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EXTREMAL KÄHLER METRICS AND HAMILTONIAN FUNCTIONS II

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Abstract. We apply a previously obtained ansatz for extremal Kähler metrics to show that if a manifold admits a Hodge metric with constant scalar curvature, then the total space of the projectivization of a line bundle with first Chern class equal to the Kähler class of the metric admits a one-parameter family of extremal Kähler metrics. This generalizes earlier constructions.

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1. Introduction. The notion of extremal Kähler metrics was introduced by Calabi [**2**]. On a compact complex manifold M^{2m}, consider the functional $\mathcal{S}(\Omega) = \int_M s^2\Omega^m$, where Ω is a Kähler form in a fixed Kähler class $[\Omega] \in H^2(M, \mathbb{R})$, and s is the scalar curvature of Ω. Kähler metrics corresponding to critical points of $\mathcal{S}$ are called *extremal* Kähler metrics. If g is a Kähler metric, then g is extremal if and only if $grad\, s$ is a real holomorphic vector field. This is equivalent to $(\bar{\partial}s)^\sharp$ being a holomorphic $(1, 0)$ vector field (we use $\sharp$ for raising indices and $\flat$ for lowering indices).

In section 2, we review and refine the ansatz for extremal Kähler metrics obtained in [**3**]. Proposition 2.1, Theorem 2.4, and Proposition 2.5 in [**3**] give an ansatz for extremal Kähler metrics with torus symmetry assuming that the Kähler quotient metric is of a very special form, namely $q = q_{\mu\nu}(dx^\mu dx^\nu + dy^\mu dy^\nu)$, where $q_{\mu\nu}$ is real. In this work the restriction on the Kähler quotient is removed.

In section 3, we solve the equations from the ansatz in a special case and obtain the main result which states that if a manifold admits a Hodge metric with constant scalar curvature, then the total space of the projectivization of a line bundle with first Chern class equal to the Kähler class of the metric admits a one-parameter family of extremal Kähler metrics. Note that this result has constructions from [**2**, **3**, **6**, **7**, **15**, **17**] as special cases , but it also gives us many more new examples.

In section 4, we restrict our attention to the case $m = 2$. We then consider the metrics among the solutions from above which are locally conformal to an Einstein metric (wherever the scalar curvature does not vanish). These metrics are Bach flat, which means that they are extremal points of the conformally invariant functional $\mathcal{W}$ defined as the (square of) the L^2-norm of the Weyl curvature. We observe that the metrics give rise to a sequence $\{g_t\}$ of Bach flat metrics on the trivial (product) ruled surface of any genus such that $\lim_{t\to\infty} \mathcal{W}([g_t]) = +\infty$.

2. An ansatz for extremal Kähler metrics. In this section, assuming the existence of a real torus acting through holomorphic isometries on a Kähler manifold, we construct an ansatz for extremal Kähler metrics.

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2.1. The moment map construction of Kähler metrics. Following [**14**] we consider the situation of a real torus T^N acting freely on the Kähler manifold M^{2m} through holomorphic isometries.

PROPOSITION 1 [**14**] *Let* (w_{ij}), $i, j = 1, \dots, N$ *be a positive definite symmetric matrix and* $(h_{\mu\nu})$, $\mu, \nu = 1, \dots, m-N$ *a positive definite hermitian matrix of smooth functions on an open set* U *in* $\mathbb{C}^{m-N} \times \mathbb{R}^N$ *with coordinates* (ξ^μ, z^i). *Assume that the* 2*-form*

$$\Omega_h := \frac{\sqrt{-1}}{2} h_{\mu\nu} d\xi^\mu \wedge d\overline{\xi^\nu}$$

is a Kähler form on an open set in $\mathbb{C}^{m-N}$ *with corresponding Kähler metric* h. *Let* M *be a* T^N*-bundle over* U *with connection* 1*-form* $\omega = (\omega_1, \dots, \omega_N)$. *Suppose that*

$$\frac{\partial^2 h_{\mu\nu}}{\partial z^i \partial z^j} + 4\frac{\partial^2 w_{ij}}{\partial \xi^\mu \partial \overline{\xi^\nu}} = 0, \tag{1}$$

$$\frac{\partial w_{ij}}{\partial z^k} = \frac{\partial w_{ik}}{\partial z^j} \tag{2}$$

and assume the torus bundle has curvature

$$F_i = \frac{\sqrt{-1}}{2}\frac{\partial h_{\mu\nu}}{\partial z^i} d\xi^\mu \wedge d\overline{\xi^\nu} + \sqrt{-1}\frac{\partial w_{ij}}{\partial \xi^\mu} dz^j \wedge d\xi^\mu - \sqrt{-1}\frac{\partial w_{ij}}{\partial \overline{\xi^\mu}} dz^j \wedge d\overline{\xi^\mu}. \tag{3}$$

