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VALUE PATTERNS OF MULTIPLICATIVE FUNCTIONS AND RELATED SEQUENCES

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Abstract

We study the existence of various sign and value patterns in sequences defined by multiplicative functions or related objects. For any set A whose indicator function is 'approximately multiplicative' and uniformly distributed on short intervals in a suitable sense, we show that the density of the pattern $n+1 \in A, n+2 \in A, n+3 \in A$ is positive as long as A has density greater than $\frac{1}{3}$. Using an inverse theorem for sumsets and some tools from ergodic theory, we also provide a theorem that deals with the critical case of A having density exactly $\frac{1}{3}$, below which one would need nontrivial information on the local distribution of A in Bohr sets to proceed. We apply our results first to answer in a stronger form a question of Erdős and Pomerance on the relative orderings of the largest prime factors $P^+(n)$, $P^+(n+1)$, $P^+(n+2)$ of three consecutive integers. Second, we show that the tuple $(\omega(n+1), \omega(n+2), \omega(n+3)) \pmod 3$ takes all the 27 possible patterns in $(\mathbb{Z}/3\mathbb{Z})^3$ with positive lower density, with $\omega(n)$ being the number of distinct prime divisors. We also prove a theorem concerning longer patterns $n+i \in A_i$, $i = 1, \ldots, k$ in approximately multiplicative sets A_i having large enough densities, generalizing some results of Hildebrand on his 'stable sets conjecture'. Finally, we consider the sign patterns of the Liouville function λ and show that there are at least 24 patterns of length 5 that occur with positive upper density. In all the proofs, we make extensive use of recent ideas concerning correlations of multiplicative functions.

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1. Introduction

For any function $a\colon \mathbb{N} \to S$ with finite range S and any $k \in \mathbb{N} := \{1, 2, \ldots\}$, we may define the length k *value patterns* of a to be the tuples $s \in S^k$ that are of the form

$$s = (a(n+1), a(n+2), \dots, a(n+k))$$

for some $n \in \mathbb{N}$. (One could also include the $n = 0$ case here if one wished, although it will not affect our main results.) We further say that the function a *attains a pattern s with positive lower density* (respectively *upper density*) if the set

$$\{n \in \mathbb{N} : (a(n+1), a(n+2), \dots, a(n+k)) = s\}$$

has positive lower density (respectively upper density). (For the precise definitions of the various densities used in this paper as well as for the standard arithmetic functions and asymptotic notation, see Section 1.6.) In the case $S = \{-1, +1\}$, we will refer to value patterns as *sign patterns*. In this paper, we will mostly be interested in whether or not a given pattern is attained with positive lower density.

The occurrence of various value patterns for an arithmetic function a has attracted particular interest in the case where $a\colon \mathbb{N} \to \mathbb{D}$ is *multiplicative*, that is to say $a(1) = 1$ and $a(mn) = a(m)a(n)$ whenever m and n are coprime natural numbers. Here, $\mathbb{D} := \{z \in \mathbb{C} : |z| \leqslant 1\}$ is the unit disc of the complex plane. Indeed, the interaction of multiplicative functions with their shifts is the subject of many conjectures, including those of Chowla [4] and Elliott [9, 30]. In particular, for the Liouville function $\lambda(n)$ and the Möbius function $\mu(n)$, the existence of various sign or value patterns has been actively studied due to connections to the aforementioned conjectures.

Chowla's conjecture [4] for the Liouville function states that the autocorrelations

$$\frac{1}{x}\sum_{n \leqslant x} \lambda(n+h_1)\dots\lambda(n+h_k)$$

of the Liouville function λ converge to 0 as $x \to \infty$, for any $k \geqslant 1$ and distinct natural numbers $h_1, \dots, h_k$. (Unless otherwise stated, all variables such as n appearing in summations are understood to be restricted to the natural numbers, with the exception of variables named p (or p_1, p_2 and so on) which are understood to be restricted to the primes.) This conjecture easily implies that $\lambda(n)$ attains all the 2^k sign patterns in $\{-1, +1\}^k$ for any k infinitely often, and in fact the conjecture is equivalent to each of these length k patterns occurring with asymptotic density 2^{-k}. The analogous version of Chowla's conjecture for the Möbius function μ implies that the function μ attains every *admissible* value pattern in $(\varepsilon_1, \dots, \varepsilon_k) \in \{-1, 0, +1\}^k$ infinitely often, where we call a pattern admissible if for every prime p, there exists $b \in [0, p^2-1]$ such that $\varepsilon_{p^2 j+b} = 0$ for all j satisfying $1 \leqslant p^2 j + b \leqslant k$. (By using the identities $\lambda(n) = \sum_{d^2|n} \mu(n/d^2)$

and $\mu(n) = \sum_{d^2|n} \mu(d)\lambda(n/d^2)$ and exploiting the absolute convergence of the sum $\sum_d 1/d^2$, one can easily show that the Chowla conjectures for the Liouville and Möbius functions are equivalent if one generalizes the distinct linear forms $n+h_1, \ldots, n+h_k$ to nonparallel affine forms $a_1n+h_1, \ldots, a_kn+h_k$; we omit the details.) Nevertheless, Chowla's conjecture (for either λ or μ) remains unsolved once $k \geqslant 2$, and thus these implications are only conditional. In Section 1.4, we will give an account of the unconditional results on sign patterns of the Liouville function as well as state our new result on length 5 patterns.

