J. Aust. Math. Soc. **99** (2015), 237–249
doi:10.1017/S1446788715000129

BOUNDED TOEPLITZ AND HANKEL PRODUCTS ON THE WEIGHTED BERGMAN SPACES OF THE UNIT BALL

MAŁGORZATA MICHALSKA and PAWEŁ SOBOLEWSKI✉

(Received 2 July 2012; accepted 10 February 2015; first published online 5 June 2015)

Communicated by I. Doust

Abstract

Let A^p_α be the weighted Bergman space of the unit ball in C^n, $n \geq 2$. Recently, Miao studied products of two Toeplitz operators defined on A^p_α. He proved a necessary condition and a sufficient condition for boundedness of such products in terms of the Berezin transform. We modify the Berezin transform and improve his sufficient condition for products of Toeplitz operators. We also investigate products of two Hankel operators defined on A^p_α, and products of the Hankel operator and the Toeplitz operator. In particular, in both cases, we prove sufficient conditions for boundedness of the products.

2010 *Mathematics subject classification*: primary 47B35; secondary 32A36.

Keywords and phrases: Toeplitz operators, Hankel operators, weighted Bergman spaces.

1. Introduction

Let dv denote the Lebesgue measure in the unit ball $\mathcal{B}$ in C^n $(n \geq 2)$ normalized so that the volume of the unit ball is equal to 1, and let $\alpha > -1$. We define the weighted Lebesgue measure in $\mathcal{B}$ as follows:

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha \, dv(z),$$

where $c_\alpha = \Gamma(n + 1 + \alpha)/(n!\Gamma(\alpha + 1))$. Such measure is also normalized, that is, $v_\alpha(\mathcal{B}) = 1$.

For $0 < p < \infty$, the weighted Bergman space A^p_α consists of all holomorphic functions on $\mathcal{B}$ for which

$$\|f\|_{L^p} = \left(\int_{\mathcal{B}} |f(z)|^p \, dv_\alpha(z)\right)^{1/p} < \infty.$$

Clearly, A^p_α is a closed linear subspace of the Lebesgue space $L^p_\alpha := L^p(\mathcal{B}, dv_\alpha)$.

Let P denote the orthogonal projection from L^2_α onto A^2_α, given by

$$Pf(w) = \int_{\mathcal{B}} \frac{f(z)\, dv_\alpha(z)}{(1 - \langle w, z\rangle)^{n+1+\alpha}}, \quad w \in \mathcal{B},$$

where the function $z \mapsto (1 - \langle z, w\rangle)^{-(n+1+\alpha)}$ defined on $\mathcal{B}$ is the reproducing kernel function for A^2_α and will be denoted by K_w. The above definition of the projection P can be extended as a bounded linear operator from L^p_α onto A^p_α if and only if p is greater than 1 (see, for example, [17, page 47]).

We now assume that $1 < p < \infty$ and we recall some useful facts concerning A^p_α. First, observe that A^q_α with $1/p + 1/q = 1$ is the dual space of A^p_α under the pairing

$$\langle f, g\rangle_\alpha = \int_{\mathcal{B}} f(z)\overline{g(z)}\, dv_\alpha(z), \quad f \in A^p_\alpha,\ g \in A^q_\alpha.$$

In view of this formula, for any f in L^p_α we get the representation

$$P(f)(w) = \langle f, K_w\rangle_\alpha,$$

where K_w is the kernel function defined above.

Moreover, the space L^p_α has a decomposition (see, for example, [9, Theorem 5.16])

$$L^p_\alpha = A^p_\alpha \oplus (A^q_\alpha)^\perp, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{1.1}$$

where

$$(A^q_\alpha)^\perp = \{f - P(f) : f \in L^p_\alpha\}$$

is the annihilator of the space A^q_α.

Now we recall the definition of the automorphism of the unit ball. Let $w \in \mathcal{B}$ and $s_w = (1 - |w|^2)^{1/2}$. The automorphism φ_w of the unit ball is given by the formula

$$\varphi_w(z) = \frac{w - P_w(z) - s_w Q_w(z)}{1 - \langle z, w\rangle},$$

where $P_w(z) = \langle z, w\rangle w/|w|^2$ if $w \neq 0$, $P_0(z) = 0$ and $Q_w = I - P_w$ (see, for example, [10, 17] for the definition and some properties of the automorphism group of the unit ball).

For a function $f \in L^\infty(\mathcal{B})$, we define the Toeplitz operator T_f on A^p_α by

$$T_f(h)(z) = P(fh)(z)$$

and the Hankel operator H_f on A^p_α by the formula

$$H_f(h)(z) = f(z)h(z) - P(fh)(z).$$

In the case when f belongs to L^1_α, we define the above operators densely on the space A^p_α.

The aim of this paper is to find the conditions for products of Toeplitz operators and products of Hankel operators to be bounded on the weighted Bergman space A^p_α in the unit ball. Our study is motivated by the results obtained for the Hardy space H^2 in the unit disk $\mathcal{D}$. Treil gave the following necessary condition for boundedness of $T_f T_{\bar{g}}$ defined on the Hardy space:

$$\sup_{w \in \mathcal{D}} \langle |f|^2 \widetilde{k}_w, \widetilde{k}_w\rangle \langle |g|^2 \widetilde{k}_w, \widetilde{k}_w\rangle < \infty,$$

https://doi.org/10.1017/S1446788715000129 Published online by Cambridge University Press

where $\widetilde{k_w}(z) = (1-|w|^2)^{1/2}/(1-\bar{w}z)$ is the normalized reproducing kernel for H^2. It was conjectured by Sarason [11] that this condition is also sufficient. Cruz-Uribe [2] gave support for Sarason's conjecture. Cruz-Uribe characterized the outer functions f and g for which the product $T_f T_{\bar{g}}$ is bounded on H^2. Unfortunately, Sarason's conjecture turned out to be false in general (see Nazarov's counterexample [6]). A slightly stronger sufficient condition was given by Zheng [16].