Then

$$g = h + w_{ij} dz^i dz^j + w^{ij}\omega_i\omega_j, \tag{4}$$

where $w^{ij} = (w^{-1})_{ij}$, *is a Kähler metric on* M. *Conversely, any Kähler metric with a torus acting freely through Poisson commuting holomorphic isometries can locally be constructed as above.*

Proof. The proof is straightforward and we just make some remarks concerning the second part of the proposition. Let M be a T^N-symmetric Kähler manifold with metric g, Kähler form Ω, and complex structure J. Let $(X_1, \dots, X_N)$ be the Hamiltonian vector fields generated by the torus action, and let $dz^j = -i_{X_j}\Omega$ define the Hamiltonian functions z^j. Then the metric is given as in equation (4), where h is a Kähler metric in the quotient space of each level set of the Hamiltonians. Note that $w^{ij} = g(X_i, X_j)$ and $\omega_i = w_{ij}X_j^\flat$, so $J\omega_i = -w_{ij}dz^j$ and $\Omega = dz^i \wedge \omega_i + \Omega_h$, where Ω_h is the Kähler form of the Kähler quotient. As J is integrable, the exterior derivative $d\varphi_i$ of the $(1, 0)$ forms $\varphi_i = w_{ij}dz^j + \sqrt{-1}\omega_i$ must have no $(0, 2)$ part. Also, for g to be Kähler, we need $d\Omega = 0$. These conditions are captured by equation (2) and by the equation $d\omega_i = F_i$, with F_i as in (3)[1]. Then equation (1) is just the integrability condition $dF_i = 0$. □

[1]To be absolutely precise, the pull-back of F_i with respect to the bundle projection is given by dw_i.

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2.2. Extremal Kähler metrics. Now, let (M^{2m}, g) be a T^N-symmetric Kähler manifold as above. We look for the condition on the scalar curvature, s, so that the metric is extremal. We have that $\bar{\partial} s = \frac{1}{2}(ds - iJds)$ is given by

$$\bar{\partial} s = \frac{\partial s}{\partial \bar{\xi}^\mu} d\bar{\xi}^\mu + \frac{1}{2}\frac{\partial s}{\partial z^k}(dz^k - iw^{kl}\omega_l). \tag{5}$$

Therefore we get

$$(\bar{\partial} s)^\sharp = \frac{\partial s}{\partial \bar{\xi}^\mu}(d\bar{\xi}^\mu)^\sharp - \frac{i}{2}\frac{\partial s}{\partial z^k}(X_k - iJX_k).$$

We need to spell out the conditions for the vector field $(\bar{\partial} s)^\sharp$ to be holomorphic.

LEMMA 1. *There exist smooth functions $F_{k\mu}$ of (ξ^μ, z^l) such that the forms $\Phi_k = F_{k\mu} d\xi^\mu + w_{kl} dz^l + i\omega_k$ together with $d\xi^\mu$, $\mu = 1, \ldots, m-1$ and $k = 1, \ldots, N$, are a local basis of holomorphic $(1,0)$-forms.*

Proof. We refer to [**3**] noting that Φ_k is holomorphic if and only if

$$\frac{\partial F_{k\mu}}{\partial z^j} - 2\frac{\partial w_{kj}}{\partial \xi^\mu} = 0, \tag{6}$$

$$\frac{\partial h_{\mu\nu}}{\partial z^k} + 2\frac{\partial F_{k\mu}}{\partial \bar{\xi}^\nu} = 0. \tag{7}$$

The integrability condition for system (6) and (7) is satisfied due to (1), (2) and the fact that h is a Kähler metric. □

We are now ready to prove our ansatz. We refer to Proposition 1 for the notation.