In this paper, we study the appearance of value patterns in more sequences that have some multiplicative structure. Let $f : \mathbb{N} \to \mathbb{D}$ be a completely multiplicative function and assume that the range $f(\mathbb{N})$ is a finite set so that it is meaningful to talk about the sign patterns of f. (We say that f is *completely multiplicative* if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$ and $f(1) = 1$.) Then actually $f(\mathbb{N}) = \mu_m$ or $f(\mathbb{N}) = \mu_m \cup \{0\}$ for some m, where $\mu_m := \{z \in \mathbb{C} : z^m = 1\}$ is the set of roots of unity of order m. The case of $f(\mathbb{N}) = \{-1, +1\}$ is rather similar to the case of the Liouville function λ, and in fact it follows easily from [41, Corollary 1.6; Proof of Corollary 7.2] that if f is *not weakly pretentious*, by which we mean that

$$\sum_{p \leqslant x} \frac{1 - \mathrm{Re}(f(p)\bar{\chi}(p))}{p} \gg_\chi \log\log x$$

for any Dirichlet character χ, then f attains all the 16 possible length 4 sign patterns with positive lower density. (In [41], the proof was written only for $f = \lambda$, but the exact same argument works for any completely multiplicative bounded f that is not weakly pretentious.) At the opposite extreme, the case of f being *pretentious* in the sense that

$$\sum_{p \leqslant x} \frac{1 - \mathrm{Re}(f(p)\bar{\chi}(p))}{p} \ll 1$$

for some Dirichlet character χ was recently considered by Klurman and Mangerel [28]. We also remark that if $f(\mathbb{N}) \subset \mu_m$ and if f satisfies the nonpretentiousness condition

$$\sum_{p \leqslant x} \frac{1 - \mathrm{Re}(f(p)^d \bar{\chi}(p))}{p} \xrightarrow{x\to\infty} \infty$$

for all $1 \leqslant d \leqslant m-1$ and every Dirichlet character χ, then Elliott's conjecture [9, 30] on correlations of multiplicative functions would imply that f attains every value pattern in μ_m^k with equal asymptotic density m^{-k}. (Indeed, if we use the expansion $1_{f(n)=e(a/m)} = 1/m \sum_{j=0}^{m-1} f(n)^j e(-aj/m)$, we immediately reduce

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the study of the value patterns of f to bounding its correlations, which can be shown to be negligible assuming Elliott's conjecture.) For $k = 2$ (and if one uses logarithmic density instead of asymptotic density), this follows unconditionally from [37, Theorem 1.5], and from [41, Corollary 1.6], we can deduce various special cases for higher values $k \geqslant 3$ (again using logarithmic density in place of asymptotic density).

In what follows, we will mostly be studying the case $f(\mathbb{N}) = \{0, 1\}$, and we only make the weaker assumption that f is 'approximately multiplicative' in a precise sense defined in Section 1.3 (there we call this notion of approximate multiplicativity 'weak stability'). In this case, it is natural to write $f(n) = 1_A(n)$ for some set $A \subset \mathbb{N}$ and to say that the set A itself is 'approximately multiplicative'. The occurrence of patterns in such sets is not covered by Elliott's conjecture. It turns out that the class of *genuinely* multiplicative sets of positive asymptotic density are not a particularly interesting class of sets (a typical example being the set $\{n : \mu^2(n) = 1\}$ of square-free numbers, the patterns of which are well-understood from basic sieve theory), but the wider class of *approximately* multiplicative sets instead does include various interesting sets related to the largest prime factors of integers or to the number of prime divisors of an integer. For instance, if $P^+(n)$ denotes the largest prime factor of a natural number n (and $P^+(1) := 1$), the sets

$$Q_{\alpha,\beta} := \{n \in \mathbb{N} : n^\alpha < P^+(n) < n^\beta\} \tag{1.1}$$

with $0 \leqslant \alpha < \beta \leqslant 1$ turn out to be sufficiently close to being multiplicative that our results in Section 1.3 apply, and we will present several applications of our results to patterns in the sets $Q_{\alpha,\beta}$. See also Sections 1.1 and 1.2 for more applications of our results to value patterns of approximately multiplicative sets.

We also investigate the case $f(\mathbb{N}) = \mu_3$ and, more specifically, the case $f(n) := e(\omega(n)/3)$, where $\omega(n)$ is the number of prime factors of n without multiplicities, and $e(\theta) := e^{2\pi i\theta}$. In this case, the prior knowledge on length 3 value patterns was very limited since the fact that $f^3 = 1$ makes the result in [41, Corollary 1.6] on 3-point correlations of multiplicative functions inapplicable. The functions $n \mapsto e(\omega(n)/q)$ can be thought of as generalizations of the Liouville or Möbius functions which take values in the qth roots of unity rather than in $\{-1, +1\}$, and their value patterns are in one-to-one correspondence with those of the sequence $\omega(n) \pmod q$. (For $q = 2$, the function $n \mapsto (-1)^{\omega(n)}$ is of course not quite equal to either the Liouville function or the Möbius function but is very closely connected to both since it takes the value -1 at all the primes.)

Before stating our results on patterns in general approximately multiplicative sets, we state the corollaries of our results for the sets $Q_{\alpha,\beta}$ and $\{n \in \mathbb{N} : \omega(n) \pmod 3\}$ mentioned above.