The studies of boundedness of Toeplitz products seem to be more interesting in the case of the Bergman spaces, since there exist bounded Toeplitz operators on the Bergman space A^2 in the unit disk with unbounded symbols. In [11], Sarason asked the question: for which functions f and g, analytic in the unit disk, is the product $T_f T_{\bar{g}}$ a bounded operator on A^2? Although a partial answer to this question is known, the problem posed by Sarason is still open. Stroethoff and Zheng [12] gave a necessary condition and a slightly stronger sufficient condition for boundedness of such products. They also obtained analogous results for the Bergman space in the polydisk [13], for the weighted Bergman spaces in the unit disk [15] and for the weighted Bergman spaces in the unit ball [14]. Similar conditions for the weighted Bergman spaces in the unit ball were obtained by Park [7], while in [8] Pott and Strouse gave the related results for the space A^2_α in the unit disk. Recently, Miao [4] generalized the results of Stroethoff and Zheng to the weighted Bergman spaces A^p_α for all $p > 1$.

Stroethoff and Zheng [12] also obtained some conditions for boundedness of the products of Hankel operators $H_f H_g^*$, $f, g \in L^2(\mathcal{D}, dA)$, densely defined on $(A^2)^\perp$. Lu and Liu [3] gave analogous results for A^2_α in the unit ball. In [5], Michalska *et al.* obtained slightly weaker sufficient conditions for products of Toeplitz operators and products of Hankel operators on A^2_α.

In this paper we give sufficient conditions for boundedness of the products of Toeplitz operators $T_f T_{\bar{g}}$ and Hankel operators $H_f H_g^*$ on the weighted Bergman spaces A^p_α, which are analogous to those obtained in [5]. Moreover, our condition for the product of two Toeplitz operators is weaker than the one obtained by Miao in [4].

To state our main theorems we use the modified Berezin transform B^p_ϵ defined as follows. Let $\epsilon > 0$. For $u \in L^1_\alpha$ and $1/p + 1/q = 1$, we define

$$B^p_\epsilon[u](w) = \int_{\mathcal{B}} (u \circ \varphi_w)(z) \log^{p(1+\epsilon)/q}(1/(1-|z|))\, dv_\alpha(z), \quad w \in \mathcal{B}.$$

We prove the following result.

Theorem 1.1. *Let $1/p + 1/q = 1$, $f \in A^p_\alpha$ and $g \in A^q_\alpha$. If there exists a positive constant ϵ such that*

$$\sup_{w\in\mathcal{B}} \{B^p_\epsilon[|f k_w^{1-2/p}|^p](w)\}^{1/p} \{B^q_\epsilon[|g k_w^{1-2/q}|^q](w)\}^{1/q} < \infty,$$

then the operator $T_f T_{\bar{g}}$ is bounded on A^p_α.

https://doi.org/10.1017/S1446788715000129 Published online by Cambridge University Press

THEOREM 1.2. *Let* $1/p + 1/q = 1$, $f \in L^p_\alpha$ *and* $g \in L^q_\alpha$. *If there exists a positive constant* ϵ *such that*

$$\sup_{w\in\mathcal{B}}\{\|[(fk_w^{1-2/p}) \circ \varphi_w - P((fk_w^{1-2/p}) \circ \varphi_w)] \log^{(1+\epsilon)/q}(1/(1-|z|))\|_{L^p} \times \|[(gk_w^{1-2/q}) \circ \varphi_w - P((gk_w^{1-2/q}) \circ \varphi_w)] \log^{(1+\epsilon)/p}(1/(1-|z|))\|_{L^q}\} < \infty,$$

then the operator $H_f H_g^*$ *is bounded on* $(A^q_\alpha)^\perp$.

We also present a necessary condition for the mixed Hankel and Toeplitz products $H_g T_{\bar{f}}$ to be bounded on the spaces A^p_α.

THEOREM 1.3. *Let* $1/p + 1/q = 1$ *and* $f \in A^q_\alpha$, $g \in L^p_\alpha$. *If the operator* $H_g T_{\overline{f}}$ *is bounded on* A^p_α, *then*

$$\sup_{w\in\mathcal{B}} \|(fk_w)^{1-2/q} \circ \varphi_w\|_{L^q} \|(gk_w)^{1-2/p} \circ \varphi - P((gk_w)^{1-2/p} \circ \varphi_w)\|_{L^p} < \infty.$$

Similarly, we give a sufficient condition for the mixed Hankel and Toeplitz products $H_g T_{\bar{f}}$, analogous to those in Theorems 1.1 and 1.2.

THEOREM 1.4. *Suppose that* $1/p + 1/q = 1$, $f \in L^q_\alpha$, $g \in L^p_\alpha$ *and* f *is a holomorphic function on* $\mathcal{B}$. *If there exist positive constants* ϵ_1 *and* ϵ_2 *such that*

$$\sup_{w\in\mathcal{B}}\{B^q_{\epsilon_1}[|fk_w^{1-2/q}|^q]\}^{1/q} \|((gk_w^{1-2/p}) \circ \varphi_w - P((gk_w^{1-2/p}) \circ \varphi_w)) \log^{(1+\epsilon_2)/q}\|_{L^p} < \infty,$$

then the operator $H_g T_{\overline{f}}$ *is bounded on* A^p_α.

