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ON α-POLYNOMIAL REGULAR FUNCTIONS, WITH APPLICATIONS TO ORDINARY DIFFERENTIAL EQUATIONS

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Abstract This research deals with properties of polynomial regular functions, which were introduced in a recent study concerning Wiman–Valiron theory in the unit disc. The relation of polynomial regular functions to a number of function classes is investigated. Of particular interest is the connection to the growth class G_α, which is closely associated with the theory of linear differential equations with analytic coefficients in the unit disc. If the coefficients are polynomial regular functions, then it turns out that a finite set of real numbers containing all possible maximum modulus orders of solutions can be found. This is in contrast to what is known about the case when the coefficients belong to G_α.

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1. Introduction and notation

Let $\mathcal{H}(\mathbb{D})$ be the set of all analytic functions in the unit disc $\mathbb{D}$, and let $a_0, \dots, a_k \in \mathcal{H}(\mathbb{D})$. According to a unit disc counterpart [**18**] of the classical theorem of Wittich, the coefficients $a_0(z), \dots, a_{k-1}(z)$ then belong to the Korenblum space $\mathcal{A}^{-\infty}$ [**23**] if and only if all solutions f of

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0 \tag{1.1}$$

belong to $\mathcal{H}(\mathbb{D})$, and are of finite M-order in the sense that

$$\sigma_M(f) = \limsup_{r\to 1^-} \frac{\log^+ \log^+ M(r,f)}{-\log(1-r)} < \infty.$$

The existing literature contains several papers refining the above result. One possible approach is to assume that $a_j \in G_{\alpha_j}$ for $j = 0, \ldots, k-1$, where the growth class G_α for $\alpha \geqslant 0$ consists of functions $g \in \mathcal{H}(\mathbb{D})$ such that

$$\limsup_{r \to 1^-} \frac{\log M(r, g)}{-\log(1-r)} = \alpha$$

(see [**9**] and the references therein). Note that $\mathcal{A}^{-\infty} = \bigcup_{\alpha \geqslant 0} G_\alpha$. It has been proved in [**9**] that the M-orders for solutions f of (1.1) can be restricted to certain intervals determined by the growth parameters α_j. To the best of our knowledge, determining whether there is a finite list for possible M-orders is still an open problem. This is in contrast to the corresponding plane case, where a finite set of rational numbers, containing all possible orders of solutions, can be obtained from simple arithmetic with the degrees of the polynomial coefficients $a_0(z), \ldots, a_{k-1}(z)$ [**17**].

Meanwhile, a strong version of Wiman–Valiron theory, valid for functions analytic in the unit disc $\mathbb{D}$, was recently developed by Fenton *et al.* [**13, 14**]. The strength of the upgraded Wiman–Valiron theory was demonstrated in [**13**] by finding the possible M-orders of growth for solutions of

$$f'' + a_1(z) f' + a_0(z) f = 0, \tag{1.2}$$

where the coefficients $a_0(z)$ and $a_1(z)$ are α-polynomial regular functions. For $\alpha > 0$, a function $g \in \mathcal{H}(\mathbb{D})$ is called α-polynomial regular, and we write that $g \in \mathcal{P}_\alpha$ if there exists a set $F \subset [0, 1)$ of positive lower density such that

$$\lim_{\substack{|z| \to 1^- \\ |z| \in F}} \frac{\log |g(z)|}{-\log(1-|z|)} = \alpha.$$

For a measurable set $E \subset [0, 1)$, the upper and lower densities are defined as

$$\bar{d}(E) = \limsup_{r \to 1^-} \frac{m(E \cap [r, 1))}{1-r} \quad \text{and} \quad \underline{d}(E) = \liminf_{r \to 1^-} \frac{m(E \cap [r, 1))}{1-r},$$

respectively, where $m(F)$ is the Lebesgue measure of F. The classes $\mathcal{P}_\alpha$ were introduced in [**13**], where $\mathcal{P}_\alpha$ is shown to be non-empty for every $\alpha > 0$.

Our first objective is to study how $\mathcal{P}_\alpha$ is related to other function classes. Referring to the discussion above, the first obvious question is how G_α and $\mathcal{P}_\alpha$ are related to each other. We show that $\mathcal{P}_\alpha \subset g_\alpha \cap G_\alpha$, where g_α consists of functions $h \in \mathcal{H}(\mathbb{D})$ satisfying

$$\liminf_{r \to 1^-} \frac{\log M(r, h)}{-\log(1-r)} = \alpha.$$

We then proceed to show that, just as polynomials are dense in the set of entire functions, $\mathcal{P}_\alpha$ is dense in $\mathcal{H}(\mathbb{D})$ for every $\alpha > 0$. Finally, we investigate the distribution and clustering of a-points of functions in $\mathcal{P}_\alpha$. Our results and their proofs make use of the fact that functions in $\mathcal{P}_\alpha$ are (strongly) annular.

Our second objective is to find the possible M-orders for solutions of (1.1) under the assumption that $a_j \in \mathcal{P}_{\alpha_j}$ for $j = 0, \ldots, k-1$. This completes the result in [**13**] related to the solutions of (1.2).

2. Properties of α-polynomial regular functions

2.1. Comparison with other function classes and spaces

A function $g \in \mathcal{H}(\mathbb{D})$ is called annular if there exists a sequence $\{J_n\}$ of Jordan curves in $\mathbb{D}$ such that

(i) each J_n lies in the interior of J_{n+1},

(ii) $\min\{|z| : z \in J_n\} \to 1^-$ as $n \to \infty$,

(iii) $\min\{|g(z)| : z \in J_n\} \to \infty$ as $n \to \infty$.

If there exists an increasing sequence $\{r_n\} \subset [0,1)$ such that (i)–(iii) hold when $J_n = \{z : |z| = r_n\}$, then g is called strongly annular. We denote the class of such functions by $\mathcal{S}$. In particular, if $g \in \mathcal{P}_\alpha$ and if F is the associated set of positive lower density, then a sequence $\{r_n\} \subset F$ can be found such that $\lim_{n\to\infty} L(r_n, g) = \infty$, where $L(r,g)$ is the minimum modulus of g on $|z| = r$. This yields $\mathcal{P}_\alpha \subset \mathcal{S}$ for every $\alpha > 0$. The following result complements this elementary observation.

Theorem 2.1. *We have $\mathcal{P}_\alpha \subset g_\alpha \cap G_\alpha$ for every $\alpha > 0$.*

The proof of Theorem 2.1 involves dealing with exceptional sets and uses the following lemma, which gives an extension of a result of Bank [**2**, Lemma C] when the exceptional set E is of positive upper density.

