\lambda(S) = \lambda(I)$.

To see that λ normal implies (iii), let A be a graded ring and J a graded ideal such that $A/J \simeq \mathbf{Z}$ with trivial grading. Then,

$$\lambda_{\mathrm{ref}}(J) \# G^* = \lambda(J \# G^*) = (J \# G^*) \cap \lambda(A \# G^*) \text{ by } (ii')$$
$$= (J \# G^*) \cap (\lambda_{\mathrm{ref}}(A) \# G^*) = (J \cap \lambda_{\mathrm{ref}}(A)) \# G^*,$$

and the statement follows. ∎

2. **The reflected radical of a normal radical**. Throughout this section, λ will denote a normal radical.

PROPOSITION 2.1. *For A a G-graded ring and λ a normal radical,* $\lambda_{\mathrm{ref}}(A)\cap A_e = \lambda(A_e)$.

PROOF. If A has an identity, then the proof follows as in [1, Corollary 3.3]. Otherwise, embed A as a G-graded ideal in A^1. Then by Lemma 1.1,

$$\begin{aligned}\lambda_{\mathrm{ref}}(A)\cap A_e &= (\lambda_{\mathrm{ref}}(A^1)\cap A)\cap A_e, \text{ since } A^1/A \simeq \mathbf{Z}, \text{ trivially graded}\\ &= \lambda(A^1_e)\cap A_e \text{ since } A^1 \text{ has a 1}\\ &= \lambda(A_e), \text{ since } A^1_e/A_e \simeq \mathbf{Z}.\end{aligned}$$

■

COROLLARY 2.2. *$\lambda(A_e)$ is G-invariant where G acts on the lattice of ideals of A_e by* ${}^gI = A_gIA_{g^{-1}}$. ■

Note that Sands [12] has recently shown both Corollary 2.2 and the converse; a radical λ is normal if and only if $\lambda(A_e)$ is G-invariant for all groups G and G-graded A. Corollary 2.2 generalizes [10, 1.3.32] to the class of normal radicals and to graded rings, not necessarily strongly graded, and possibly without identity.

LEMMA 2.3. *If A is a G-graded ring, and $A\#G^*$ is semiprime, then the G-grading on A is nondegenerate.*

PROOF. Suppose a_g is a nonzero element of A_g. Then

$$L = \{(na_g + ba_g)p_e : n \in \mathbf{Z},\ b \in A\}$$

is a nonzero left ideal of $A\#G^*$ and therefore $L^2 \neq 0$. Thus there exists a nonzero homogeneous element b of A such that $a_gp_e(ba_gp_e) \neq 0$. Then $b \in A_{g^{-1}}$, and a_gb and ba_g are nonzero. ■

THEOREM 2.4. *If λ is a normal supernilpotent radical, A a G-graded ring, then the following are equivalent:*

i). $\lambda(A\#G^*) = 0$ (or equivalently $\lambda_{\mathrm{ref}}(A) = 0$).
ii). $\lambda(A_e) = 0$ and the G-grading on A is nondegenerate.

PROOF. The implication i) $\Rightarrow$ ii) follows from Proposition 2.1 and Lemma 2.3. Conversely, by Proposition 2.1, ii) implies that $\lambda_{\mathrm{ref}}(A)_e = 0$ and then nondegenerate grading implies that $\lambda_{\mathrm{ref}}(A) = 0$. ■

COROLLARY 2.5. *For λ, A as in the theorem, the G-grading on $A' = A/\lambda_{\mathrm{ref}}(A)$ is nondegenerate.*

PROOF. $\lambda(A'\#G^*) = \lambda_{\mathrm{ref}}(A')\#G^* = 0$. ■

COROLLARY 2.6. *For λ, A as in the theorem, the reflected radical, $\lambda_{\text{ref}}(A)$, is the largest graded ideal I of A such that $I \cap A_e = \lambda(A_e)$.*

PROOF. Let I be the largest graded ideal of A such that $I \cap A_e = \lambda(A_e)$. By Proposition 2.1, $\lambda_{\text{ref}}(A) \subseteq I$. Suppose the inclusion is proper. Let I' be the image of I in $A' = A/\lambda_{\text{ref}}(A)$. Then I' is a nonzero graded ideal of A' but with $I' \cap A'_e = 0$; by Corollary 2.5, this is a contradiction. ∎

The results above yield a further characterization of $\lambda_{\text{ref}}(A)$.