THEOREM 1. *Let M^{2m} be a T^N-symmetric Kähler manifold of scalar curvature s. The the metric is extremal if and only if*

$$\frac{\partial (h^{\mu\nu}\frac{\partial s}{\partial \bar{\xi}^\nu})}{\partial z^k} = 0, \tag{8}$$

$$\frac{\partial (h^{\mu\nu}\frac{\partial s}{\partial \bar{\xi}^\nu})}{\partial \bar{\xi}^\lambda} = 0, \tag{9}$$

$$4h^{\mu\nu}\frac{\partial w_{kl}}{\partial \xi^\mu}\frac{\partial s}{\partial \bar{\xi}^\nu} + \frac{\partial^2 s}{\partial z^k \partial z^l} = 0. \tag{10}$$

Note that if $h_{\mu\nu}$ is real, then (8), (9) and (10) are equivalent to (13), (14), (15), (16), (17), and (18) in Theorem 2.4 of [**3**].

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Proof of the theorem. The $(1,0)$ vector field $(\overline{\partial}s)^\sharp$ is holomorphic if and only if $d\xi^\mu((\overline{\partial}s)^\sharp) = 2h^{\mu\nu}\frac{\partial s}{\partial\overline{\xi}^\nu}$, and $\Phi_k((\overline{\partial}s)^\sharp) = 2F_{k\lambda}h^{\lambda\nu}\frac{\partial s}{\partial\overline{\xi}^\nu} + \frac{\partial s}{\partial z^k}$ are holomorphic functions for all μ and k. □

In order to work with the above ansatz we need an expression for the scalar curvature.

Proposition 2 [**3**] *Let M^{2m} be a symmetric Kähler manifold as in Proposition 1 and let $u = \log\det h - \log\det w$. Then the scalar curvature s satisfies*

$$-s = \{4\frac{\partial^2 u}{\partial\xi^\mu\partial\overline{\xi}^\nu} + \left(w^{kl}\frac{\partial u}{\partial z^k}\right)\frac{\partial h_{\mu\nu}}{\partial z^l}\}h^{\mu\nu} + \frac{\partial}{\partial z^l}\left(w^{kl}\frac{\partial u}{\partial z^k}\right).$$

3. Construction of new extremal Kähler metrics. In this section, we consider the case $N = 1$. By solving the differential equations from the ansatz in a special case, we find new compact extremal Kähler metrics. The work here generalizes the work in Section 3 of [**3**] and makes up for the unnecessarily complicated presentation of the hypotheses in Theorem 3.1 in [**3**] (see footnote 2).

First, we give the details on the special case in which we solve the equations. Then, we apply the ansatz from the previous section.

3.1. The assumptions. Let (B, g_B) be a $(m-1)$-dimensional compact Kähler manifold with constant scalar curvature s_B. Assume that the Kähler form Ω_B is such that the deRham class $[\frac{\Omega_B}{2\pi}]$ is contained in the image of $H^2(B,\mathbb{Z}) \to H^2(B,\mathbb{R})$. Let L be a holomorphic line bundle such that $c_1(L) = [\frac{-\Omega_B}{2\pi}]$. On the total space M of $(L-0) \xrightarrow{\pi} B$, we can form an S^1-symmetric Kähler metric

$$g = zg_B + wdz^2 + w^{-1}\omega^2,$$

where z, being the coordinate of $(a,b) \subset (0,\infty]$, becomes the moment map of g with the obvious S^1 action on L, w is a positive function depending only on z, and ω is the connection one-form of the connection induced by g on the S^1-bundle

$$(L-0) \overset{(\pi,z)}{\to} B \times (a,b).$$

That is,

$$d\omega = \Omega_B.$$

Notice that equations (1) and (2) are satisfied. The complex structure J on M is given by the complex structure on B and

$$J\omega = -wdz.$$

The Kähler form is given by

In [**3**], at first glance, it does look as if we are constructing compact metrics on projective bundles over a product of negative Kähler-Einstein metrics. However, coinidentally, the metrics on each factor were chosen such that the product was itself a Kähler-Einstein manifold. It was the realization of this fact that motivated Theorem 3.