Lemma A (Heittokangas [19, Lemma 2]). *Let $g(r)$ and $h(r)$ be monotone increasing real-valued functions on $[0,1)$ such that $g(r) \leqslant h(r)$ for every $r \in [0,1) \setminus E$, where $\bar{d}(E) < 1$. There then exist constants $b \in (0,1)$ and $r_0 \in [0,1)$ such that $g(r) \leqslant h(1 - b(1-r))$ for all $r \in [r_0, 1)$.*

Lemma 2.2. *Let $g(r)$ be a monotone increasing real-valued function on $[0,1)$. Suppose that there exist a constant $\alpha > 0$ and a set $F \subset [0,1)$ with $\underline{d}(F) > 0$ such that $g(r) = (-\alpha + o(1)) \log(1-r)$ as $r \to 1^-$ through F. Then,*

$$\lim_{r\to 1^-} \frac{g(r)}{-\log(1-r)} = \alpha.$$

Proof. Define $E = [0,1) \setminus F$. By the additivity of the Lebesgue measure on disjoint sets, we have that

$$1 = \frac{m([r,1))}{1-r} = \frac{m((E \cap [r,1)) \cup (F \cap [r,1)))}{1-r} = \frac{m(E \cap [r,1))}{1-r} + \frac{m(F \cap [r,1))}{1-r}.$$

Since $\underline{d}(F) > 0$, we get that

$$\liminf_{r\to 1^-} \left(1 - \frac{m(E \cap [r,1))}{1-r}\right) > 0,$$

that is, $\bar{d}(E) < 1$. Let $\varepsilon \in (0, \min\{1, \alpha\})$. By the assumption we find an $R \in [0,1)$ such that

$$(-\alpha + \varepsilon) \log(1-r) \leqslant g(r) \leqslant (-\alpha - \varepsilon) \log(1-r), \quad r \in F \cap [R,1).$$

By Lemma A there exist constants $b \in (0,1)$ and $r_0 \in [0,1)$ such that

$$\alpha - \varepsilon \leqslant \frac{g(1-b(1-r))}{-\log(1-r)}, \quad r \in [r_0, 1), \tag{2.1}$$

and

$$\limsup_{r\to 1^-} \frac{g(r)}{-\log(1-r)} \leqslant \alpha + \varepsilon. \tag{2.2}$$

We may let $\varepsilon \to 0^+$ in (2.2). As for (2.1), we observe that

$$\log \frac{1}{1-r} = \log \frac{1}{b(1-r)} - \log \frac{1}{b} > (1-\varepsilon) \log \frac{1}{b(1-r)}$$

for all r sufficiently close to 1. Hence, (2.1) yields

$$(\alpha-\varepsilon)(1-\varepsilon) \leqslant \liminf_{r\to 1^-} \frac{\log g(1-b(1-r))}{-\log b(1-r)} = \liminf_{t\to 1^-} \frac{\log g(t)}{-\log(1-t)},$$

where we may let $\varepsilon \to 0^+$. The assertion is now proved. □

Proof of Theorem 2.1. Let $\alpha > 0$, let $f \in \mathcal{P}_\alpha$, and let $F \subset [0,1)$ be the set of positive lower density related to f. We have that $\log |f(z)| = (-\alpha + o(1)) \log(1-|z|)$ as $|z| \to 1^-$ through F. This clearly implies that

$$\log M(r,f) = (-\alpha + o(1)) \log(1-r), \quad r \to 1^-, \ r \in F.$$

By Lemma 2.2 we have that

$$\lim_{r\to 1^-} \frac{\log M(r,f)}{-\log(1-r)} = \alpha, \tag{2.3}$$

which yields the result. □

Let $\mathcal{N}$ denote the Nevanlinna class (functions of bounded characteristic in the unit disc). For $p > 0$ and $q > -1$, the weighted Bergman space A^p_q consists of functions $g \in \mathcal{H}(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |g(z)|^p (1-|z|^2)^q \, \mathrm{d}m(z) < \infty,$$

where $\mathrm{d}m(z)$ is the standard Euclidean area measure. In [**27**] it is shown that every $A^p = A^p_0$ space contains strongly annular functions, while the classical Hardy spaces do not. Correspondingly, we ask whether there are any α-polynomial regular functions in $\mathcal{N}$ or in A^p_q. Since every function $g \in \mathcal{N}$ has finite radial limits almost everywhere on $\partial\mathbb{D}$, and since ∞ is the only possible radial limit (or even asymptotic value; see § 2.4) for a function $g \in \mathcal{P}_\alpha$, we conclude that $\mathcal{N} \cap \mathcal{P}_\alpha = \emptyset$ for every $\alpha > 0$. Functions in the weighted Bergman spaces, however, need not have radial limits. It follows quite easily by Theorem 2.1 that $\mathcal{P}_\alpha \subset A^p_q$, provided that $\alpha < (q+1)/p$.

2.2. Topological properties

The topology on $\mathcal{H}(\mathbb{D})$ is that of uniform convergence on compact subsets of $\mathbb{D}$ [**8**, pp. 142–148] induced by the metric

$$d(f,g)=\sum_{n=1}^{\infty}2^{-n}\min\Big\{1,\max_{z\in K_n}|f(z)-g(z)|\Big\},\quad f,g\in\mathcal{H}(\mathbb{D}),$$

where $K_n=\{z\colon |z|\leqslant 1-1/n\}$. Note that $d(f,g)\leqslant\sum_{n=1}^{\infty}2^{-n}=1$ for any $f,g\in\mathcal{H}(\mathbb{D})$. Theorem 2.3 shows that α-polynomial regular functions are dense in $\mathcal{H}(\mathbb{D})$. This is by no means a surprise, since

$$\mathcal{P}_\alpha\subset\mathcal{S}\cap g_\alpha\cap G_\alpha \tag{2.4}$$

and $\mathcal{S}$ is residual in $\mathcal{H}(\mathbb{D})$ in the sense of category [**5**, **7**, **21**].