PROPOSITION 2.7. *Let λ be a normal supernilpotent radical and A a G-graded ring. Then*

$$\lambda_{\text{ref}}(A) = \{a \in A : a_g A_{g^{-1}} \subseteq \lambda(A_e) \text{ for all } g \in G\}$$
$$= \{a \in A : A_{g^{-1}} a_g \subseteq \lambda(A_e) \text{ for all } g \in G\}.$$

PROOF. Let $T_1 = \{a \in A : a_g A_{g^{-1}} \subseteq \lambda(A_e) \text{ for all } g \in G\}$, $T_2 = \{a \in A : A_{g^{-1}} a_g \subseteq \lambda(A_e) \text{ for all } g \in G\}$. First we show that $T_1 = T_2$. Since λ is supernilpotent, $A_e/\lambda(A_e)$ is semiprime. But for $a \in T_1, A_{g^{-1}} a_g$ is a left ideal of A_e whose square lies in $\lambda(A_e)$ by Corollary 2.2. Thus $A_{g^{-1}} a_g \subseteq \lambda(A_e)$ and $a \in T_2$. Similarly $T_2 \subseteq T_1$; let T denote $T_1 = T_2$.

T is a graded ideal of A since if $a \in T_g$, $b \in A_h$, then $abA_{(gh)^{-1}} \subseteq aA_{g^{-1}} \subseteq \lambda(A_e)$, so that $ab \in T$. Similarly $ba \in T$.

Let $a \in T \cap A_e$ and let I be the right ideal of A_e, $I = \{na+ab : n \in \mathbf{Z},\ b \in A_e\}$. Then $I^2 \subseteq \lambda(A_e)$ by the definition of T. Since $A_e/\lambda(A_e)$ has no nilpotent ideals, $I \subseteq \lambda(A_e)$; therefore $T \cap A_e \subseteq \lambda(A_e)$. The reverse inclusion is clear. Thus $T \subseteq \lambda_{\text{ref}}(A)$ by Corollary 2.6. Conversely, if $a \in (\lambda_{\text{ref}}(A))_g$, then $aA_{g^{-1}} \subseteq (\lambda_{\text{ref}}(A))_e = \lambda(A_e)$ by Proposition 2.1. ∎

Finally we remark that it is certainly not true for all λ that $\lambda_{\text{ref}}(A)$ is the largest graded ideal with intersection with A_e equal to $\lambda(A_e)$. For example, consider A the infinite cyclic group ring $k[t, t^{-1}]$, k a field, $\mathbf{Z}$-graded in the usual way, and λ either the strongly prime radical s or the Brown-McCoy radical G. (Here $G(A)$ is graded by [8, Theorem 6] and $s(A)$ is graded by [11, Corollary 2].) In both cases, it was shown in [2] that $0 = \lambda(A) = \lambda(A)_G = \lambda(A_e)$ but that $\lambda_{\text{ref}}(A) = A$.

3. **Graded versions of the prime radical for rings graded by an infinite group**. The purpose of this section is to investigate the reflected prime radical B_{ref} of [2] and graded prime radical B_G of [3] for G infinite.

A graded ideal P of a graded ring A is called graded prime if for I, J graded ideals of A, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. In [9] a graded left A-module M is called gr-prime if for every nonzero graded submodule N of M and every graded ideal I of A, $IN = 0$ implies $I \subseteq ann_A(M)$.

DEFINITION 3.1. *The graded prime radical of A, $B_G(A)$, may be defined equivalently as*

i). [3] the intersection of the graded prime ideals of A,
or ii). [9] the intersection of the annihilators of the gr-prime A-modules.

It is straightforward to check that (i) and (ii) above are equivalent. For if M is a *gr*-prime A-module, then $ann_A(M)$ is a graded prime ideal, and if P is a graded prime ideal of A, then A/P is *gr*-prime.

Recall that $B_G(A) = B(A)_G$, the largest graded ideal contained in $B(A)$ [3, Lemma 5.1].