2. Sufficient conditions for boundedness of Toeplitz and Hankel products

We begin by recalling the fractional radial derivative $R^{s,t}$ of a holomorphic function f on $\mathcal{B}$. Suppose that f has the homogeneous expansion

$$f(z) = \sum_{k=0}^{\infty} f_k(z).$$

If for any real parameters s, t neither $n + s$ nor $n + s + t$ is a negative integer, then

$$R^{s,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+s)\Gamma(n+1+s+k+t)}{\Gamma(n+1+s+t)\Gamma(n+1+s+k)} f_k(z)$$

is called the fractional radial derivative. In the case $\alpha > -1$ and $t > 0$, the derivative $R^{\alpha,t}$ can be written as

$$R^{\alpha,t} f(z) = \lim_{r\to 1^-} \int_{\mathcal{B}} \frac{f(rw)\, dv_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+\alpha+t}}.$$

In particular, if $f \in A^1_\alpha$, then

$$R^{\alpha,t} f(z) = \int_{\mathcal{B}} \frac{f(w)\, dv_\alpha(w)}{(1-\langle z, w\rangle)^{n+1+\alpha+t}}.$$

The following two results are needed in the proof of Lemma 2.3.

https://doi.org/10.1017/S1446788715000129 Published online by Cambridge University Press

Lemma 2.1 [17, Example 2.19, page 77]. *Suppose that $t > 0,\ b > 0$. Then there exists a function $F(z, w)$, holomorphic in z, conjugate holomorphic in w, and bounded in $\mathcal{B}\times\mathcal{B}$, such that for all $z, w \in \mathcal{B}$,*

$$R^{\alpha,t}\Big[\frac{1}{(1-\langle z,w\rangle)^b}\Big] = \frac{F(z,w)}{(1-\langle z,w\rangle)^{b+t}}.$$

Lemma 2.2 [4, Lemma 3.1]. *Let $s > 0,\ t > 0$ and $1/p + 1/q = 1$. Then, for all $f \in A^p_\alpha$ and $g \in A^q_\alpha$,*

$$\langle f, g\rangle_\alpha = \langle R^{\alpha,s} f, R^{\alpha+s,t} g\rangle_{s+t+\alpha}.$$

In the next lemma we give the estimates of the fractional radial derivative of the Toeplitz and the Hankel operators.

Lemma 2.3. *Let $1/p + 1/q = 1$ and $\epsilon > 0$. Suppose that $\beta > -1$ and $t > 0$. Then:*

(i) *for all functions $f \in A^q_\alpha, h \in A^p_\alpha$ and $w \in \mathcal{B}$,*

$$|R^{\beta,t}T_{\bar f}h(w)| \le \frac{C}{(1-|w|^2)^l}\{B^q_\epsilon[|fk_w^{1-2/q}|^q](w)\}^{1/q} \times\Big\{\int_{\mathcal{B}}\frac{|h(z)|^p}{|1-\langle w,z\rangle|^t}\log^{-(1+\epsilon)}(1/(1-|\varphi_w(z)|))\,dv_\alpha(z)\Big\}^{1/p},$$

where $l = (2(n+1+\alpha+t) + (q-2)(n+1+\alpha))/(2q)$;

(ii) *for $g \in L^p_\alpha$, $u \in (A^p_\alpha)^\perp$ and $w \in \mathcal{B}$,*

$$|R^{\beta,t}H^*_g u(w)| \le \frac{C}{(1-|w|^2)^l}\|[(gk_w^{1-2/p})\circ\varphi_w - P((gk_w^{1-2/p})\circ\varphi_w)]\log^{(1+\epsilon)/q}(1/(1-|z|))\|_{L^p} \times\Big\{\int_{\mathcal{B}}\frac{|u(z)|^q}{|1-\langle w,z\rangle|^t}\log^{-(1+\epsilon)}(1/(1-|\varphi_w(z)|))\,dv_\alpha(z)\Big\}^{1/q},$$

where $l = (2(n+1+\alpha+t) + (p-2)(n+1+\alpha))/(2p)$.

Proof. (i) The definition of the Toeplitz operator and Lemma 2.1 give the inequality

$$|R^{\beta,t}T_{\bar f}h(w)| \le \frac{C}{(1-|w|^2)^{t/q}}\int_{\mathcal{B}}\frac{|f(z)|}{|1-\langle w,z\rangle|^{n+1+\alpha}}\frac{|h(z)|}{|1-\langle w,z\rangle|^{t/p}}\,dv_\alpha(z).$$

Now, applying Hölder's inequality and change-of-variable formula,

$$|R^{\beta,t}T_{\bar f}h(w)| \le C\frac{\{B^q_\epsilon[|fk_w^{1-2/q}|^q](w)\}^{1/q}}{(1-|w|^2)^l}\Big\{\int_{\mathcal{B}}\frac{|h(z)|^p\log^{-(1+\epsilon)}(1/(1-|\varphi_w(z)|))}{|1-\langle w,z\rangle|^t}\,dv_\alpha(z)\Big\}^{1/p},$$

where $l = (2(n+1+\alpha+t) + (q-2)(n+1+\alpha))/(2q)$.

(ii) Let $F(w, z)$ be the function described in Lemma 2.1. Then, for all $g \in L^p_\alpha$, the function

$$h_w(z) = \frac{\overline{F(w,z)}k_w^{2/p-1}(z)P((gk_w^{1-2/p})\circ\varphi_w)\circ\varphi_w(z)}{(1-\langle z,w\rangle)^{n+1+\alpha+t}}$$

https://doi.org/10.1017/S1446788715000129 Published online by Cambridge University Press

belongs to A^p_α. Thus, for $u \in (A^p_\alpha)^\perp$,

$$\langle u, h_w\rangle_\alpha = \int_{\mathcal{B}} \frac{u(z)F(w,z)\overline{k_w^{2/p-1}(z)P((gk_w^{1-2/p})\circ\varphi_w)\circ\varphi_w(z)}}{(1-\langle w,z\rangle)^{n+1+\alpha+t}}\,d\upsilon_\alpha(z) \equiv 0.$$