Theorem 2.3. *Let $\alpha>0$, and let $f\in\mathcal{H}(\mathbb{D})$. There then exists a sequence $\{g_k\}$ of functions in $\mathcal{P}_\alpha$ such that $d(f,g_k)\to 0$ as $k\to\infty$.*

Proof. Let $\varepsilon\in(0,1)$ be a constant. We may write $f(z)=\sum_{n=0}^{\infty}a_nz^n$. Choose $N\in\mathbb{N}$ such that $N>\log_2(3/\varepsilon)$, so that

$$\sum_{n=N+1}^{\infty}2^{-n}=2^{-N}<\frac{\varepsilon}{3}. \tag{2.5}$$

Then choose $\nu=\nu(\varepsilon)\in\mathbb{N}$ large enough such that

$$\sum_{n=\nu+1}^{\infty}|a_n||z|^n<\frac{\varepsilon}{3},\quad z\in K_N, \tag{2.6}$$

where, as above, $K_N=\{z\in\mathbb{D}\colon |z|\leqslant 1-1/N\}$. Next, let $b(z)=\sum_{n=3}^{\infty}\lambda^{\alpha n}z^{\lambda^n}$, where λ is a large positive integer. It is known that $b\in\mathcal{P}_\alpha$ (see [**13**] and Example 2.9). But, in fact, slightly more than this is shown in [**13**]. Indeed, following [**13**, pp. 143–144], let

$$r_n=1-D\lambda^{-n}\quad\text{and}\quad r_n'=1-D'\lambda^{-n},$$

where

$$D=\log((\lambda^\alpha-1)/5)\quad\text{and}\quad D'=(\lambda-1)^{-1}\log(6\lambda^\alpha).$$

Depending on the constant $\alpha>0$, the integer λ should be chosen large enough such that the intervals $[r_n,r_n']$ are pairwise disjoint. Define $F=\bigcup_{n=1}^{\infty}[r_n,r_n']$. The reasoning in [**13**] then shows that $\underline{d}(F)\geqslant(D-D')/(D'(\lambda-1))>0$ and (see (48) and the subsequent displayed equation in [**13**]) that

$$C_1(1+o(1))(1-|z|)^{-\alpha}\leqslant|b(z)|\leqslant C_2(1+o(1))(1-|z|)^{-\alpha},\quad |z|\in F, \tag{2.7}$$

where $C_1=\frac{1}{2}D'^{\alpha}\mathrm{e}^{-D}$ and $C_2=\frac{3}{2}D^{\alpha}\mathrm{e}^{-D'}$. (We remark that there is a typographical mistake in the line after [**13**, (48)]: it should read $D'<\theta<D$.) Choose $\mu=\mu(\varepsilon)\in\mathbb{N}$ large enough that

$$\sum_{n=\mu+2}^{\infty}\lambda^{\alpha n}|z|^{\lambda^n}<\frac{\varepsilon}{3},\quad z\in K_N. \tag{2.8}$$

Define $g_k = P_k + R_k$, where

$$P_k(z) = \sum_{n=0}^{k} a_n z^n \quad \text{and} \quad R_k(z) = \sum_{n=k+2}^{\infty} \lambda^{\alpha n} z^{\lambda^n}.$$

Choose $N_0 = \max\{\nu, \mu\}$. It follows by (2.6) and (2.8) that if $k \geqslant N_0$, then

$$|f(z) - g_k(z)| \leqslant |f(z) - P_k(z)| + |R_k(z)| < \frac{2\varepsilon}{3}, \quad z \in K_N.$$

Since $K_1 \subset K_2 \subset \cdots \subset K_N \subset \cdots$, we have, by (2.5), that

$$d(f, g_k) = \sum_{n=1}^{N} 2^{-n} \min\Big\{1, \max_{z \in K_n} |f(z) - g_k(z)|\Big\} + \sum_{n=N+1}^{\infty} 2^{-n} < \varepsilon, \quad k \geqslant N_0.$$

It suffices to prove that $g_k \in \mathcal{P}_\alpha$. Define $F_m = \bigcup_{n=m}^{\infty} [r_n, r_n']$, so that $F_1 = F$. The calculation in [**13**, (49)] shows that $\underline{d}(F_m) = \underline{d}(F) > 0$ for every fixed $m \in \mathbb{N}$. Moreover, reminiscent of (2.7), for every k we can find an integer $m_0 = m_0(\alpha, \lambda, k)$ such that

$$C_1(1 + o(1))(1 - |z|)^{-\alpha} \leqslant |R_k(z)| \leqslant C_2(1 + o(1))(1 - |z|)^{-\alpha}, \quad |z| \in F_{m_0}.$$

Since P_k is a polynomial and, hence, in H^∞, it is now clear that $g_k \in \mathcal{P}_\alpha$. □

2.3. Asymptotic properties

In the following, $N(r, f, a)$ stands for the integrated counting function of the a-points of f. Furthermore, $g(r) \sim h(r)$ means that $g(r)/h(r) \to 1$ as $r \to 1^-$.

Theorem 2.4. *Let $\alpha > 0$ and let $f \in \mathcal{P}_\alpha$. Then*

$$T(r, f) \sim \log M(r, f) \sim N(r, f, a) \sim \alpha \log \frac{1}{1 - r} \tag{2.9}$$

for every $a \in \mathbb{C}$.

Proof. The asymptotic property $\log M(r, f) \sim -\alpha \log(1 - r)$ was already proved in (2.3). Let $F \subset [0, 1)$ be the set of positive lower density associated with f. From the definition of the $\mathcal{P}_\alpha$-class we obtain that

$$\log L(r, f) = (1 + o(1)) \log M(r, f), \quad r \to 1^-, \ r \in F. \tag{2.10}$$

By Jensen's theorem, it follows that

$$N(r, f, 0) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta + O(1),$$

where the last term depends on the behaviour of f at $z = 0$. Since $N(r, f, 0)$ is increasing in r, the asymptotic relation $N(r, f, 0) \sim -\alpha \log(1 - r)$ follows from (2.10) and Lemma 2.2. The asymptotic relation for $N(r, f, a)$, where $a \in \mathbb{C} \setminus \{0\}$, is proved similarly by considering an auxiliary function $g(z) = f(z) - a$. Evidently, $g \in \mathcal{P}_\alpha$ and $N(r, f, a) = N(r, g, 0)$.

Since f is analytic, we have that $\log L(r, f) \leqslant T(r, f) = m(r, f) \leqslant \log M(r, f)$ for all $r \in [0, 1)$. Finally, since $T(r, f)$ is an increasing function of r, we have, by Lemma 2.2, that $T(r, f) \sim -\alpha \log(1 - r)$. □

Since the asymptotic behaviour of $N(r,f,a)$ for $f \in \mathcal{P}_\alpha$ is known, we can deduce bounds for the non-integrated counting function.