1.1. Comparison of largest prime factors of consecutive integers. In what follows, let

$$d_-(A) := \liminf_{x\to\infty} \frac{|A \cap [1, x]|}{x}$$

denote the lower density of a set $A \subset \mathbb{N}$.

In 1978, Erdős and Pomerance [11] studied the orderings of the largest prime factors of consecutive integers and showed that

$$d_-(\{n \in \mathbb{N} : P^+(n+1) < P^+(n+2)\}) \geqslant c_0 > 0 \tag{1.2}$$

for some explicit c_0. They also showed that the set

$$\{n \in \mathbb{N} : P^+(n+1) < P^+(n+2) < P^+(n+3)\} \tag{1.3}$$

is infinite by looking at the explicit sequence $n = p^{2^{k_p}} - 2$ with k_p suitably chosen for every odd p and raised the problem of proving that also the set

$$\{n \in \mathbb{N} : P^+(n+1) > P^+(n+2) > P^+(n+3)\} \tag{1.4}$$

corresponding to the opposite ordering was infinite. This was eventually solved by Balog [2], who showed that there are infinitely many solutions having the specific form $n = m^2 - 2$. It is clear, however, that both the construction of Erdős and Pomerance and that of Balog only produce sparse sequences of $n \leqslant x$ that belong to the set (1.3) or the set (1.4); for (1.3), we get $\ll \sqrt{x}$ elements up to x (since certainly we must have $k_p \geqslant 1$), and for (1.4), Balog's proof gives $\asymp \sqrt{x}$ elements up to x.

Our main theorem in Section 1.3 will be seen in Section 6 to imply the following strengthenings of the above results, in which the sets (1.3) and (1.4) are shown to have positive lower density, and also give some limited comparison with $P^+(n+4)$, or with various powers n^α, n^β.

THEOREM 1.1 (Orderings of largest prime factors). *We have*

$$d_-(\{n \in \mathbb{N} : P^+(n+1) < P^+(n+2) < P^+(n+3) > P^+(n+4)\}) > 0$$

and

$$d_-(\{n \in \mathbb{N} : P^+(n+1) > P^+(n+2) > P^+(n+3) < P^+(n+4)\}) > 0.$$

THEOREM 1.2 (Largest prime factors of three consecutive integers). *Let $0 < \alpha < \beta < 1$ be real numbers such that $\rho(1/\alpha) + \rho(1/\beta) \neq 1$, where ρ is the Dickman function (see [26]). Then we have*

$$d_-(\{n \in \mathbb{N} : P^+(n+1) < n^\alpha < P^+(n+2) < n^\beta < P^+(n+3)\}) > 0$$

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and

$$d_-(\{n \in \mathbb{N} : P^+(n+3) < n^\alpha < P^+(n+2) < n^\beta < P^+(n+1)\}) > 0.$$

As mentioned above, either of Theorem 1.1 and Theorem 1.2 immediately imply the new result that both set (1.3) and set (1.4) have positive lower density. The condition $\rho(1/\alpha) + \rho(1/\beta) \neq 1$ should be removable, but this seems to be beyond the methods in this paper (unless there is substantial progress on understanding local Fourier uniformity of multiplicative functions or indicator functions of weakly stable sets).

We remark that the study of the largest prime factors of *two* consecutive integers has been taken up by several authors. In particular, the original value of $c_0 = 0.0099$ in (1.2) by Erdős and Pomerance was improved by de la Bretèche, Pomerance and Tenenbaum [7] to $c_0 = 0.05544$, and the current record is held by Wang [43] with $c_0 = 0.1356$. It was conjectured in the correspondence of Erdős and Turán [35] (and repeated by Erdős in [10]) that the set of n with $P^+(n) < P^+(n+1)$ has asymptotic density equal to $1/2$, as one would naturally expect. In [42], it was shown that the *logarithmic* density of this set indeed equals $1/2$. For orderings of longer strings of consecutive values of $P^+(n)$, little is known, but Wang [43] showed that either of

$$P^+(n+i) < \min_{\substack{j \leqslant J \\ j \neq i}} P^+(n+j) \quad \text{and} \quad P^+(n+i) > \max_{\substack{j \leqslant J \\ j \neq i}} P^+(n+j)$$

happens with positive lower density for any $J \geqslant 3$. For completely arbitrary orderings of the largest prime factors at consecutive integers, there is a natural conjecture of de Koninck and Doyon [6], which states that for any permutation $\{a_1, \ldots, a_k\}$ of $\{1, \ldots, k\}$ we have

$$d(\{n \in \mathbb{N} : P^+(n+a_1) < \cdots < P^+(n+a_k)\}) = \frac{1}{k!}.$$

This, however, seems to be far out of reach, and even for $k = 2$, we only know lower bounds for the asymptotic density and we know the correct value for the logarithmic density but do not know that the asymptotic density exists to start with.

1.2. Patterns of the number of prime factors modulo 3. In Section 6, we will also utilize our main theorem stated in Section 1.3 to prove the following result about the sign patterns of $\omega(n) \pmod 3$.

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THEOREM 1.3 (Value patterns of $\omega \pmod 3$). *The function $\omega(n) \pmod 3$ attains each of the 27 possible length 3 value patterns with positive lower density. In other words, we have*

$$d_-(\{n \in \mathbb{N} : \omega(n+1) \equiv a \pmod 3, \quad \omega(n+2) \equiv b \pmod 3, \\ \omega(n+3) \equiv c \pmod 3\}) > 0$$

for all $a, b, c \in \mathbb{Z}/3\mathbb{Z}$. The same holds for $\Omega(n)$, the number of prime factors of n counting multiplicities, in place of $\omega(n)$.