We call a G-graded ring A graded prime if (0) is a graded prime ideal of A, and graded semiprime if $B_G(A) = 0$.

Recall that for G finite, $B_G = B_{\text{ref}}$ [2], but that in general the inclusion $B_G(A) \subseteq B_{\text{ref}}(A)$ may be proper as illustrated by the following example from [2]. (Note that [9, 3.7] is in error.)

EXAMPLE 3.2. [2, Example 2.4] Let $A = k[t]$, the polynomial ring in an indeterminate t over a field k, $\mathbf{Z}$-graded in the usual way. Then A is prime and so graded prime. However, by Corollary 2.6, $B_{\text{ref}}(A) = tA$. ∎

We note that proper inclusion is possible even for G a locally finite group and A a strongly graded ring; this is illustrated by Example 3.3 below.

EXAMPLE 3.3. Let k be a field, and $R = k[X_1, X_2, \ldots]$, the polynomial ring in countably many commuting indeterminates. Let I be the ideal of R generated by the X_i^2, let $S = R/I$, and let x_i be the image of X_i in S. Let G be the permutations of $\{1, 2, 3, \ldots\}$ which leave all but finitely many elements fixed. Then G is a locally finite group acting as a group of automorphisms on S. Note that each of the x_i generates a nilpotent ideal so that $B(S)$ is the ideal generated by the x_i, $i = 1, 2, \ldots$. Now let A be the skew group ring $S * G$. If I is a graded ideal of A, then for arbitrarily large $N \in \mathbf{Z}$, I contains elements $s * g$ where s is a polynomial in the x_i, $i > N$. Hence A is graded prime, so $B_G(A) = 0$. However, by Proposition 2.7, $B_{\text{ref}}(A) = B(S) * G$. ∎

Note that Example 3.3 also shows that we may have $B_G(A) \cap A_e$ properly contained in $B_{\text{ref}}(A) \cap A_e$.

The following theorem is a restatement of Theorem 2.4 with $\lambda = B$, and provides an analogue to [3, Theorem 2.9] for infinite groups G. (An analogue to [3, Theorem 2.10] for infinite groups has been proved by the second author and will appear elsewhere.)

THEOREM 3.4. *Consider the following conditions:*

i) $A \# G^*$ is semiprime.
ii) A_e is semiprime, and the grading on A is nondegenerate.
iii) A is graded semiprime.

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Then (i) is equivalent to (ii), either implies (iii) but the reverse implication does not hold.

PROOF. The implications follow from Theorem 2.4 and the fact that $B_G(A) \subseteq B_{\text{ref}}(A)$. Example 3.3 shows that (iii) does not imply the other conditions even if A is strongly graded and graded prime. ∎

Example 3.2 shows that if the grading is degenerate, A_e may be prime but $A \# G^*$ not semiprime. The next example shows that if the grading is degenerate, A_e may be prime but A not graded semiprime, i.e. (iii) may fail.

EXAMPLE 3.5. Let $R = k[Y]/I$ be the polynomial ring in one indeterminate over a field k mod I, the ideal generated by Y^2. Let y be the image of Y in R; the ideal N of R generated by y is nilpotent. Let $A \subseteq R[X]$ be the subring of polynomials of $R[X]$ whose constant coefficient is in k and whose remaining coefficients are from N. Grade A by $G = \mathbf{Z}$ in the usual way. Then $A_0 = k$ is prime, but $B(A) = B_G(A)$ is the set of all polynomials in A with 0 constant term.

REMARK 3.6. Although, in general, (iii) of Theorem 3.4 does not imply the equivalent conditions (i) and (ii), if for every ideal I of A_e, IA is an ideal of A, then A graded semiprime implies A_e semiprime. For let A be graded semiprime and let I be a nilpotent ideal of A_e. If $IA \neq 0$, then IA is a nonzero graded nilpotent ideal of A. Thus $IA = 0$. But then $AI + I$ is a nonzero graded nilpotent ideal of A. Thus $I = 0$, and A_e is semiprime. If, as well, the grading is nondegenerate, then $B_G(A) = 0$ implies $B_{\text{ref}}(A) = 0$ by Theorem 3.4.