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$$\Omega = z\Omega_B + dz \wedge \omega.$$

If X is the Hamiltonian vector field generated by the S^1 action, then

$$dz = \Omega(-X, \cdot) = (-JX)^\flat,$$

$$w^{-1} = g(X, X),$$

and

$$\omega = \frac{g(X, \cdot)}{g(X, X)} = wX^\flat.$$

The Ricci form ρ is given by

$$\rho = \rho_B - i\partial\bar{\partial}\log(\frac{z^{m-1}}{w})$$

which implies that the scalar curvature s is given by

$$s = \frac{s_B}{z} - \frac{(\frac{z^{m-1}}{w})_{zz}}{z^{m-1}}.$$

If w^{-1} (by which we mean $1/w$) is such that $w^{-1}(a) = 0$ and $(w^{-1})'(a) = 2$, then we can add a copy of B at $z = a$ and extend the Kähler metric g over the zero section of the bundle $L \to B$. If, moreover, $b < \infty$, $w^{-1}(b) = 0$ and $(w^{-1})'(b) = -2$, then we can add another copy of B at $z = b$ and extend g to a Kähler metric on the total space of the $\mathbb{CP}_1$-bundle $\mathbb{P}(\mathcal{O} \oplus L)$. We refer to [**9**,**10**] for the details.

3.2. Applying the ansatz. In this special case, the only equation remaining from the ansatz is

$$s_{zz} = 0.$$

Integrating and using the above formula for s we get the equation

$$\frac{z^{m-1}}{w} = P(z),$$

where

$$P(z) = \frac{s_B}{(m-1)m} z^m - C_1 z^{m+2} - C_2 z^{m+1} - C_3 z - C_4.$$

The endpoint conditions on w^{-1} for compactification are equivalent to the following conditions on $P(z)$:

$$P(1) = P(b) = 0;$$

$$P'(1) = 2;$$

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$$P'(b) = -2b^{m-1}.$$

For convenience, we have assumed that $a = 1$. This can easily be achieved by rescaling. These conditions determine the coefficients of $P(z)$. Moreover, since w is a positive function, we need $P(z) > 0$ in the interval $(1, b)$. For a given b, the coefficients C_1, C_2, C_3 and C_4 are given as follows:

$$\begin{aligned}
C_1 &= \frac{n_1}{d};\\
C_2 &= \frac{n_2}{d};\\
C_3 &= -(m+2)C_1 - (m+1)C_2 + \frac{s_B}{(m-1)} - 2;\\
C_4 &= -C_1 - C_2 - C_3 + \frac{s_B}{(m-1)m};
\end{aligned}$$

where

$$\begin{aligned}
n_1 &= \frac{s_B}{(m-1)}\Big(\frac{-1}{m}b^{2m} + mb^{m+1} + 2\Big(\frac{1}{m} - m\Big)b^m + mb^{m-1} - \frac{1}{m}\Big)\\
&\quad + 2(b^{2m} - mb^{m+1} + mb^{m-1} - 1),\\
n_2 &= \frac{s_B}{(m-1)}\Big(\frac{2}{m}(b^{2m+1} + 1) - (m+1)(b^{m+2} + b^{m-1}) + \Big(m + 1 - \frac{2}{m}\Big)(b^{m+1} + b^m)\Big)\\
&\quad + 2((1 - b^{2m+1}) + (m+1)(b^{m+2} - b^{m-1}) + (m+2)(b^m - b^{m+1})),
\end{aligned}$$

and

$$d = b^{2m+2} - (m+1)^2 b^{m+2} + 2m(m+2)b^{m+1} - (m+1)^2 b^m + 1.$$

3.3. Case $m = 2$. When $s_B > 0$, we have Calabi's extremal Kähler metrics [**2**] on (non-trivial) Hirzebruch surfaces. When $s_B < 0$, we have extremal Kähler metrics on pseudo-Hirzebruch surfaces [**17**]. The case $s_B = 0$, which appears in Hwang's construction of extremal Kähler metrics [**7**], has, to the author's knowledge, not yet been considered explicitly for $m = 2$. In this case, we have

$$\begin{aligned}
C_1 &= \frac{2(b+1)}{(b-1)(b^2+4b+1)},\\
C_2 &= \frac{-2(b^2+1)}{(b-1)(b^2+4b+1)},\\
C_3 &= \frac{-2b(b^2+1)}{(b-1)(b^2+4b+1)},\\
C_4 &= \frac{2b^2(b+1)}{(b-1)(b^2+4b+1)}.
\end{aligned}$$