The value patterns of $\Omega(n) \pmod 2$ have of course been an active subject of study since they are in one-to-one correspondence with sign patterns of the Liouville function; see [23, 31, 41] for some works studying the number of these sign patterns. Showing that $\Omega(n) \pmod 3$ attains all the value patterns of length 3 with positive lower density is evidently harder than showing the same for $\Omega(n) \pmod 2$ (which was shown by Matomäki, Radziwiłł and the first author in [31]) since the number of possible patterns for $\Omega(n) \pmod 3$ is 27, meaning that each pattern should conjecturally have a rather small asymptotic density of $1/27$, as opposed to the much larger asymptotic density of $1/8$ corresponding to the patterns of length 3 for $\Omega(n) \pmod 2$. Perhaps surprisingly, it is much easier to deal with the longer patterns $(\Omega(n+1) \pmod{q_1}, \dots, \Omega(n+k) \pmod{q_k})$ for various choices of *distinct* q_j. Namely, if $q_1, \dots, q_k$ are all pairwise coprime, the authors showed in [41, Theorem 1.13] that each of the $q_1 \cdots q_k$ possible patterns occurs with logarithmic density $1/q_1 \cdots q_k$. The fact that the patterns with coprime q_j are easier stems from the result towards the Elliott conjecture in [41], which applies to correlations

$$\frac{1}{\log x} \sum_{n \leqslant x} \frac{g_1(n+1) \cdots g_k(n+k)}{n} \tag{1.5}$$

of 1-bounded multiplicative functions whenever the *product* $g_1 \cdots g_k$ is not 'weakly pretentious'. If we expand $1_{\Omega(n) \equiv a_j \pmod{q_j}}$ as a linear combination of the multiplicative functions $n \mapsto e(b\Omega(n)/q_j)$, then the logarithmic density of the sign pattern can be written as a linear combination of correlations like (1.5), but without the assumption of the q_j being coprime, the result in [41, Corollary 1.6] on the correlations (1.5) is not directly applicable. Of course, assuming the full Elliott conjecture and applying the same strategy, one would see that each of the $q_1 \cdots q_k$ value patterns is attained with asymptotic density $1/(q_1 \cdots q_k)$ without any restrictions on the q_j, but, needless to say, even for $q_1 = \cdots = q_k = 2$, proving this is out of reach.

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1.3. Results on weakly stable sets. Theorems 1.1, 1.2 and 1.3 will all be deduced from our main results concerning patterns in sets that are 'approximately multiplicative' in a suitable sense. The notion of approximate multiplicativity that we want to consider is called stability. In what follows, we use the expectation notation

$$\mathbb{E}_{n\in A} f(n) := \frac{1}{|A|}\sum_{n\in A} f(n)$$

for any finite, nonempty set $A \subset \mathbb{N}$ and for any function $f : A \to \mathbb{C}$.

DEFINITION 1.4 [1]. We say that a set $A \subset \mathbb{N}$ is *stable* if for every prime p, we have

$$\lim_{x\to\infty} \mathbb{E}_{n\leqslant x}|1_A(n) - 1_A(pn)| = 0.$$

Equivalently, A is stable if and only if $d(A \triangle p^{-1}A) = 0$ for every prime p, where $\triangle$ denotes the symmetric difference, and $p^{-1}A := \{n \in \mathbb{N} : pn \in A\}$.

An important class of stable sets is given by

$$Q_{\alpha,\beta} := \{n \in \mathbb{N} : n^{\alpha} < P^{+}(n) < n^{\beta}\},$$

where $0 \leqslant \alpha < \beta \leqslant 1$. By the classical result of Dickman [8], this set has asymptotic density $\rho(1/\beta) - \rho(1/\alpha) > 0$, where ρ is the Dickman function. The stability of $Q_{\alpha,\beta}$ then follows easily from the continuity of the Dickman function.

A completely different class of stable sets is

$$A_{\alpha,\beta} := \left\{n \in \mathbb{N} : \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} \in [\alpha, \beta]\right\}$$

for $-\infty < \alpha < \beta < \infty$. By the Erdős–Kac theorem, this set has a positive asymptotic density as well.

Stable sets were first introduced by Balog in [1], where he conjectured that if $A \subset \mathbb{N}$ is stable with $d_{-}(A) > 0$, then the pattern $n + 1 \in A, n + 2 \in A$ occurs with positive lower density or, equivalently, that

$$d_{-}((A - 1) \cap (A - 2)) > 0.$$

This conjecture was settled by Hildebrand [22] using an elementary but intricate method. Hildebrand [25] himself later posed a conjecture that generalizes Balog's conjecture to length k patterns.