Now, by the definition of the Hankel operator and Lemma 2.1,

$$\begin{aligned}|R^{\beta,t}H_g^* u(w)| = |R^{\beta,t}H_g^* u(w) - \langle u, h_w\rangle_\alpha| &\le \frac{C}{(1-|w|^2)^{t/p}} \\ &\times \int_{\mathcal{B}} \frac{|g(z) - k_w^{2/p-1}(z)P((gk_w^{1-2/p})\circ\varphi_w)\circ\varphi_w(z)|}{|1-\langle w,z\rangle|^{n+1+\alpha}} \\ &\times \frac{|u(z)|}{|1-\langle w,z\rangle|^{t/q}}\,d\upsilon_\alpha(z).\end{aligned}$$

Finally, the same argument as in the proof of (i) implies that

$$\begin{aligned}&|R^{\beta,t}H_g^* u(w)| \\ &\le \frac{C}{(1-|w|^2)^l}\|[(gk_w^{1-2/p})\circ\varphi_w - P((gk_w^{1-2/p})\circ\varphi_w)]\log^{(1+\epsilon)/q}(1/(1-|z|))\|_{L^p} \\ &\quad\times\bigg\{\int_{\mathcal{B}} \frac{|u(z)|^q}{|1-\langle w,z\rangle|^t}\log^{-(1+\epsilon)}(1/(1-|\varphi_w(z)|))\,d\upsilon_\alpha(z)\bigg\}^{1/q},\end{aligned}$$

where $l = (2(n+1+\alpha+t)+(p-2)(n+1+\alpha))/(2p)$, as desired. □

Proof of Theorem 1.1. With no loss of generality, we may assume that $0<\epsilon<1$. We show that for $u \in A^p_\alpha$, $v \in A^q_\alpha$ the following inequality holds:

$$|\langle T_f T_{\bar g} u, v\rangle_\alpha| \le C\|u\|_{L^p}\|v\|_{L^q}.$$

Using Lemmas 2.2 and 2.3(i), we obtain the estimate

$$\begin{aligned}|\langle T_f T_{\bar g} u, v\rangle_\alpha| &= |\langle R^{\alpha,s}T_{\bar g}u, R^{\alpha+s,t}T_{\bar f}v\rangle_{s+t+\alpha}| \\ &\le C\sup_{w\in\mathcal{B}}\{B^p_\epsilon[|f|^p](w)\}^{1/p}\{B^q_\epsilon[|g|^q](w)\}^{1/q} \\ &\quad\times\int_{\mathcal{B}}\bigg\{(1-|w|^2)^{s-n-1-\alpha}\bigg\{\int_{\mathcal{B}}\frac{|u(z)|^p\log^{-(1+\epsilon)}(1/(1-|\varphi_w(z)|))}{|1-\langle w,z\rangle|^t}\,d\upsilon_\alpha(z)\bigg\}^{1/p} \\ &\quad\times\bigg\{\int_{\mathcal{B}}\frac{|v(z)|^q\log^{-(1+\epsilon)}(1/(1-|\varphi_w(z)|))}{|1-\langle w,z\rangle|^t}\,d\upsilon_\alpha(z)\bigg\}^{1/q}\bigg\}\,d\upsilon_\alpha(w).\end{aligned}$$

Putting $t=s=n+1+\alpha>0$ and applying Hölder's inequality,

$$\begin{aligned}|\langle T_f T_{\bar g} u, v\rangle_\alpha| &\le C\sup_{w\in\mathcal{B}}\{B^p_\epsilon[|fk_w^{1-2/p}|^p](w)\}^{1/p}\{B^q_\epsilon[|gk_w^{1-2/q}|^q](w)\}^{1/q} \\ &\quad\times\bigg\{\int_{\mathcal{B}}\int_{\mathcal{B}}\frac{|u(z)|^p\log^{-(1+\epsilon)}(1/(1-|\varphi_w(z)|))}{|1-\langle w,z\rangle|^{n+1+\alpha}}\,d\upsilon_\alpha(z)\,d\upsilon_\alpha(w)\bigg\}^{1/p} \\ &\quad\times\bigg\{\int_{\mathcal{B}}\int_{\mathcal{B}}\frac{|v(z)|^q\log^{-(1+\epsilon)}(1/(1-|\varphi_w(z)|))}{|1-\langle w,z\rangle|^{n+1+\alpha}}\,d\upsilon_\alpha(z)\,d\upsilon_\alpha(w)\bigg\}^{1/q}.\end{aligned} \tag{2.1}$$

https://doi.org/10.1017/S1446788715000129 Published online by Cambridge University Press

Now, to complete the proof, we need to show that the integrals in (2.1) are bounded. By Fubini's theorem, change-of-variable formula and integration in polar coordinates,

$$\begin{aligned} I &:= \int_{\mathcal{B}}\int_{\mathcal{B}} \frac{|u(z)|^p}{|1-\langle z,w\rangle|^{n+1+\alpha}} \log^{-(1+\epsilon)}(1/(1-|\varphi_w(z)|))\, dv_\alpha(z)\, dv_\alpha(w) \\ &= \int_{\mathcal{B}} |u(z)|^p 2nc_\alpha \Big\{ \int_0^1 r^{2n-1}(1-r^2)^\alpha \log^{-(1+\epsilon)}(1/(1-r)) \\ &\qquad \times \int_S \frac{d\sigma(\zeta)}{|1-\langle r\zeta, z\rangle|^{n+1+\alpha}}\, dr \Big\}\, dv_\alpha(z). \end{aligned}$$

Since

$$\int_S \frac{1}{|1-\langle r\zeta, z\rangle|^{n+1+\alpha}}\, d\sigma(\zeta) \le \frac{C}{(1-r)^{1+\alpha}}$$

(see, for example, [17, Theorem 1.12]),

$$I \le C \int_{\mathcal{B}} |u(z)|^2\, dv_\alpha(z) \int_0^1 \frac{r}{1-r} \log^{-(1+\epsilon)}(1/(1-r))\, dr.$$

It is easy to check that the above integral is convergent for $0<\epsilon<1$. Thus,

$$I \le C\|u\|^p_{L^p} \tag{2.2}$$

and, consequently,

$$|\langle T_f T_{\bar{g}} u, v\rangle_\alpha| \le C\|u\|_{L^p}\|v\|_{L^q}. \qquad \square$$

The proof of Theorem 1.2 is analogous.