Hence

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$$P(z) = \frac{2}{(b-1)(b^2+4b+1)}(z-1)(b-z)((b+1)z^2+2bz+b^2+b).$$

Thus, for any given $b > 1$, $P(z)$ satisfies the boundary conditions and is positive in the interval $(1, b)$. The geometric picture is as follows. Let g_B be a scalar flat Kähler metric on a compact Riemann surface (of genus one). By rescaling we can assume that the class $[\frac{\Omega_B}{2\pi}]$ is integral. Let L be a line bundle on B such that $c_1(L) = [\frac{-\Omega_B}{2\pi}]$. Any negative line bundle can be obtained in this way. The above calculations show that the ruled surface $\mathbb{P}(\mathcal{O} \oplus L)$ has a one-parameter family of extremal Kähler metrics. The parameter b determines the Kähler class, and one can check (using the same ideas as in [**17**]) that varying b and rescaling (varying a) sweeps out the whole Kähler cone. Thus any Kähler class on $\mathbb{P}(\mathcal{O} \oplus L)$ has an extremal Kähler metric.

THEOREM 2. *Let B be a compact Riemann surface of genus one. Let L be a non-trivial holomorphic line bundle on B. Then any Kähler class on the ruled surface $\mathbb{P}(\mathcal{O} \oplus L)$ admits an extremal Kähler metric.*

3.4. Case $m \geq 2$. We want to find $b > 1$ such that $P(z)$ both satisfies the boundary conditions and is positive in the interval $(1, b)$. Given that the boundary conditions are satisfied, this would hold if $P''(z) < 0$ on the interval.

LEMMA 2. *Let the coefficients of $P(z)$ be such that the boundary conditions are satisfied. There exists $\beta > 1$ such that for $b \in (1, \beta)$, $P''(z)$ is negative in the interval $[1, b]$.*

Proof. We can write $P''(z) = z^{m-2} S_m(z)$ where

$$S_m(z) = -C_1(m+2)(m+1)z^2 - C_2 m(m+1)z + s_B.$$

Recall the general formula for the coefficients C_1 and C_2 and consider n_1, n_2 and d as functions of b. Firstly, observe that

$$d(1) = d'(1) = d''(1) = d'''(1) = 0$$

and

$$d''''(1) > 0.$$

Secondly, observe that

$$n_1(1) = n_1'(1) = n_1''(1) = 0$$

and

$$n_1'''(1) > 0.$$

Setting $h = mn_2 + (m+2)n_1$ we also have that

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$$h(1) = h'(1) = h''(1) = 0$$

and

$$h'''(1) > 0.$$

Finally, setting $f = -(m+1)h + s_B d$, we have that

$$f(1) = f'(1) = f''(1) = 0$$

and

$$f'''(1) < 0.$$

From the above we see that there exists a $\beta > 1$ such that if $b \in (1, \beta)$, then $d > 0$, $C_1 > 0$, $h > 0$, and $f < 0$. In this case, S_m is concave down and the apex

$$z = \frac{1}{2}\frac{-mC_2}{(m+2)C_1} = \frac{1}{2}\frac{-mn_2}{(m+2)n_1}$$

is less than $\frac{1}{2}$. Moreover,

$$S_m(1) = -C_1(m+2)(m+1) - C_2 m(m+1) + s_B = \frac{f}{d} < 0.$$

This tells us that there are no roots to the right of $z = 1$ and consequently for $b \in (1, \beta)$, $S_m(z) < 0$ for $z \geq 1$. In particular, S_m and $P''(z)$ are negative in the interval $[1, b]$. □

Thus we have the following result.

THEOREM 3. *Let B be a compact Kählerian manifold which admits a Hodge metric with constant scalar curvature. Let L be a holomorphic line bundle on B such that that first Chern class of L is given by (±) the Kähler class of the metric. Then the total space M of* $\mathbb{P}(\mathcal{O} \oplus L) \to B$ *admits an extremal Kähler metric.*

Notice that a manifold satisfying the conditions in Theorem 3 must be a projective algebraic manifold.