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CONJECTURE 1.5 (Hildebrand's stable sets conjecture [25]). *Let $k \geqslant 2$ and let $A \subset \mathbb{N}$ be any stable set with $d_-(A) > 0$. Then we have*

$$d_-((A-1) \cap (A-2) \cap \cdots \cap (A-k)) > 0.$$

For higher values of k, Conjecture 1.5 is certainly a deep one since it implies, for any $\varepsilon > 0$, that both the sets

$$\{n \in \mathbb{N} : P^+(n+j) < n^{\varepsilon} \text{ for all } 1 \leqslant j \leqslant k)\} \tag{1.6}$$

and

$$\{n \in \mathbb{N} : P^+(n+j) > n^{1-\varepsilon} \text{ for all } 1 \leqslant j \leqslant k)\} \tag{1.7}$$

have positive lower density. Remarkably, Balog and Wooley [3] were able to prove that the set (1.6) is always infinite, but their construction gives a very sparse set of such n. For the set (1.7), in turn, it is not even known that it is infinite, except for $k = 2$ (which follows from [22]).

It follows from a trivial pigeonholing argument that the stable sets conjecture holds when $d_-(A) > 1 - 1/k$. Hildebrand [24] extended this range to $d_-(A) > 1 - 1/(k-1)$ when $k \geqslant 3$; thus, for instance, he established the $k = 3$ case of the conjecture for $d_-(A) > \frac{1}{2}$.

We make progress on a variant of the stable sets conjecture for all $k \geqslant 3$, where we have a somewhat different set of assumptions. First, our theorem applies to k *distinct* sets $A_1, \ldots, A_k \subset \mathbb{N}$, whereas the method of Hildebrand in [22] appears difficult to adapt to this setting. Second, the notion of stability that we need is weaker than in Definition 1.4; see Definition 1.6. On the other hand, we need a stronger density assumption for the A_i. It turns out that a stable set is always uniformly distributed in arithmetic progressions in the sense that

$$d_-(A \cap \{n \in \mathbb{N} : n \equiv b \pmod{q}\}) = \frac{1}{q} d_-(A)$$

for any $b, q \in \mathbb{N}$; see [24]. What we need in our main theorem is that a similar statement holds when A is restricted to almost all short intervals. In all of our applications, this stronger condition will be satisfied by the Matomäki–Radziwiłł theorem [29] or some variant thereof.

We now define the precise concepts that we need for the main theorem.

DEFINITION 1.6 (Weakly stable sets). We say that a set $A \subset \mathbb{N}$ is *weakly stable* if for every $x \geqslant 1$, there is a set $B_x \subset \mathbb{N}$ such that for every prime p, we have

$$\lim_{x \to \infty} \mathbb{E}_{\substack{n \leqslant x \\ p \nmid n}} |1_A(n) - 1_{B_x}(pn)| = 0. \tag{1.8}$$

In addition, we say that the sequence (B_x) *corresponds to* A.

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It is clear that if A is stable, then A is also weakly stable (with $B_x = A$ in this case). Importantly for us, the class of weakly stable sets also contains interesting sets that do not satisfy the usual definition of stability; for example, the sets $A = \{n \in \mathbb{N} : \omega(n) \equiv a \pmod q\}$ are weakly stable but not stable for any $a \in \mathbb{N}$, $q \geqslant 2$; the point is that the sets B_x need to be taken here to equal $\{n \in \mathbb{N} : \omega(n) \equiv a + 1 \pmod q\} \neq A$. It is because of applications to such sets that we want to have the condition $p \nmid n$ in (1.8); without that condition, these sets would not be weakly stable.

Another definition that we need is that of uniform distribution in short intervals.

DEFINITION 1.7 (Uniform distribution in short intervals). We say that a set $A \subset \mathbb{N}$ is *uniformly distributed in short intervals with asymptotic density* δ if we have

$$\lim_{H\to\infty} \limsup_{x\to\infty} \frac{1}{x} \int_0^x \left| \frac{|A \cap [y, y+H] \cap (q\mathbb{Z} + b)|}{H} - \frac{\delta}{q} \right| dy = 0$$

for all $b, q \in \mathbb{N}$.

With this notation, we can prove the following results.

THEOREM 1.8 ($k = 3$ main theorem, large density). *Let $A_1, A_2, A_3 \subset \mathbb{N}$ be weakly stable and uniformly distributed in short intervals with densities δ_1, δ_2, $\delta_3 > 0$, respectively. Suppose that $\delta_1 + \delta_2 + \delta_3 > 1$. Then*

$$d_-((A_1 - 1) \cap (A_2 - 2) \cap (A_3 - 3)) > 0.$$

THEOREM 1.9 ($k = 3$ main theorem, critical density). *Let $A_1, A_2, A_3 \subset \mathbb{N}$ be weakly stable and uniformly distributed in short intervals with densities δ_1, δ_2, $\delta_3 > 0$, respectively. Suppose that $\delta_1 + \delta_2 + \delta_3 = 1$. Then for every $c \in \{0, 1, 2\}$, we have*

$$d_-\Bigg(\bigcup_{\substack{c_1,c_2,c_3 \in \{0,1,2\} \\ c_1+c_2+c_3 \equiv c \pmod 3}} (A_{c_1} - 1) \cap (A_{c_2} - 2) \cap (A_{c_3} - 3) \Bigg) > 0.$$

Further, if $\delta_1 \neq \delta_3$ and $d(A_1 \cup A_2 \cup A_3) = 1$, then

$$d_-((A_1 - 1) \cap (A_2 - 2) \cap (A_3 - 3)) > 0.$$

THEOREM 1.10 ($k > 3$ main theorem). *Let $k \geqslant 4$ and let $A_1, \ldots, A_k \subset \mathbb{N}$ be weakly stable and uniformly distributed in short intervals with densities $\delta_1, \ldots,$*