Corollary 2.5. *Let $a \in \mathbb{C}$, let $\alpha > 0$, and let $f \in \mathcal{P}_\alpha$. Then*

$$n(r,f,a) = o\biggl(\frac{1}{1-r}\log\frac{1}{1-r}\biggr)$$

and

$$\limsup_{r\to 1^-} n(r,f,a)(1-r) \geqslant \alpha. \tag{2.11}$$

Proof. Theorem 2.4 shows that $N(r,f,a) = (-\alpha + o(1))\log(1-r)$ as $r \to 1^-$. Since $n(r,f,a)$ is an increasing function of r, a standard calculation gives us that

$$\begin{aligned}(1-r)n(r,f,a) &= 2n(r,f,a)\int_r^{(1+r)/2} \mathrm{d}t \leqslant 2\int_r^{(1+r)/2} \frac{n(t,f,a)}{t}\,\mathrm{d}t \\ &= 2\biggl(N\biggl(\frac{1+r}{2},f,a\biggr) - N(r,f,a)\biggr) = o\biggl(\log\frac{1}{1-r}\biggr)\end{aligned}$$

as $r \to 1^-$. Assume, contrary to (2.11), that there exist $R \in [0,1)$ and $\varepsilon > 0$ such that $n(r,f,a) < (\alpha-\varepsilon)/(1-r)$ for all $r \in [R,1)$. Consequently, we have that

$$N(r,f,a) \leqslant \int_R^r \frac{n(t,f,a)}{t}\,\mathrm{d}t + O(1) \leqslant (\alpha-\varepsilon)\log\frac{1}{1-r} + O(1),$$

which contradicts (2.9) as $r \to 1^-$. The claim follows. □

2.4. Distribution and clustering of a-points

The only possible asymptotic value for a function $f \in \mathcal{P}_\alpha$ is ∞. Hence, every $f \in \mathcal{P}_\alpha$ has infinitely many a-points for any $a \in \mathbb{C}$; see [**6**, Property 3.2] for a proof based on the minimum modulus theorem and on the fact that $\mathcal{P}_\alpha \subset \mathcal{S}$. Note also that this can be deduced from Theorem 2.4, since $N(r,f,a)$ is unbounded for all $a \in \mathbb{C}$. In particular, there are no universal zero divisors in $\mathcal{P}_\alpha$. This means that if the zeros of a function $f \in \mathcal{P}_\alpha$ are divided out, the resulting function does not belong to $\mathcal{P}_\alpha$. Moreover, the sequence $\{z_n(a)\}$ of a-points of $f \in \mathcal{P}_\alpha$ cannot satisfy the Blaschke condition

$$\sum_{n=1}^{\infty}(1-|z_n(a)|) < \infty$$

for any $a \in \mathbb{C}$, since f is an annular function [**4**, p. 23]. Again, Theorem 2.4 gives an alternative proof, since $N(r,f,a)$ is unbounded for all $a \in \mathbb{C}$. Note that if $f \in \mathcal{P}_\alpha$, then $f \in A^p = A^p_0$ for $p \in (0,1/\alpha)$, and, hence, all a-points of f on one ray emanating from the origin constitute a Blaschke sequence [**12**, p. 116–117]. Simple modification of the proof shows that the same is true for all a-points in any disc $D(\zeta, 1-|\zeta|)$ where $\zeta \in \mathbb{D}$ and $|\zeta| > 1/2$.

Lemma B gives a detailed treatment of the convergence of the Blaschke sum for a-points, and it is based on the fact that the exact asymptotic growth of $N(r, f, a)$ is known for $f \in \mathcal{P}_\alpha$. The statement is a trivial modification of the original result in [**25**]. Indeed, if τ is the function in Lemma B, then

$$\int_1^R \frac{x\tau'(x)}{\tau^2(x)}\,\mathrm{d}x = \int_1^R \frac{1}{\tau(x)}\,\mathrm{d}x - \frac{R}{\tau(R)} + \frac{1}{\tau(1)} \tag{2.12}$$

for all $R > 1$, where $x/\tau(x) = O(1)$ as $x \to \infty$. Consequently, both integrals in (2.12) either converge or diverge as $R \to \infty$.

Lemma B (Peláez and Rättyä [25, Lemma 3.9]). *Suppose that $R \in [0, \infty)$. Let $\alpha > 0$, let $f \in \mathcal{P}_\alpha$, and let $z_n(a)$ be the sequence of a-points of f. Furthermore, let $\tau\colon [R, \infty) \to [0, \infty)$ be an increasing function such that $x = O(\tau(x))$ and $\tau'(x) = O(\tau(x) + 1)$ as $x \to \infty$. Then*

$$\int_R^\infty \frac{1}{\tau(x)}\,\mathrm{d}x < \infty \quad \textit{if and only if} \quad \sum_{n=1}^\infty \frac{1 - |z_n(a)|}{\tau(\log(\mathrm{e}^R/(1 - |z_n(a)|)))} < \infty.$$

By choosing $R = 1$ and $\tau(x) = x^{1+\varepsilon}$, $\varepsilon > 0$, in Lemma B, we deduce the result in [**12**, p. 95] for polynomial regular functions. In contrast, $\tau(x) = x$ implies divergence. Even more precise statements can be made. For example, if $R = e$ and $\tau(x) = x(\log x)^{1+\varepsilon}$, $\varepsilon > 0$, then

$$\sum_{n=1}^\infty \frac{1 - |z_n(a)|}{\log(\mathrm{e}^e/(1 - |z_n(a)|))(\log\log(\mathrm{e}^e/(1 - |z_n(a)|)))^{1+\varepsilon}} \tag{2.13}$$

converges. The choice $\tau(x) = x \log x$ shows that (2.13) diverges for $\varepsilon = 0$.

For a function $f \in \mathcal{H}(\mathbb{D})$ and for $a \in \mathbb{C}$, let $z(f, a) \subset \mathbb{D}$ denote the set of a-points of f, and let $z'(f, a) \subset \partial\mathbb{D}$ be the limit points of $z(f, a)$. Note that $z(f, a)$ may, in general, be a finite set or even an empty set, while, if $z(f, a)$ contains infinitely many points, $z'(f, a) \neq \emptyset$. If $z'(f, a) \neq \partial\mathbb{D}$, then a is called a singular value for f. We denote the set of all singular values for f by $S(f)$.

If $f \in \mathcal{S}$, then the set $S(f)$ is at most countable [**6**, Theorem 4.4]. The same is clearly true for $f \in \mathcal{P}_\alpha$. If $S \subset \mathbb{C}$ is a non-empty and at most countable set, then there exists a function $f \in \mathcal{S}$ such that $S(f) = S$ by the construction in [**24**]. The next natural question is whether this is true for $f \in \mathcal{P}_\alpha$. We note that, for every $a \in \mathbb{C}$, the lacunary series $b(z)$ given in the proof of Theorem 2.3 has infinitely many a-points in every sector $\theta_1 < \arg z < \theta_2$ by [**15**, p. 1], and hence $S(b) = \emptyset$. Next, we show that functions in $\mathcal{P}_\alpha$ do not have singular values.