We should mention that Theorem 1.1 extends the results obtained by Miao [4] and Stroethoff and Zheng [12]. Namely, we have the following result.

Lemma 2.4. *Let $1/p+1/q=1$ and $f\in A^p_\alpha$, $g\in A^q_\alpha$. Then, for $\epsilon>0$ and $w\in\mathcal{B}$,*

$$\begin{aligned} &\{B^p_\epsilon[|fk_w^{1-2/p}|^p](w)\}^{1/p}\{B^q_\epsilon[|gk_w^{1-2/q}|^q](w)\}^{1/q} \\ &\qquad \le C\{B[|fk_w^{1-2/p}|^{2+\epsilon}](w)\}^{1/(p+\epsilon)}\{B[|gk_w^{1-2/q}|^{2+\epsilon}](w)\}^{1/(q+\epsilon)}. \end{aligned}$$

Proof. Let $w\in\mathcal{B}$ be fixed. Using Hölder's inequality,

$$\begin{aligned} &\{B^p_\epsilon[|fk_w^{1-2/p}|^p](w)\}^{1/p} \\ &\quad = \Big\{\int_{\mathcal{B}} |f(z)k_w^{1-2/p}(z)|^p \log^{p(1+\epsilon)/q}(1/(1-|\varphi_w(z)|)) \\ &\qquad \times \frac{(1-|w|^2)^{n+1+\alpha}}{|1-\langle w,z\rangle|^{2n+2+2\alpha}}\, dv_\alpha(z)\Big\}^{1/p} \\ &\quad = \{B[|fk_w^{1-2/p}|^{2+\epsilon}](w)\}^{1/(p+\epsilon)} \\ &\qquad \times \Big\{\int_{\mathcal{B}} \log^{p(p+\epsilon)/(q\epsilon)}(1/(1-|z|))\, dv_\alpha(z)\Big\}^{\epsilon/((p+\epsilon)p)}. \end{aligned}$$

The convergence of the last integral implies the desired result. $\square$

https://doi.org/10.1017/S1446788715000129 Published online by Cambridge University Press

3. Conditions for boundedness of mixed Hankel and Toeplitz products

In this section we investigate products of the Hankel operator and the Toeplitz operator.

First, we introduce the so-called dual Toeplitz operator. Let $f \in L^\infty(\mathcal{B})$. In view of the decomposition (1.1), the multiplication operator $M_f g = fg$ on L^p_α can be written as follows:

$$M_f = \begin{bmatrix} T_f & H^*_{\overline{f}} \\ H_f & S_f \end{bmatrix}.$$

The operator $S_f : (A^q_\alpha)^\perp \to (A^q_\alpha)^\perp$, given by the formula

$$S_f(h)(z) = f(z)h(z) - P(fh)(z),$$

is called the dual Toeplitz operator. The above representation of M_f on L^p_α is analogous to the representation of the multiplication operator defined on L^2_α (see [3, 12]). In particular, we have $T^*_f = T_{\overline{f}}$ and $S^*_f = S_{\overline{f}}$. We should mention that in the case when $f \in L^1_\alpha$, the operators introduced above are densely defined on A^p_α. The next lemma gives some properties of the operator M_f.

Lemma 3.1. *Let $\psi \in L^\infty$ and $\phi \in H^\infty$. Then*

$$S_\phi H_\psi = H_\psi T_\phi \quad \textit{and} \quad H^*_\psi S_{\overline{\phi}} = T_{\overline{\phi}} H^*_\psi,$$

where S is the dual Toeplitz operator.

Proof. The proof proceeds analogously as for the space A^2_α (see [12, page 297]). □

Let $1/p + 1/q = 1$. For $f \in L^q_\alpha$, $g \in L^p_\alpha$, we define an operator $f \otimes g$ on L^q_α by the formula

$$(f \otimes g)h = \langle h, g \rangle_\alpha f.$$

One can show that $\|f \otimes g\| = \|f\|_{L^q} \|g\|_{L^p}$. If $f \in A^p_\alpha$, then $(g - P(g)) \otimes f$ can be seen as an operator on A^p_α, which has the following representation.

Lemma 3.2. *Let $1/p + 1/q = 1$ and $f \in A^q_\alpha$, $g \in L^p_\alpha$.*

(i) *If $\alpha \neq 0, 1, 2, \ldots$, then*

$$(g - P(g)) \otimes f = \sum_{k=0}^{\infty} \frac{\Gamma(k - n - 1 - \alpha)}{k!\Gamma(-n - 1 - \alpha)} \sum_{|s|=k} \frac{k!}{s!} S_{z^s} H_g T_{\overline{f}} T_{\overline{z}^s}.$$

(ii) *If $\alpha = 0, 1, 2, \ldots$, then*

$$(g - P(g)) \otimes f = \sum_{k=0}^{n+1+\alpha} \frac{(-1)^k (n + 1 + \alpha)!}{k!(n + 1 + \alpha - k)!} \sum_{|s|=k} \frac{k!}{s!} S_{z^s} H_g T_{\overline{f}} T_{\overline{z}^s}.$$

https://doi.org/10.1017/S1446788715000129 Published online by Cambridge University Press