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$\delta_k > 0$, *respectively. Define the constants* c_k *by*

$$c_4 := \frac{3+\sqrt{2}}{7} = 0.6306\ldots$$
$$c_5 := \frac{9+2\sqrt{6}}{19} = 0.7315\ldots$$

and, more generally, $c_k \in (0, 1)$ *is the largest root of the quadratic equation*

$$\left(\frac{9}{2}\binom{k}{3} + (6-4a_k)\binom{k}{2}\right)(1-X)^2 + (a_k^2 - a_k)k(1-X) - a_k(a_k-1) = 0,$$

where $a_k := \lceil (3k+2)/4 \rceil$. *Suppose that* $\delta_i > c_k$ *for all* $i \leqslant k$. *Then*

$$d_-((A_1 - 1) \cap (A_2 - 2) \cap \cdots \cap (A_k - k)) > 0.$$

REMARK 1.11. Inspecting the proof of Theorem 1.10 in Section 4, we see that it works equally well for $k = 3$ with $c_3 = 1/3$. However, since this is a special case of Theorem 1.8 (namely the case $\delta_1, \delta_2, \delta_3 > 1/3$), we confine ourselves to $k \geqslant 4$ in Theorem 1.10.

We remark that a routine but tedious calculation yields the asymptotic

$$c_k = 1 - \frac{1}{k - \frac{4}{3} + \eta_k},$$

where η_k goes to zero as k goes to infinity. For instance, one can calculate

$$\eta_4 = 0.04044\ldots$$
$$\eta_5 = 0.05808\ldots$$
$$\eta_{10} = 0.04143\ldots$$
$$\eta_{100} = 0.00435\ldots$$
$$\eta_{1000} = 0.00071\ldots.$$

The value of c_k should be compared with the value $1 - 1/(k-1)$, which is the threshold in Hildebrand's result about Conjecture 1.5. It turns out that our value of c_k is smaller (or equivalently, that $\eta_k < 1/3$) for every $k \geqslant 4$.

When it comes to our applications stated as Theorems 1.2 and 1.3, we want to apply our main theorems to the triples of sets

$$\{n \in \mathbb{N} : P^+(n) < n^\alpha\}, \quad \{n \in \mathbb{N} : n^\alpha < P^+(n) < n^\beta\}, \quad \{n \in \mathbb{N} : P^+(n) > n^\beta\}$$

or

$$\{n \in \mathbb{N} : \omega(n) \equiv a \pmod 3\}, \quad \{n \in \mathbb{N} : \omega(n) \equiv b \pmod 3\},$$
$$\{n \in \mathbb{N} : \omega(n) \equiv c \pmod 3\}.$$

In either case, the sum of the densities of these sets will be *exactly* 1, so we are in the critical case $\delta_1+\delta_2+\delta_3 = 1$ where Theorem 1.8 no longer applies. It turns out that the case $\delta_1+\delta_2+\delta_3 = 1$ is much more delicate than the case $\delta_1+\delta_2+\delta_3 > 1$ since for $\delta_1+\delta_2+\delta_3 < 1$, our method based on the study of sumsets in abelian groups breaks down. In addition, as soon as $\delta_1+\delta_2+\delta_3 \leqslant 1$, all the A_i could theoretically be 'local Bohr sets' in the sense that, for any slowly growing function $H = H(X)$ tending to infinity, we would have

$$A_i \cap [x, x+H] = \{n \in [x, x+H] : n\alpha_{i,x} \in U_i\}$$

for almost all x and for some irrational numbers $\alpha_{i,x} \in \mathbb{R}/\mathbb{Z}$ and open sets $U_i \subset \mathbb{R}/\mathbb{Z}$ of measure δ_i. Such sets are certainly uniformly distributed in short intervals, and it may happen that $(A_1+A_3)\cap 2A_2 = \emptyset$ when $\delta_1+\delta_2+\delta_3 < 1$ (see Remark 2.5) so that certainly $(A_1-1)\cap(A_2-2)\cap(A_3-3) = \emptyset$. Of course, we do not expect any such sets to be stable, but even showing that such local Bohr sets cannot be linear combinations of multiplicative functions appears very difficult. Even in the special case of $A = \{n \in \mathbb{N} : \Omega(n) \equiv 0 \pmod 2\}$, it has not been shown that A does not correlate with local Bohr sets, as that would amount to showing that

$$\frac{1}{X}\int_X^{2X} \sup_{\alpha\in\mathbb{R}} |\mathbb{E}_{x\leqslant n\leqslant x+H}\lambda(n)e(\alpha n)|\,dx = o(1) \tag{1.9}$$

for any $H = H(X)$ tending to infinity, which is the Fourier uniformity conjecture from [38]. See, however, [32] for recent progress on this. The sup norm estimate (1.9) is open for slowly growing functions $H = H(X) = X^{o(1)}$, and it is in fact closely connected to Chowla's conjecture (see [38] for this connection). Nevertheless, it is still possible to deploy tools from additive combinatorics to be able to establish results like Theorem 1.9 (and hence Theorems 1.2 and 1.3) even if the weakly stable sets involved behave like Bohr sets, thus allowing us to avoid having to establish unproven results such as (1.9).

Both Theorems 1.8 and 1.10 can be applied to the sets $Q_{\alpha,\beta}$ defined in (1.1), and they yield the following results about the largest prime factors of consecutive integers.