Proof of the theorem. If $\tilde{g}_B$ is a Hodge metric with $\tilde{s}_B =$constant, then $[\tilde{\Omega}_B]$ sits in the image of $H^2(B, \mathbb{Z}) \to H^2(B, \mathbb{R})$. By setting $g_B = 2\pi\tilde{g}_B$, we have that

$$c_1(L) = [\pm\tilde{\Omega}_B] = [\frac{\pm\Omega_B}{2\pi}].$$

Since $\mathbb{P}(\mathcal{O} \oplus L) \cong \mathbb{P}(\mathcal{O} \oplus L^{-1})$, we may assume that

$$c_1(L) = [\frac{-\Omega_B}{2\pi}].$$

Now let β be as in Lemma 2. For $b \in (1, \beta)$, $z \in (1, b)$, $\frac{z^{m-1}}{w} = P(z)$, and $P(z)$ satisfying the boundary conditions, the metric

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$$g = zg_B + wdz^2 + w^{-1}\omega^2$$

is an extremal Kähler metric on the total space of $(L - 0) \to B$ which extends smoothly to an extremal Kähler metric on the total space of $\mathbb{P}(\mathcal{O} \oplus L)$. □

The scalar curvature of the metric is given by

$$s = (m+1)((m+2)C_1 z + mC_2).$$

For $b \in (1, \beta)$, where β is as in the proof of Lemma 2, we have that $C_1 > 0$, and, at the point $z = 1$, $s = \frac{h}{d} > 0$. Thus s is positive on M.

If (B, g_B) is a product of non-negative Kähler-Einstein manifolds, then these metrics have been constructed by Hwang [**7**], (generalizing Calabi's construction [**2**] for $B = \mathbb{CP}_{m-1}$). See also Guan's paper [**6**]. If (B, g_B) is a Kähler-Einstein manifold with $s_B = -2(m-1)$, then the metrics were constructed in [**3**][2]. However, the above theorem includes many more new examples. For instance, (B, g_B) could be a product of Kähler-Einstein manifolds, not necessarily with the same sign of curvature.

Let B be a projective algebraic manifold. Assume that $H^2(B, \mathbb{R}) = H^{1,1}(B, \mathbb{R})$. Then $H^2(B, \mathbb{Q})$ is dense in $H^{1,1}(B, \mathbb{R})$. The set of Kähler classes of extremal Kähler metrics on B is open in $H^{1,1}(B, \mathbb{R})$ [**12**]. Thus, if there exists a constant scalar curvature Kähler metric on B, then there exists an extremal Kähler metric on B whose Kähler class sits in $H^2(B, \mathbb{Q})$. By a suitable rescaling, we have an extremal Hodge metric. If B does not have any non-trivial holomorphic vector fields of gradient type, this metric has constant scalar curvature.

THEOREM 4. *Let B be a compact projective algebraic manifold such that $H^2(B, \mathbb{R}) = H^{1,1}(B, \mathbb{R})$. Assume that B has no non-trivial holomorphic vector fields of gradient type. If B admits a constant scalar curvature Kähler metric, then there exists a holomorphic line bundle L on B such that the total space of $\mathbb{P}(\mathcal{O} \oplus L) \to B$ admits an extremal Kähler metric.*

EXAMPLE 1. A Kähler surface which satisfies the conditions in the above theorem can be obtained by blowing-up a ruled surface of genus at least two sufficiently many times [**11**].

4. Bach flat metrics. In this section, we restrict our attention to the case where the complex dimension is equal to two and discuss the metrics among the solutions from the last section which are locally conformal to an Einstein metric (apart from where the scalar curvature vanishes).

4.1. Case $m = 2$ revisited. Let $m = 2$ and consider the extremal Kähler metrics constructed in the previous section. In this case, B is a compact Riemann surface of constant scalar curvature s_B. When $s_B \neq 0$, then $c_1(L) = \frac{2}{s_B} c_1(K)$, where K is the canonical bundle over B. If $s_B > 0$, then the possible values of s_B are $s_B = \frac{4}{k}$, where $k \in \mathbb{N}$. Then $L = K^{\frac{k}{2}} = \mathcal{O}(-k)$. If $s_B < 0$, we will, for simplicity, assume that $s_B = \frac{-4}{k}$,

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$k \in \mathbb{N}$. Thus $c_1(L) = -k(\mathbf{g} - 1)$, where $\mathbf{g}$ denotes the genus of B, and $L = K^{\frac{-k}{2}}$ up to diffeomorphism.