However, Example 3.7 shows that even if IA is an ideal of A for every ideal I of A_e and the grading is nondegenerate, $B_G(A)$ may be properly contained in $B_{\text{ref}}(A)$. (This is because the G-grading on $A/B_G(A)$ may be degenerate.)

EXAMPLE 3.7. Let S be the commutative ring defined by $S = k[X_\alpha : \alpha \in (0,1)]/I$, where k is a field, the X_α are commuting indeterminates, and I is the ideal generated by $\{X_\alpha X_\beta - X_{\alpha+\beta} : \alpha + \beta < 1\} \cup \{X_\alpha X_\beta : \alpha + \beta \geqq 1\}$. Let x_α be the image of X_α in S. $B(S)$ is the union of the nilpotent ideals generated by the x_α, $\alpha \in (0,1)$. This is the nil non-nilpotent Zassenhaus ring of [4, Example 3 p. 19]; note that for any $0 \neq y \in B(S)$, $yB(S) \neq 0$. Now let $A \subset S[t, t^{-1}]$, the infinite cyclic group ring, $A = \{\sum a_i t^i : a_i \in B(S) \text{ for } i < 0\}$. For $G = \mathbf{Z}$, A has a G-grading induced by the grading on the group ring, and this is a nondegenerate (but not strong) grading. Clearly $K = \{\sum a_i t^i : a_i \in B(S) \text{ for all } i\} \subseteq B_G(A)$, and since A/K is isomorphic to the prime ring $k[t]$, $B_G(A) = K$. From Corollary 2.6 or Proposition 2.7, $B_{\text{ref}}(A) = \{\sum a_i t^i : a_i \in B(S) \text{ for } i \leqq 0\}$. ∎

However, we have the following proposition.

PROPOSITION 3.8. *Suppose A is strongly graded and IA is an ideal of A for every ideal I of A_e. Then $B_G(A) = B_{\text{ref}}(A)$.*

PROOF. Since A is strongly graded, so is A/I for any graded ideal I of A. In particular, $A' = A/B_G(A)$ has a strong, and therefore nondegenerate, grading. Now it follows from

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Remark 3.6, replacing A by A', that $B_{\text{ref}}(A') = 0$. Thus $B_{\text{ref}}(A) \subseteq B_G(A)$, and therefore $B_{\text{ref}}(A) = B_G(A)$. ∎

REFERENCES

1. M. Beattie, *A generalization of the smash product of a graded ring*, J. Pure & Appl. Alg. **52** (1988), 219–226.

2. M. Beattie and P. Stewart, *Graded radicals of graded rings*, Acta. Math. Hung., to appear.

3. M. Cohen and S. Montgomery, *Group graded rings, smash products and group actions*, Trans. Amer. Math. Soc. **282** (1984), 237–258, Addendum, Trans. Amer. Math. Soc. **300** (1987), 810–811.

4. N. J. Divinsky, *Rings and Radicals*, University of Toronto Press, 1965.

5. M. Jaegermann, *Morita contexts and radicals*, Bull. Acad. Polon. Sci. **20** (1972), 619–623.

6. M. Jaegermann, *Normal radicals of endomorphism rings of free and projective modules*, Fund. Math. **86** (1975), 237–250.

7. M. Jaegermann, *Normal radicals*, Fund. Math. **95** (1977), 147–155.

8. E. Jespers and E. Puczylowski, *The Jacobson and Brown-McCoy radical of rings graded by free groups*, Comm. Alg., to appear.

9. Liu S.-X. and F. Van Oystaeyen, *Group graded rings, smash products and additive categories*, in Perspectives in Ring Theory, 1988, Kluwer Academic Press, 299–310.

10. C. Nastasescu and F. Van Oystaeyen, *Graded Ring Theory*, North-Holland Mathematical Library 28 (North Holland, Amsterdam, 1982).

11. C. Nastasescu and F. Van Oystaeyen, *The strongly prime radical of graded rings*, Bull. Soc. Math. Belgique, Ser. B. **36** (1984), 243–251.

12. A. D. Sands, On invariant radicals, Canad. Math. Bull. **32** (1989), 255–256.

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