Theorem 2.6. *Let $\alpha > 0$, and let $f \in \mathcal{P}_\alpha$. Then $S(f) = \emptyset$.*

Proof. It is clear by Theorem 2.1 that f satisfies the Hornblower condition

$$\int_0^1 \log^+ \log^+ M(r, f)\,\mathrm{d}r < \infty.$$

Hence, f belongs to the MacLane class, which means that f has asymptotic values at each point of a dense subset of $\partial\mathbb{D}$. The assertion then follows by either [**6**, Theorem 1.1] or [**1**, pp. 340–341]. $\square$

2.5. Derivatives

In general, the derivative of an annular function need not be annular [**3**]. However, differentiation behaves more regularly in the class $g_\alpha \cap G_\alpha$.

Lemma 2.7. *Let $\alpha > 0$. We have $f \in g_\alpha \cap G_\alpha$ if and only if $f' \in g_{\alpha+1} \cap G_{\alpha+1}$. In particular, $f \in \mathcal{A}^{-\infty}$ if and only if $f' \in \mathcal{A}^{-\infty}$.*

The proof of Lemma 2.7 is a simple modification of [**11**, Theorem 5.5], relying on Cauchy's formula for derivatives and on the inequality

$$|f(re^{i\theta})| \leqslant |f(0)| + \int_0^r |f'(se^{i\theta})|\, ds,$$

valid for every $f \in \mathcal{H}(\mathbb{D})$. The details are omitted.

Theorem 2.1 and Lemma 2.7 lead us to study how differentiation is reflected in the $\mathcal{P}_\alpha$-classes.

Theorem 2.8. *Let $\alpha > 0$. Then $f \in \mathcal{P}_\alpha$ if and only if $f' \in \mathcal{P}_{\alpha+1}$.*

We rely on recent developments of Wiman–Valiron theory [**13**, **14**]. Assume that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$. The maximum term is defined as

$$\mu(r) = \mu(r, f) = \max_{n \geqslant 0} |a_n| r^n,$$

while the central index, denoted by $\nu(r) = \nu(r, f)$, is the largest integer n for which the maximum is attained.

Proof of Theorem 2.8. Suppose first that $f \in \mathcal{P}_\alpha$. The proof of [**13**, Theorem 2] then shows that

$$|f'(z)| = (1 + o(1))\nu(|z|)|f(z)| = (1 + o(1))\nu(|z|)M(|z|, f), \quad |z| \to 1^-,\ |z| \in F, \tag{2.14}$$

where $F \subset [0, 1)$ is of positive lower density. We note that the right-hand side of (2.14) is independent of $\arg(z)$, and so we obtain that

$$M(r, f') = (1 + o(1))\nu(r)M(r, f), \quad r \to 1^-,\ r \in F.$$

Hence, (2.14) gives us that

$$|f'(z)| = (1 + o(1))M(|z|, f'), \quad |z| \to 1^-,\ |z| \in F. \tag{2.15}$$

Moreover, [**22**, (1.5.6)] and the trivial inequality $\mu(r) \leqslant M(r, f)$ yield

$$\limsup_{r \to 1^-} \frac{\log \nu(r)}{-\log(1 - r)} \leqslant 1. \tag{2.16}$$

Applying Theorem 2.4 and (2.16) in (2.14), we conclude that

$$\log|f'(z)| \leqslant (-\alpha - 1 + o(1))\log(1-|z|), \quad |z| \to 1^-,\ |z| \in F. \tag{2.17}$$

Suppose that there exists an $\varepsilon > 0$ such that

$$\log|f'(z)| \leqslant (-\alpha - 1 + \varepsilon)\log(1-|z|), \quad |z| \to 1^-,\ |z| \in F.$$

By (2.15) this would imply that

$$\log M(|z|, f') \leqslant (-\alpha - 1 + \varepsilon/2)\log(1-|z|), \quad |z| \to 1^-,\ |z| \in F.$$

Noting that $E = [0,1) \setminus F$ satisfies $\bar{d}(E) < 1$, an application of Lemma A gives us $f' \notin G_{\alpha+1}$. However, the assumption that $f \in \mathcal{P}_\alpha$ yields $f \in G_\alpha$ by Theorem 2.1, and so $f' \in G_{\alpha+1}$ by Lemma 2.7, which is a contradiction. Hence, the equality in (2.17) holds, giving us $f \in \mathcal{P}_{\alpha+1}$.

Conversely, suppose that $f' \in \mathcal{P}_{\alpha+1}$. The first equality in (2.14) then gives us that

$$M(|z|, f') = (1 + o(1))\nu(|z|)|f(z)|, \quad |z| \to 1^-,\ |z| \in F,$$

where $F \subset [0,1)$ is of positive lower density. Theorem 2.4 and (2.16) now imply that

$$(-\alpha + o(1))\log(1-|z|) \leqslant \log|f(z)|, \quad |z| \to 1^-,\ |z| \in F. \tag{2.18}$$

Suppose that there exists an $\varepsilon > 0$ such that

$$(-\alpha - \varepsilon)\log(1-|z|) \leqslant \log|f(z)|, \quad |z| \to 1^-,\ |z| \in F.$$

This gives us that

$$(-\alpha - \varepsilon)\log(1-r) \leqslant \log M(r, f), \quad r \to 1^-,\ r \in F.$$

An application of Lemma A and the fact that $\log(1-r) = (1+o(1))\log(1-s(r))$, where $s(r) = 1 - b(1-r)$, results in $f \notin g_\alpha$. However, the assumption that $f' \in \mathcal{P}_{\alpha+1}$ yields $f' \in g_{\alpha+1}$ by Theorem 2.1, and so $f \in g_\alpha$ by Lemma 2.7, which is a contradiction. Hence, the equality in (2.18) holds, giving us $f \in \mathcal{P}_\alpha$. □

2.6. Examples of α-polynomial regular functions

Examples of $\mathcal{P}_\alpha$-functions in the literature are given in terms of lacunary series.

Example 2.9. Let $b(z) = \sum_{j=3}^{\infty} \lambda^{\alpha j} z^{\lambda^j}$, where $\alpha > 0$ and λ is a large positive integer. It is known that $b \in \mathcal{P}_\alpha$ [**13**]. For similar computations without reference to exceptional sets of positive lower densities, see [**26**, Proposition 5.4] and [**29**, Theorem 2.1.1].

Let $\alpha > 0$ and define $f(z) = b(z) + \log(1/(1-z))$. Then $f \in \mathcal{P}_\alpha$, but, of course, f is far from being lacunary, since $\log(1/(1-z)) = \sum_{j=1}^{\infty}(1/j)z^j$.