If $s_B \geq 0$, then for any $b > 1$, that is, in any Kähler class [**17**], we have an extremal Kähler metric

$$g = zg_B + \frac{z}{P(z)}dz^2 + \frac{P(z)}{z}\omega^2,$$

where

$$P(z) = \frac{s_B}{2}z^2 - C_1 z^4 - C_2 z^3 - C_3 z - C_4,$$

$$C_1 = \frac{\frac{s_B}{2}(1-b) + 2(b+1)}{(b-1)(b^2+4b+1)},$$

$$C_2 = \frac{s_B(b^2-1) - 2(b^2+1)}{(b-1)(b^2+4b+1)},$$

$$C_3 = -4C_1 - 3C_2 + s_B - 2,$$

and

$$C_4 = -C_1 - C_2 - C_3 + \frac{s_B}{2}.$$

If $s_B < 0$, then there exists a $\tilde{b} > 1$ such that for any b, $1 < b < \tilde{b}$, we have an extremal Kähler metric as above [**17**]. The bound $\tilde{b}$ is the unique solution, greater than one, of the equation

$$\tilde{b}^4 - 4(k^2 + 3k + 1)\tilde{b}^3 + 2(3 - 2k^2)\tilde{b}^2 - 4(k^2 - 3k + 1)\tilde{b} + 1 = 0.$$

One checks easily that $\tilde{b} > 6$ for any $k \in \mathbb{N}$.

4.2. Extremal Kähler metrics which are locally conformal to Einstein metrics. Let g be an extremal metric as in the above subsection. It is well known, [**5**], that if

$$s^2 - 6s\Delta s - 12|ds|^2 = 0,$$

then the metric $s^{-2}g$ is an Einstein metric on $M \setminus \{s = 0\}$. Since $s = 6(2C_1 z + C_2)$, this equation reduces to the equation

$$4C_1C_4 = C_2C_3,$$

which in turn becomes an equation in b:

$$s_B^2(-b^4 + b^3 + b - 1) + 4s_B(b^4 - 2b^3 + 2b - 1) + 4(-b^4 + 4b^3 + 6b^2 + 4b - 1) = 0. \tag{11}$$

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For each solution $\hat{b}$ to equation (11), we have a corresponding metric $\hat{g}$ of the type described in subsection 4.1 such that $\hat{s}^{-2}\hat{g}$ is Einstein where defined.

4.2.1. Case $s_B=0$. In this case equation (11) becomes

$$12b^2 - (b-1)^4 = 0,$$

and $\hat{b} = 1 + \sqrt{3} + \sqrt{3+\sqrt{12}} \approx 5.275$ solves the equation.

4.2.2. Case $s_B>0, s_B \neq 2$. Since the left hand side of equation (11) is positive at $b=1$ and the limit as b goes to $+\infty$ is equal to $-\infty$, there exists a $\hat{b}>1$ solving (11). Moreover, $\lim_{s_B\to 0}\hat{b} = 1+\sqrt{3}+\sqrt{3+\sqrt{12}}$.

4.2.3. Case $s_B=2$. Equation (11) has no solutions greater than one.

4.2.4. Case $s_B<0$. Since the left hand side of equation (11) is positive at $b=1$ and negative at $b=6$, there exists a $\hat{b}$ solving (11) such that $1<\hat{b}<6<\tilde{b}$. Again, $\lim_{s_B\to 0}\hat{b} = 1+\sqrt{3}+\sqrt{3+\sqrt{12}}$.

We conclude with the following proposition.

PROPOSITION 3. *Let B be a compact Riemann surface of genus* **g**.