PROOF. For all functions in A^p_α we have the atomic decomposition

$$f(z) = \sum_{k=0}^{\infty} c_k (1 - |w_k|^2)^{(n+1+\alpha)(1-1/p)} K_{w_k}(z),$$

where $\{w_k\}_{k=0}^{\infty}$ is a sequence in $\mathcal{B}$, $\{c_k\}_{k=0}^{\infty}$ belongs to l^p and the series converges in the norm of A^p_α (see [17, Theorem 2.30]). Thus, in order to show the equality of two bounded operators on A^p_α, it is enough to show that they are the same on K_w for all $w \in \mathcal{B}$. Clearly,

$$((g - P(g)) \otimes f)(K_w) = \overline{f(w)}(g - P(g)).$$

On the other hand, for any multi-index s,

$$S_{z^s} H_g T_{\overline{f}} T_{\overline{z}^s} K_w = \overline{w}^s \overline{f(w)} S_{z^s}(H_g K_w) = \overline{w}^s \overline{f(w)} (z^s g K_w - P(z^s g K_w)).$$

Using the identity

$$\langle z, w \rangle^k = \sum_{|s|=k} \frac{k!}{s!} z^s \overline{w}^s,$$

$$\sum_{|s|=k} \frac{k!}{s!} S_{z^s} H_g T_{\overline{f}} T_{\overline{z}^s} K_w(z) = \overline{f(w)} g(z) K_w(z) \langle z, w \rangle^k - P(\overline{f(w)} g K_w \langle z, w \rangle^k)(z)$$

and, consequently,

$$\sum_{k=0}^{\infty} \frac{\Gamma(k-n-1-\alpha)}{k!\Gamma(-n-1-\alpha)} [\overline{f(w)} g(z) K_w(z) \langle z, w \rangle^k - P(\overline{f(w)} g K_w \langle z, w \rangle^k)(z)]$$
$$= \overline{f(w)}(g(z) - P(g)(z)).$$

This completes the proof of (i). The proof of (ii) is analogous. □

To prove Theorem 1.3 we also need a few technical lemmas. The first one can be obtained by proceeding analogously to Miao's proof of [4, Lemma 2.2].

LEMMA 3.3. *There exists a positive constant C such that for any nonnegative integer k:*

(i) *if $1 < p < 2$, then, for all functions $u \in A^p_\alpha$ and $v \in (A^q_\alpha)^\perp$, $1/p + 1/q = 1$,*

$$\sum_{|s|=k} \left(\frac{k!}{s!}\right)^{p/2} \|T_{\overline{z}^s} u\|^p_{L^p} \le C(k+1)^{(n-1)(1-p/2)} \|u\|_{L^p},$$

$$\sum_{|s|=k} \left(\frac{k!}{s!}\right)^{p/2} \|S^*_{z^s} v\|^p_{L^p} \le C(k+1)^{(n-1)(1-p/2)} \|v\|_{L^p};$$

https://doi.org/10.1017/S1446788715000129 Published online by Cambridge University Press

(ii) *if* $2 \le p < \infty$, *then, for all functions* $u \in A^p_\alpha$ *and* $v \in (A^q_\alpha)^\perp$, $1/p + 1/q = 1$,

$$\sum_{|s|=k} \Big(\frac{k!}{s!}\Big)^{p/2} \|T_{\overline{z}^s} u\|^p_{L^p} \le C\|u\|_{L^p},$$

$$\sum_{|s|=k} \Big(\frac{k!}{s!}\Big)^{p/2} \|S^*_{z^s} v\|^p_{L^p} \le C\|v\|_{L^p}.$$

Now, using Lemma 3.3, we prove the following result.

Lemma 3.4. *Let* $1/p + 1/q = 1$ *and* $f \in A^q_\alpha$, $g \in L^p_\alpha$. *Then there exists a positive constant* C *such that*

$$\|f\|_{L^q}\|g - P(g)\|_{L^p} \le C\|H_g T_{\overline{f}}\|.$$

Proof. Suppose that $u \in A^p_\alpha$ and $v \in (A^p_\alpha)^\perp$. Then, by Lemma 3.2 and the triangle inequality,

$$\begin{aligned} &|\langle((g - P(g)) \otimes f)u, v\rangle_\alpha| \\ &\quad = \Big|\sum_{k=0}^{\infty} \frac{\Gamma(k-n-1-\alpha)}{k!\Gamma(-n-1-\alpha)} \sum_{|s|=k} \frac{k!}{s!} \langle S_{z^s} H_g T_{\overline{f}} T_{\overline{z}^s} u, v\rangle_\alpha\Big| \\ &\quad \le \sum_{k=0}^{\infty} \Big|\frac{\Gamma(k-n-1-\alpha)}{k!\Gamma(-n-1-\alpha)}\Big| \sum_{|s|=k} \frac{k!}{s!} \|H_g T_{\overline{f}}\| \, \|T_{\overline{z}^s} u\|_{L^p}\| \, \|S^*_{z^s} v\|_{L^q}. \end{aligned}$$

Using Hölder's inequality and Lemma 3.3,

$$\sum_{|s|=k} \frac{k!}{s!} \|T_{\overline{z}^s} u\|_{L^p}\| \, \|S^*_{z^s} v\|_{L^q} \le C(k+1)^{(n-1)/2} \|u\|_{L^p}\|v\|_{L^q}.$$

To complete the proof, we observe that Gauss's formula (see, for example, [1, page 178]) guarantees the convergence of the series

$$\sum_{k=0}^{\infty} \Big|\frac{\Gamma(k-n-1-\alpha)}{k!\Gamma(-n-1-\alpha)}\Big| (k+1)^{(n-1)/2}. \qquad \square$$

Finally, we describe the commutative property of the Hankel operator. Let $w \in \mathcal{B}$ be fixed, and let the mapping U_w be defined by the formula

$$U_w h = (h \circ \varphi_w) k_w \quad h \in L^p_\alpha, \ 1 < p < \infty.$$

Then we have the following result.