The study of the geometric distribution of zeros is very difficult if the point of departure is a function given in series form. The following example is concerned with an analytic function having regularly spaced zeros in the unit disc, and the starting point is now an infinite product (see [**16**, **20**, **25**]).

Example 2.10. Let $0<\alpha<\infty$ and let $a=2^\alpha$, and define

$$f(z)=\prod_{k=1}^{\infty}F_k(z),\quad \text{where } F_k(z)=\frac{1+az^{2^k}}{1+a^{-1}z^{2^k}} \tag{2.19}$$

for all $z\in\mathbb{D}$. The product in (2.19) defines an analytic function in $\mathbb{D}$, since the functions F_k are bounded in $\mathbb{D}$, and

$$\sum_{k=1}^{\infty}|F_k(z)-1|\leqslant\left(a-\frac{1}{a}\right)\sum_{k=1}^{\infty}\frac{|z|^{2^k}}{1-a^{-1}|z|^{2^k}}<\frac{a+1}{1-|z|}$$

converges uniformly on compact subsets of $\mathbb{D}$ (see [**28**, Theorem 15.4]). Evidently, f has equally spaced zeros in $\mathbb{D}$, which are exactly the solutions of $1+az^{2^k}=0$ for $k\in\mathbb{N}$. The following discussion, which is given in two parts, proves that $f\in\mathcal{P}_\alpha$.

(1) We prove that f satisfies

$$|f(z)|\leqslant\frac{O(1)}{(1-|z|)^\alpha},\quad z\in\mathbb{D}. \tag{2.20}$$

Set $r_n=\mathrm{e}^{-2^{-n}}$ for $n\in\mathbb{N}$, and note that

$$|f(z)|=\left|\prod_{k=1}^{n}a\frac{a^{-1}+z^{2^k}}{1+a^{-1}z^{2^k}}\right|\left|\prod_{j=1}^{\infty}\frac{1+az^{2^{n+j}}}{1+a^{-1}z^{2^{n+j}}}\right|. \tag{2.21}$$

By using the triangle inequality for the pseudo-hyperbolic metric [**12**, p. 39] and the fact that $(1+ax)/(1+a^{-1}x)$ is increasing in x, we obtain that

$$\left|\frac{1+az^{2^{n+j}}}{1+a^{-1}z^{2^{n+j}}}\right|=a\left|\frac{a^{-1}+z^{2^{n+j}}}{1+a^{-1}z^{2^{n+j}}}\right|\leqslant a\frac{a^{-1}+|z|^{2^{n+j}}}{1+a^{-1}|z|^{2^{n+j}}}\leqslant\frac{1+a(1/e)^{2^j}}{1+a^{-1}(1/e)^{2^j}}, \tag{2.22}$$

assuming that $|z|\leqslant r_n$ and $j,n\geqslant 1$. Now (2.22) yields

$$\left|\prod_{j=1}^{\infty}\frac{1+az^{2^{n+j}}}{1+a^{-1}z^{2^{n+j}}}\right|\leqslant\prod_{j=1}^{\infty}\frac{1+a(1/e)^{2^j}}{1+a^{-1}(1/e)^{2^j}}=A<\infty,\quad |z|\leqslant r_n,\ n\geqslant 1. \tag{2.23}$$

So, using (2.21), (2.23) and the inequality $\mathrm{e}^{-x}\geqslant 1-x$, $x\geqslant 0$, we obtain that

$$|f(z)|\leqslant A\prod_{k=1}^{n}a\leqslant\frac{O(1)}{(1-r_n)^\alpha},\quad |z|\leqslant r_n,\ n\geqslant 1. \tag{2.24}$$

Now let $|z|\geqslant 1/\sqrt{\mathrm{e}}$ be given and fix $n\in\mathbb{N}$ such that $r_n\leqslant|z|<r_{n+1}$. Then, (2.24) and the inequality $1-x\leqslant\mathrm{e}^{-x}\leqslant 1-\frac{1}{2}x$, $x\in[0,1]$, give that

$$|f(z)|\leqslant\frac{O(1)}{(1-r_{n+1})^\alpha}\leqslant\frac{O(1)}{(1-|z|)^\alpha}.$$

(2) We consider the lower bound for the minimum modulus of f in a set $|z| \in F \subset [0,1)$, where

$$F := \bigcup_{n=1}^{\infty} [a^{-(7/8)(1/2^n)}, a^{-(5/8)(1/2^n)}).$$

We aim to prove that for values $|z| \in F$ this lower bound agrees with the upper bound in (2.20). Note that the lower density of F is strictly positive, and, in particular,

$$\underline{d}(F) \geqslant \lim_{n\to\infty} \frac{a^{-(5/8)(1/2^{n+1})} - a^{-(7/8)(1/2^{n+1})}}{1 - a^{-(5/8)(1/2^n)}} = \frac{1}{5} > 0$$

by L'Hôpital's rule for sequences. Let $z \in F$ be fixed. It follows that there exists $n \in \mathbb{N}$ such that $a^{-7/8} \leqslant |z|^{2^n} < a^{-5/8}$. Evidently,

$$|z|^{2^{n+j}} \leqslant (|z|^{2^n})^2 \leqslant \frac{1}{a^{5/4}} < \frac{1}{a}$$

for all $j \in \mathbb{N}$, and, hence, by [**12**, p. 39],

$$\left|\frac{1 + az^{2^{n+j}}}{1 + a^{-1}z^{2^{n+j}}}\right| = a\left|\frac{a^{-1} + z^{2^{n+j}}}{1 + a^{-1}z^{2^{n+j}}}\right| \geqslant a\frac{a^{-1} - |z|^{2^{n+j}}}{1 - a^{-1}|z|^{2^{n+j}}} = \frac{1 - a|z|^{2^{n+j}}}{1 - a^{-1}|z|^{2^{n+j}}}. \tag{2.25}$$

Since the function $(1 - ax)/(1 - a^{-1}x)$ is decreasing in x, we deduce from (2.25) that

$$\prod_{j=1}^{\infty}\left|\frac{1 + az^{2^{n+j}}}{1 + a^{-1}z^{2^{n+j}}}\right| \geqslant \prod_{j=1}^{\infty} \frac{1 - a(a^{-5/8})^{2^j}}{1 - a^{-1}(a^{-5/8})^{2^j}} = B_1 \in (0,\infty). \tag{2.26}$$

In view of (2.21), we aim to prove that

$$\left|\prod_{k=1}^{n} a\frac{a^{-1} + z^{2^k}}{1 + a^{-1}z^{2^k}}\right| = a^n \prod_{k=1}^{n}\left|\frac{a^{-1} + z^{2^k}}{1 + a^{-1}z^{2^k}}\right| \geqslant B_2 a^n, \quad B_2 \in (0,\infty). \tag{2.27}$$