(1) [**1**, **4**, **8**] *If* $\mathbf{g}=0$, *then on any complex manifold of the type $M=\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(k))\to B$, $k\in\mathbb{N}\setminus\{2\}$, there exists an extremal Kähler metric $\hat{g}$ of the type described in subsection 4.1 such that $\hat{s}^{-2}\hat{g}$ is an Einstein metric on $M\setminus\{s=0\}$.*

(2) *If* $\mathbf{g}=1$, *then on any complex manifold of the type $M=\mathbb{P}(\mathcal{O}\oplus L)\to B$, where L is a non-flat holomorphic line bundle, there exists an extremal Kähler metric $\hat{g}$ of the type described in subsection 4.1 such that $\hat{s}^{-2}\hat{g}$ is an Einstein metric on $M\setminus\{s=0\}$.*

(3) *If* $\mathbf{g}\geq 2$, *then on any complex manifold of the type $M=\mathbb{P}(\mathcal{O}\oplus L)\to B$, where L is a holomorphic line bundle such that $L\overset{C^\infty}{\cong}K^{\frac{-k}{2}}$, $k\in\mathbb{N}$ and K is the canonical line bundle on B, there exists an extremal Kähler metric $\hat{g}$ of the type described in subsection 4.1 such that $\hat{s}^{-2}\hat{g}$ is an Einstein metric on $M\setminus\{s=0\}$.*

Notice that $\{s=0\}$ is a real smooth submanifold of M. Unless $\mathbf{g}=0$ and $k=1$ (and $\hat{s}^{-2}\hat{g}$ is the Page metric [**4**]) this submanifold is never the empty set.

The fact that $\hat{g}$ is extremal and locally conformally Einstein implies that it is also *strongly* extremal [**8**, **16**].

4.3. Bach flat metrics. Case 1 in the above proposition is well known [**1**, **4**, **8**], and case 3 was considered in [**3**]. However, one does not have to end the story there. As is well known, [**5**], the above metrics $\hat{g}$ have vanishing Bach tensor. This means that they are extremal points of the functional $\mathcal{W}([g]) := \int_M ||W||^2 d\mu$ defined over all conformal structures on M, where W denotes the Weyl curvature of g. Since the signature of M vanishes, we see that $\mathcal{W}([g]) = 2\int_M ||W^+||^2 d\mu$, where W^+ is the self-dual

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part of the Weyl curvature. If g is a Kähler metric, we have that $2\int_M ||W^+||^2 d\mu = \frac{1}{12}\int_M s^2 d\mu$, which, for any g as in subsection 4.1, is given by

$$-\pi^2 deg\, L\left(\frac{s_B^2(b^2-1)}{(b^2+4b+1)} + \frac{s_B(b^4+3b^3+10b^2+3b+1)}{(b^2+4b+1)^2} + \frac{8(b^4-1)}{(b-1)^2(b^2+4b+1)}\right).$$

For $s_B = 0$, recall that $\hat{g}$ corresponds to $\hat{b} = 1+\sqrt{3}+\sqrt{3+\sqrt{12}}$. It is easy to see that

$$\lim_{deg\, L\to-\infty} \mathcal{W}([\hat{g}]) = \lim_{deg\, L\to-\infty} \frac{1}{12}\int_M \hat{s}^2 d\hat{\mu} = +\infty.$$

For $s_B \neq 0$, recall that $deg\, L = \frac{4}{s_B}(\mathbf{g}-1)$ and $\lim_{s_B\to 0}\hat{b} = 1+\sqrt{3}+\sqrt{3+\sqrt{12}}$. Hence,

$$\lim_{k\to+\infty} \mathcal{W}([\hat{g}]) = \lim_{s_B\to 0} \mathcal{W}([\hat{g}]) = +\infty.$$

For each B, there are exactly two diffeomorphism classes for the manifolds $M = \mathbb{P}(\mathcal{O}\oplus L) \to B$: the product manifold and another one. If $deg\, L$ is even, then $M \overset{C^\infty}{\cong} B\times S^2$. If $deg\, L$ is odd, then M is in the other diffeomorphism class (see for example Example 4.26 in [**13**]). The above observations can then be interpreted in the following way.

Proposition 4. *For each compact Riemann surface B, we have a sequence $\{g_t\}$ of Bach-flat metrics on the 4-manifold $B\times S^2$ such that*

$$\lim_{t\to\infty} \mathcal{W}([g_t]) = +\infty.$$

The same statement is true for the other diffeomorphism class of $\mathbb{P}(\mathcal{O}\oplus L)\to B$.

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