Lemma 3.5. *For any fixed* $w \in \mathcal{B}$ *and* $g \in L^\infty(\mathcal{B})$,

$$U_w H_g = H_{g \circ \varphi_w} U_w.$$

https://doi.org/10.1017/S1446788715000129 Published online by Cambridge University Press

PROOF. For $u \in A^p_\alpha$, $v \in A^q_\alpha$, we have $U_w T_g = T_{g\circ\varphi_w} U_w$ (see [14, (2.3)]). Consequently,

$$\begin{aligned} U_w H_g u &= U_w(gu) - U_w P(gu) = (g\circ\varphi_w)(u\circ\varphi_w)k_w - T_{g\circ\varphi_w} U_w u \\ &= (g\circ\varphi_w)U_w u - P((g\circ\varphi_w)U_w u) = H_{g\circ\varphi_w} U_w u, \end{aligned}$$

which completes the proof of the lemma. □

PROOF OF THEOREM 1.3. For any fixed $w \in \mathcal{B}$, we define an operator $V^p_w : A^p_\alpha \to A^p_\alpha$ in the following way:

$$V^p_w h = P((U_w h)\overline{k}_w^{2/p-1})$$

and an operator $\widetilde{V}^q_w : (A^p_\alpha)^\perp \to (A^p_\alpha)^\perp$ as follows:

$$\widetilde{V}^q_w h = (U_w h)\overline{k}_w^{2/q-1}.$$

Let $u \in (A^q_\alpha)^\perp$ and $v \in (A^p_\alpha)^\perp$; then

$$\begin{aligned} &\langle U_w v, u\rangle_\alpha \\ &\quad = \langle \overline{k}_w^{1-2/q}(v\circ\varphi_w)k_w\overline{k}_w^{2/q-1}, u\rangle_\alpha - \langle P(\overline{k}_w^{1-2/q}(v\circ\varphi_w)k_w\overline{k}_w^{2/q-1}), u\rangle_\alpha \\ &\quad = \langle S_{\overline{k}_w^{1-2/q}}((v\circ\varphi_w)k_w\overline{k}_w^{2/q-1}), u\rangle_\alpha = \langle S_{\overline{k}_w^{1-2/q}} \widetilde{V}^q_w v, u\rangle_\alpha. \end{aligned}$$

Hence,

$$U_w v = S_{\overline{k}_w^{1-2/q}} \widetilde{V}^q_w v.$$

Moreover, it is clear that $P(\overline{\phi}P(g)) = P(\overline{\phi}g)$ for any holomorphic function ϕ. Thus, for $h \in A^p_\alpha$,

$$T_{\overline{k}_w^{1-2/p}} V^p_w h = T_{\overline{k}_w^{1-2/p}} P((h\circ\varphi_w)k_w\overline{k}_w^{2/p-1}) = P((h\circ\varphi_w)k_w) = U_w h.$$

Now let $u \in A^p_\alpha$ and $v \in (A^p_\alpha)^\perp$. Then, by Lemma 3.5,

$$\begin{aligned} &\langle H_{g_1\circ\varphi_w} T_{\overline{f}_1\circ\varphi_w} u, v\rangle_\alpha \\ &\quad = \langle H_{g_1} T_{\overline{f}_1} U_w u, U_w v\rangle_\alpha = \langle T_{\overline{f}_1} T_{\overline{k}_w^{1-2/p}} V^p_w u, H^*_{g_1} S_{\overline{k}_w^{1-2/q}} \widetilde{V}^q_w v\rangle_\alpha. \end{aligned}$$

Next, putting $f_1 = fk_w^{2/p-1} \in A^q_\alpha$ and $g_1 = gk_w^{2/q-1} \in L^p_\alpha$ and using Lemma 3.1,

$$\begin{aligned} &\langle H_{gk_w^{2/q-1}\circ\varphi_w} T_{\overline{fk_w^{2/p-1}}\circ\varphi_w} u, v\rangle_\alpha \\ &\quad = \langle T_{\overline{fk_w^{2/p-1}}} T_{\overline{k}_w^{1-2/p}} V^p_w u, H^*_{gk_w{}^{2/q-1}} S_{\overline{k}_w^{1-2/q}} \widetilde{V}^q_w v\rangle_\alpha \\ &\quad = \langle T_{\overline{f}} V^p_w u, T_{\overline{k}_w^{1-2/q}} P(\overline{gk_w^{2/q-1}} \widetilde{V}^q_w v)\rangle_\alpha = \langle H_g T_{\overline{f}} V^p_w u, \widetilde{V}^q_w v\rangle_\alpha. \end{aligned}$$

Consequently,

$$|\langle H_{(gk_w^{2/q-1})\circ\varphi_w} T_{(\overline{fk_w^{2/p-1}})\circ\varphi_w} u, v\rangle_\alpha| \le \|H_g T_{\overline{f}}\| \, \|V^p_w u\|_{L^p} \|\widetilde{V}^q_w v\|_{L^q}.$$

https://doi.org/10.1017/S1446788715000129 Published online by Cambridge University Press

Since

$$\|V_w^p u\|_{L^p} = \|P((u \circ \varphi_w) k_w \overline{k}_w^{2/p-1})\|_{L^p} \le C\|u\|_{L^p}$$

and

$$\|\widetilde{V}_w^q v\|_{L^q} = \|(v \circ \varphi_w) k_w \overline{k}_w^{2/q-1}\|_{L^q} = \|v\|_{L^q},$$

$$\|H_{g_1 \circ \varphi_w} T_{\overline{f_1 \circ \varphi_w}}\| \le C\|H_g T_{\overline{f}}\|.$$

Thus, by Lemma 3.4,

$$\|(f k_w^{1-2/q}) \circ \varphi_w\|_{L^q} \|(g k_w^{1-2/p}) \circ \varphi_w - P((g k_w^{1-2/p}) \circ \varphi)\|_{L^p} \le C\|H_g T_{\overline{f}}\|,$$

which completes the proof. □

Now we give the proof of our last theorem.