Since $a^{-1} < |z|^{2^n} < |z|^{2^k}$ for all $k = 1, \dots, n$, and the function $(x - a^{-1})/(1 - a^{-1}x)$ is increasing in x, we obtain that

$$\prod_{k=1}^{n}\left|\frac{a^{-1} + z^{2^k}}{1 + a^{-1}z^{2^k}}\right| \geqslant \prod_{k=1}^{n} \frac{|z|^{2^k} - a^{-1}}{1 - a^{-1}|z|^{2^k}} \geqslant \prod_{k=1}^{n} \frac{a^{-(7/8)(1/2^{n-k})} - a^{-1}}{1 - a^{-1}a^{-(7/8)(1/2^{n-k})}}.$$

To prove (2.27), we recall the inequalities $\log x \leqslant x - 1$ and $1 - \mathrm{e}^{-x} \leqslant x$, and conclude that

$$\left(\prod_{k=1}^{n} \frac{a^{-(7/8)(1/2^{n-k})} - a^{-1}}{1 - a^{-1}a^{-(7/8)(1/2^{n-k})}}\right)^{-1} = \exp\left(\sum_{k=1}^{n} \log \frac{1 - a^{-1}a^{-(7/8)(1/2^{n-k})}}{a^{-(7/8)(1/2^{n-k})} - a^{-1}}\right),$$

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where

$$\begin{aligned}\sum_{k=1}^{n}\log\frac{1-a^{-1}a^{-(7/8)(1/2^{n-k})}}{a^{-(7/8)(1/2^{n-k})}-a^{-1}} &\leqslant (1+a^{-1})\sum_{k=1}^{n}\frac{1-a^{-(7/8)(1/2^{n-k})}}{a^{-(7/8)(1/2^{n-k})}-a^{-1}}\\ &\leqslant O(1)\sum_{j=0}^{\infty}(1-a^{-(7/8)(1/2^{j})})\\ &\leqslant O(1)\sum_{j=0}^{\infty}\frac{1}{2^{j}}\\ &< \infty.\end{aligned}$$

Combining (2.21), (2.26) and (2.27) yields $|f(z)| \geqslant B_1 B_2 a^n \geqslant C(1-|z|)^{-\alpha}$ for $|z| \in F$, where $C \in (0,\infty)$ is a constant. This estimate, together with (1), proves that $f \in \mathcal{P}_\alpha$.

Remark 2.11. Let f be the function in Example 2.10. We consider the asymptotic growths of the counting functions $n(r,f,0)$ and $N(r,f,0)$. The zero set of f is the union of the zero sets of the functions F_k. So f has exactly 2^k zeros on the circle $\{z \in \mathbb{C} \colon |z| = a^{-2^{-k}}\}$. Clearly,

$$n(a^{-2^{-k}}, f, 0) = \sum_{j=1}^{k} 2^j = 2(2^k - 1).$$

It follows that, for all $r \in [0,1)$, we have that

$$\frac{\log a}{\log(1/r)} - 2 < n(r,f,0) \leqslant 2\frac{\log a}{\log(1/r)} - 2,$$

where equality holds provided that $r = a^{-2^{-k}}$ for some $k \in \mathbb{N}$. As a consequence, the functions $n(r,f,0)$ and $(1-r)^{-1}$ are asymptotically comparable as $r \to 1^-$. Elementary computations show that

$$\begin{aligned}N(r,f,0) &\sim \left(\frac{\log a}{\log 2} + \frac{O(1)}{\log(1/(1-r))}\right)\log\left(\frac{\log a}{\log(1/r)}\right)\\ &\sim \left(\alpha + \frac{O(1)}{\log(1/(1-r))}\right)\log\left(\frac{1}{1-r}\right)\end{aligned}$$

as $r \to 1^-$. This estimate is in line with Theorem 2.4. As a curiosity, we note that $f^{(p)} \in \mathcal{P}_{\alpha+p}$ for all $p \in \mathbb{N}$ by Theorem 2.8, which is not easy to verify directly.

In contrast to Example 2.10, note that not every infinite product having equally spaced zeros in $\mathbb{D}$ is annular, and, hence, not every such product is α-polynomial regular.

Example 2.12. Following the argument in [**12**, p. 94], we consider

$$f(z) = \prod_{k=1}^{\infty}(1 - 2z^{n_k}),$$

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assuming that $\{n_k\}$ is an increasing sequence of positive integers. This function f is analytic in $\mathbb{D}$ (see Example 2.10). It has n_k equally spaced zeros on the circle of radius $2^{-1/n_k}$, and it satisfies the growth condition

$$|f(z)| \leqslant \prod_{k=1}^{\infty}(1+2r^{n_k}) \leqslant \exp\left(2\sum_{k=1}^{\infty} r^{n_k}\right). \tag{2.28}$$

The right-hand side of (2.28) can be made to grow arbitrarily slowly as $r \to 1^-$. However, since the zeros of f do not satisfy the Blaschke condition, f cannot be bounded. On the other hand, f is bounded on the positive real axis, which shows that $f \notin \mathcal{P}_\alpha$ for any $\alpha > 0$.

3. Possible M-orders for solutions

We proceed to construct a finite ordered list of possible non-zero M-orders $\beta_1, \beta_2, \dots, \beta_p$ for solutions of (1.1) such that

$$\beta_1 > \beta_2 > \cdots > \beta_p > 0. \tag{3.1}$$

To this end, note that one of the main tools used in [**17**] is Wiman–Valiron theory, which is now available in $\mathbb{D}$ [**13**]. We modify the reasoning in [**17**] by replacing the polynomial coefficients with $a_j \in \mathcal{P}_{\alpha_j}$ for $j = 0, \dots, k-1$, and by taking advantage of [**13**]. For notational convenience, set $a_k(z) \equiv 1$ and $\alpha_k = 0$, and write $a_k \in \mathcal{P}_{\alpha_k}$.