THEOREM 1.12 (Consecutive triples with large prime factors). *Let* $\gamma_3 := e^{-1/3} = 0.7165\ldots$. *Then for any* $\gamma < \gamma_3$, *we have*

$$d_-(\{n \in \mathbb{N}\colon P^+(n+1) > n^\gamma, P^+(n+2) > n^\gamma, P^+(n+3) > n^\gamma\}) > 0.$$

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Here, γ_3 is the solution to $1-\rho(1/x)=1/3$, so the set $\{n\in\mathbb{N}: P^+(n)>n^{\gamma_3}\}$ has asymptotic density $1/3$. In [24], the same was proved with γ_3 replaced by the smaller value $e^{-1/2}=0.6065\ldots$, where this value of γ_3 solves $1-\rho(1/x)=1/2$.

We can also prove a result for longer strings of largest prime factors.

THEOREM 1.13 (Consecutive k-tuples with large prime factors). *Define*

$$\gamma_4 := 0.5322$$
$$\gamma_5 := 0.4804.$$

Then for $k=4,5$, *we have*

$$d_-(\{n\in\mathbb{N}\colon P^+(n+1)>n^{\gamma_k}, P^+(n+2)>n^{\gamma_k},\ldots,P^+(n+k)>n^{\gamma_k}\})>0.$$

Again, Hildebrand [24] proved a similar result with γ_k replaced by the smaller value $1/\rho^{-1}(1/k-1)$, where ρ^{-1} is the inverse function of the Dickman ρ function. Like his result, ours can also be applied for higher values of k, but since our value of γ_k behaves asymptotically like Hildebrand's value as $k\to\infty$, we omit the cases $k\geqslant 6$ from the theorem.

1.4. Sign patterns of the Liouville function. In Section 7, we will prove a result on length 5 sign patterns of the Liouville function. This application will not be based on Theorem 1.8 or Theorem 1.10, but nevertheless, like those theorems, it will be reduced to results about correlations of multiplicative functions. In particular, we will use what we called an 'isotopy formula' in [41, Section 1] that implies, in particular, that

$$\mathbb{E}^{\log}_{n\leqslant x}\lambda(n+h_1)\cdots\lambda(n+h_k)=\mathbb{E}^{\log}_{n\leqslant x}\lambda(n-h_1)\cdots\lambda(n-h_k)+o(1)$$

for any $h_1,\ldots,h_k\in\mathbb{N}$. We will use this to show that there are at least 24 sign patterns of length 5 for the Liouville function.

THEOREM 1.14 (Length 5 sign patterns of Liouville). *There are at least* 24 *sign patterns in* $\{-1,+1\}^5$ *that are attained by* λ *with positive upper density, including the six explicit sign patterns*

$$\pm(+1,+1,+1,+1,-1), \pm(+1,+1,+1,-1,-1), \pm(+1,-1,+1,+1,-1)$$

and their reversals

$$\pm(-1,+1,+1,+1,+1), \pm(-1,-1,+1,+1,+1), \pm(-1,+1,+1,-1,+1).$$

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If we denote by $s(k)$ the number of length k sign patterns that occur infinitely often in the Liouville function, then Theorem 1.14 implies that $s(5) \geqslant 24$. In [41, Corollary 7.2], the authors proved that $s(4) = 16$. For large values of k, our knowledge on $s(k)$ is rather weak; [41, Remark 1.12] gives the explicit bound $s(k) \geqslant 2k + 8$, whereas Frantzikinakis and Host [13, Theorem 1.2] proved that $s(k)$ grows faster than linearly with k. Very recently, this was improved by McNamara [33] to $s(k) \gg k^2$. Trivially, if we had Chowla's conjecture, then $s(k) = 2^k$ would follow.

In order to improve the bound of 24 in Theorem 1.14, one would have to improve the known bounds on the correlations of the Liouville function. Namely, if we define

$$C_A := \lim_{m\to\infty} \mathbb{E}^{\log}_{n\leqslant x_m} \prod_{j\in A} \lambda(n+j)$$

for any finite set $A \subset \mathbb{N}$, where the sequence (x_m) tending to infinity is chosen so that all the limits exist (which is possible by a diagonal argument), then from [41, Proposition 7.1], we have the bound $|C_{\{1,2,\ldots,k\}}| \leqslant 1/2$. If this bound was sharp for $k = 4$, then we could have the hypothetical scenario

$$C_{\{1,2,3,4\}} = C_{\{2,3,4,5\}} = \tfrac{1}{2}, \quad C_{\{1,2,3,5\}} = C_{\{1,2,4,5\}} = C_{\{1,2,3,5\}} = 0,$$

in which case one would easily see (using the odd order logarithmic Chowla conjecture from [41, Theorem 1.1(i)]) that there are no more than 24 sign patterns of the Liouville function that occur with positive logarithmic lower density. Thus, one would have to rule out this scenario to be able to improve on the number of length 5 sign patterns.