Proof of Theorem 1.4. It is enough to show that there exists a positive constant C such that for any $u \in A_\alpha^p$ and $v \in (A_\alpha^p)^\perp$ the following inequality holds:

$$|\langle H_g T_{\overline{f}} u, v\rangle_\alpha| \le C\|u\|_{L^p}\|v\|_{L^q}.$$

By Lemma 2.2,

$$\begin{aligned} |\langle H_g T_{\overline{f}} u, v\rangle_\alpha| &= |\langle R^{\beta,t_1} T_{\overline{f}} u, R^{\beta+t_1,t_2} H_g^* v\rangle_{t_1+t_2+\alpha}| \\ &\le \int_{\mathcal{B}} |R^{\beta,t_1} T_{\overline{f}} u(w)| \, |R^{\beta+t_1,t_2} H_g^* v| (1-|w|^2)^{t_1+t_2}\, d\upsilon_\alpha(w). \end{aligned}$$

Moreover, using Lemma 2.3 and putting $t_1 = t_2 = n + 1 + \alpha$,

$$\begin{aligned} &\int_{\mathcal{B}} |R^{\beta,t_1} T_{\overline{f}} u(w)| |R^{\beta+t_1,t_2} H_g^* v| (1-|w|^2)^{t_1+t_2}\, d\upsilon_\alpha(w) \\ &\quad \le C \sup_{w \in \mathcal{B}} \{\{B_{\epsilon_1}^q [|f k_w^{1-2/q}|^q]\}^{1/q} \\ &\qquad \times \|[(g k_w^{1-2/p}) \circ \varphi_w - P((g k_w^{1-2/p}) \circ \varphi_w)] \log^{(1+\epsilon_2)/q}(1/(1-|z|))\|_{L^p}\} \\ &\qquad \times \int_{\mathcal{B}} \bigg\{ \int_{\mathcal{B}} \frac{|u(z)|^p}{|1-\langle w, z\rangle|^{n+1+\alpha}} \log^{-(1+\epsilon_1)}(1/(1-|\varphi_w(z)|))\, d\upsilon_\alpha(z) \bigg\}^{1/p} \\ &\qquad \times \bigg\{ \int_{\mathcal{B}} \frac{|v(z)|^q}{|1-\langle w, z\rangle|^{n+1+\alpha}} \log^{-(1+\epsilon_2)}(1/(1-|\varphi_w(z)|))\, d\upsilon_\alpha(z) \bigg\}^{1/q} d\upsilon_\alpha(w). \end{aligned} \tag{3.1}$$

Now, applying Hölder's inequality and property (2.2) to the integral (3.1), we get the desired conclusion. □

https://doi.org/10.1017/S1446788715000129 Published online by Cambridge University Press

References

[1] J. B. Conway, *Functions of One Complex Variable*, 2nd edn, Graduate Texts in Mathematics, 11 (Springer, New York, Berlin, 1978).

[2] D. Cruz-Uribe, 'The invertibility of the product of unbounded Toeplitz operators', *Integral Equations Operator Theory* **20**(2) (1994), 231–237.

[3] Y. Lu and Ch. Liu, 'Toeplitz and Hankel products on Bergman spaces of the unit ball', *Chin. Ann. Math. B* **30**(3) (2009), 293–310.

[4] J. Miao, 'Bounded Toeplitz products on the weighted Bergman spaces of the unit ball', *J. Math. Anal. Appl.* **346** (2008), 305–313.

[5] M. Michalska, M. Nowak and P. Sobolewski, 'Bounded Toeplitz and Hankel products on weighted Bergman spaces of the unit ball', *Ann. Polon. Math.* **99**(1) (2010), 45–53.

[6] F. Nazarov, 'A counterexample to Sarason's conjecture', Preprint available online at www.math.msu.edu/~fedja/prepr.html.

[7] J. D. Park, 'Bounded Toeplitz products on the Bergman space of the unit ball in C^n', *Integral Equations Operator Theory* **54**(4) (2006), 571–584.

[8] S. Pott and E. Strouse, 'Products of Toeplitz operators on the Bergman spaces A^2_α', *St. Petersburg Math. J.* **18**(1) (2007), 105–118.

[9] W. Rudin, *Functional Analysis*, McGraw-Hill Series in Higher Mathematics, 397 (McGraw-Hill, New York, Düsseldorf, Johannesburg, 1973).

[10] W. Rudin, *Function Theory in the Unit Ball of C^n*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], 241 (Springer, New York, Berlin, 1980).

[11] D. Sarason, 'Products of Toeplitz operators', in: *Linear and Complex Analysis Problem Book 3*, Lecture Notes in Mathematics, 1573 (Springer, Berlin, New York, 1994), 318–319.

[12] K. Stroethoff and D. Zheng, 'Products of Hankel and Toeplitz operators on the Bergman space', *J. Funct. Anal.* **169** (1999), 289–313.

[13] K. Stroethoff and D. Zheng, 'Bounded Toeplitz products on the Bergman space of the polydisk', *J. Math. Anal. Appl.* **278**(1) (2003), 125–135.

[14] K. Stroethoff and D. Zheng, 'Bounded Toeplitz products on Bergman spaces of the unit ball', *J. Math. Anal. Appl.* **325** (2007), 114–129.

[15] K. Stroethoff and D. Zheng, 'Bounded Toeplitz products on weighted Bergman spaces', *J. Operator Theory* **59**(2) (2008), 277–308.

[16] D. Zheng, 'The distribution function inequality and products of Toeplitz and Hankel operators', *J. Funct. Anal.* **138** (1996), 477–501.

[17] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Graduate Texts in Mathematics, 226 (Springer, New York, 2005).

MAŁGORZATA MICHALSKA, Instytut Matematyki UMCS,
pl. Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland
e-mail: malgorzata.michalska@umcs.pl

PAWEŁ SOBOLEWSKI, Instytut Matematyki UMCS,
pl. Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland
e-mail: pawel.sobolewski@umcs.eu