To begin with, let $s_1 \in \{0, \dots, k-1\}$ be the smallest index satisfying

$$\frac{\alpha_{s_1}}{k-s_1} = \max_{0 \leqslant j \leqslant k-1} \frac{\alpha_j}{k-j} > 1. \tag{3.2}$$

If s_1 cannot be found, then all solutions f of (1.1) satisfy $\sigma_M(f) = 0$ by Theorem 2.1 and [**10**, Theorem 1.4]. If s_1 exists, we define further indices s_m recursively as follows. For a given s_m, $m \geqslant 1$, let $s_{m+1} \in \{0, \dots, s_m - 1\}$ be the smallest index satisfying

$$\frac{\alpha_{s_{m+1}} - \alpha_{s_m}}{s_m - s_{m+1}} = \max_{0 \leqslant j \leqslant s_m - 1} \frac{\alpha_j - \alpha_{s_m}}{s_m - j} > 1. \tag{3.3}$$

Eventually this process will stop, yielding a finite list of indices $s_1, \dots, s_p$ such that $p \leqslant k$ and

$$s_1 > s_2 > \cdots > s_p \geqslant 0.$$

Finally, we define the possible non-zero M-orders as

$$\beta_j = \frac{\alpha_{s_j} - \alpha_{s_{j-1}}}{s_{j-1} - s_j} - 1, \quad j = 1, \dots, p, \tag{3.4}$$

where $s_0 = k$ and $\alpha_{s_0} = \alpha_k = 0$.

Theorem 3.1. *Suppose that $a_j \in \mathcal{P}_{\alpha_j}$ for $j = 0, \dots, k-1$.*

(a) *If f is a solution of (1.1), then $\sigma_M(f) \in \{0, \beta_1, \dots, \beta_p\}$.*

(b) *If $s_1 \geqslant 1$ and $p \geqslant 2$, then the inequalities (3.1) are valid.*

(c) *If $s_1 = 0$, then all non-trivial solutions f of (1.1) satisfy $\sigma_M(f) = \alpha_0/k - 1$.*

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We note that Theorem 3.1 (a) is proved in [**13**] in the case $k = 2$. Regarding (b), if $s_1 \geqslant 1$ and $p = 1$, we obtain, by (a), (3.2) and (3.4), that any solution f of (1.1) satisfies

$$\sigma_M(f) \in \{0, \beta_1\} = \left\{0, \max_{0 \leqslant j \leqslant k-1} \frac{\alpha_j}{k-j} - 1\right\}.$$

We remark that the statement (b) is equivalent to [**9**, Lemma 2.4 (i)], and, hence, its proof is omitted. The statement (c) has been added for completeness, as it is just a special case of [**10**, Corollary 1.5] by means of Theorem 2.1. Thus, it suffices to prove (a) for arbitrary $k \geqslant 2$.

Proof of Theorem 3.1 (a). Let f be a solution of (1.1). As 0 is one of the possible M-orders, we may suppose that $\sigma_M(f) > 0$. Let $\nu(r) = \nu(r, f)$ be the central index of f. By [**13**, Theorem 1], we have that

$$\frac{f^{(q)}(\zeta)}{f(\zeta)} = (1 + o(1))\left(\frac{\nu(|\zeta|)}{\zeta}\right)^q, \quad q \in \mathbb{N}, \tag{3.5}$$

as $|\zeta| \to 1^-$ outside a certain exceptional set $E \subset [0, 1)$. If $F \subset [0, 1)$ is any set with $\underline{d}(F) > 0$, then [**13**, Corollary 1] shows that there exists a sequence $\{\zeta_n\}$, with $|\zeta_n| \to 1^-$ and $|\zeta_n| \in F \setminus E$, such that

$$\lim_{n\to\infty} \frac{\log \nu(|\zeta_n|)}{-\log(1 - |\zeta_n|)} = \sigma_M(f) + 1, \tag{3.6}$$

and that (3.5) holds for $\zeta = \zeta_n$.

Dividing (1.1) by f and making use of (3.5) and (3.6) together with the definition of the $\mathcal{P}_\alpha$-class, we obtain an algebraic equation where the right-hand side is equal to 0 and the left-hand side contains terms whose moduli on the circles $|z| = |\zeta_n|$ are asymptotically equal to the expressions

$$\left(\frac{1}{1 - |\zeta_n|}\right)^{j(\sigma_M(f)+1)+\alpha_j+o(1)}, \quad j = 0, \dots, k, \tag{3.7}$$

where $\alpha_k = 0$. Note that $\sigma_M(f) \leqslant \beta_1$ by Theorem 2.1 and [**10**, Theorem 1.4], while (3.1) holds by Theorem 3.1 (b). Our aim is to show that, if $\sigma_M(f) \notin \{0, \beta_1, \dots, \beta_p\}$, precisely one term in (3.7) is dominant. Evidently, this leads to a contradiction, and hence $\sigma_M(f) \in \{\beta_1, \dots, \beta_p\}$. Consequently, we consider two separate cases, which are parallel to those in [**17**, pp. 1230–1232].

Case 1. $\beta_{j+1} < \sigma_M(f) < \beta_j$ for some $j \in \{1, \dots, p-1\}$. On the one hand, by (3.3) we obtain that

$$\sigma_M(f) + 1 > \beta_{j+1} + 1 = \frac{\alpha_{s_{j+1}} - \alpha_{s_j}}{s_j - s_{j+1}} \geqslant \frac{\alpha_t - \alpha_{s_j}}{s_j - t}, \quad 0 \leqslant t < s_j,$$

or, equivalently,

$$s_j(\sigma_M(f) + 1) + \alpha_{s_j} > t(\sigma_M(f) + 1) + \alpha_t \tag{3.8}$$

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for all $0 \leqslant t < s_j$. On the other hand, by [**9**, (6.7)] with obvious modifications in notation, we have that

$$\sigma_M(f) + 1 < \beta_j + 1 = \frac{\alpha_{s_j} - \alpha_{s_{j-1}}}{s_{j-1} - s_j} \leqslant \frac{\alpha_{s_j} - \alpha_t}{t - s_j}, \quad s_j < t \leqslant k,$$

which proves that (3.8) holds for all $s_j < t \leqslant k$. We deduce that the term with the exponent $s_j(\sigma_M(f)+1) + \alpha_{s_j}$ in (3.7) is dominant.

Case 2. $\sigma_M(f) < \beta_p$. On the one hand, as in Case 1, [**9**, (6.7)] yields

$$s_p(\sigma_M(f)+1) + \alpha_{s_p} > t(\sigma_M(f)+1) + \alpha_t \tag{3.9}$$

for all $s_p < t \leqslant k$. On the other hand, the construction of the sequence $s_1, \dots, s_p$ guarantees that

$$\frac{\alpha_t - \alpha_{s_p}}{s_p - t} \leqslant 1 < 1 + \sigma_M(f), \quad 0 \leqslant t < s_p,$$

which proves that (3.9) holds for all $0 \leqslant t < s_p$. We deduce that the term with the exponent $s_p(\sigma_M(f)+1) + \alpha_{s_p}$ in (3.7) is dominant.

By Cases 1 and 2, we conclude that $\sigma_M(f) \in \{\beta_1, \dots, \beta_p\}$, and we are done. □

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