1.5. Proof strategy.

We briefly describe the ideas that go into the proofs of Theorems 1.8, 1.9 and 1.10. Consider, for example, Theorem 1.8. By an elementary argument, one sees that $d_-((A_1 - 1) \cap (A_2 - 2) \cap (A_3 - 3)) > 0$ is equivalent to the triple correlation

$$\mathbb{E}^{\log}_{x/\omega(x)\leqslant n\leqslant x} 1_{A_1}(n+1)1_{A_2}(n+2)1_{A_3}(n+3) \tag{1.10}$$

being $\gg 1$ as $x \to \infty$ for every $\omega(X) \leqslant X$ tending to infinity. The functions 1_{A_i} are not assumed to be multiplicative, but the assumption of weak stability works as a useful substitute to this since for some sets $B_{x,i}$ and all primes p, we can write $1_{A_i}(n) = 1_{B_{x,i}}(pn) + o(1)$ for most $n \leqslant x$. Using this relation, averaging (1.10) over primes, and applying the entropy decrement argument from [37, 41], we conclude that (1.10) is equal to

$$\mathbb{E}^{\log}_{p\leqslant P} \mathbb{E}^{\log}_{x/\omega(x)\leqslant n\leqslant x} 1_{B_{x,1}}(n+p)1_{B_{x,2}}(n+2p)1_{B_{x,3}}(n+3p) + o(1) \tag{1.11}$$

with $P = P(x)$ being a medium-sized parameter. Such a double average is evidently easier to analyse than a single average. The only information that we will use about the sets $B_{x,i}$ is that they are uniformly distributed in short intervals with densities $\delta_1, \delta_2, \delta_3 > 0$, respectively, as follows easily from the fact that the A_i have this property.

Appealing to the Furstenberg correspondence principle, the average (1.11) being $\gg 1$ will follow from the following ergodic-theoretic statement: For any measure-preserving system (X, μ, T) and any measurable sets $B_1, B_2, B_3 \subset X$ satisfying the uniform distribution property

$$\lim_{H\to\infty} \int_X |\mathbb{E}_{h\leqslant H} 1_{B_i}(T^{qh}x) - \delta_i| \, d\mu(x) = 0 \tag{1.12}$$

for all $q \in \mathbb{N}$ and with δ_i as in Theorem 1.8, we have

$$\mathbb{E}^{\log}_{p\leqslant P} \int_X 1_{B_1}(T^p x) 1_{B_2}(T^{2p} x) 1_{B_3}(T^{3p} x) \, d\mu(x) \gg 1. \tag{1.13}$$

By the generalized von Neumann theorem and the Gowers uniformity of the primes [18], the bound (1.13) will follow from

$$\mathbb{E}^{\log}_{d\leqslant P:\,(d,W)=1} \int_X 1_{B_1}(T^d x) 1_{B_2}(T^{2d} x) 1_{B_3}(T^{3d} x) \, d\mu(x) \gg 1, \tag{1.14}$$

where we are now averaging over integers rather than primes and $W := \prod_{p\leqslant w} p$ (with w being a slowly growing function of P). This is roughly the conclusion we reach after Section 2.

In Section 3, we make several ergodic-theoretic reductions to reduce to the case where $X = (\mathbb{R}/\mathbb{Z})^d \times (\mathbb{Z}/m\mathbb{Z})$ for some $d, m \in \mathbb{N}$, so that the problem has essentially been reduced to the same problem on a torus. Now we apply a Pollard-type inequality from [36] (which can be viewed as a quantitative version of the inequality $\mu(A + B) \geqslant \mu(A) + \mu(B)$ valid for compact subsets $A, B \subset X$ of any compact, connected abelian group, with μ being the Haar measure on X) to conclude the proof (it is here that the assumption $\delta_1 + \delta_2 + \delta_3 > 1$ is crucial).

In the case of Theorem 1.9, we proceed similarly up to the point where $X = (\mathbb{R}/\mathbb{Z})^d \times (\mathbb{Z}/m\mathbb{Z})$. Since $\delta_1+\delta_2+\delta_3$ is exactly 1, the Pollard-type inequality is no longer sufficient to conclude, but employing instead an inverse theorem for it from [39] (see Theorem 3.2), we can deduce that (1.14) holds unless B_1, B_2, B_3 (or rather their projections to $(\mathbb{R}/\mathbb{Z})^d$) are essentially Bohr sets. The case where B_1, B_2, B_3 are Bohr sets can be dealt with a bit of Fourier analysis, and we eventually conclude that (1.14) holds then as well under the conditions of Theorem 1.9.

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For Theorem 1.10, we make a similar reduction to the statement

$$\mathbb{E}^{\log}_{p\leqslant P}\int_X 1_{B_1}(T^p x)1_{B_2}(T^{2p}x)\dots 1_{B_k}(T^{kp}x)\,d\mu(x)\gg 1$$

with the B_i satisfying (1.12) as before. One easily sees from (1.12) that $\int_X 1_{B_i}(x)\,d\mu(x) = \delta_i$, $\int_X 1_{B_{i_1}}(T^{i_1 p}x)1_{B_{i_2}}(T^{i_2 p}x)\,d\mu(x) = \delta_{i_1}\delta_{i_2}$ for $1\leqslant i_1 < i_2 \leqslant k$. Using the Pollard-type inequality mentioned above, we can also get a lower bound for

$$\int_X 1_{B_{i_1}}(T^{i_1 p}x)1_{B_{i_2}}(T^{i_2 p}x)1_{B_{i_3}}(T^{i_3 p}x)\,d\mu(x)$$

for $1\leqslant i_1 < i_2 < i_3 \leqslant k$. The question is then, how large $\delta = \min_i \delta_i$ can be under these constraints if (1.14) fails. This is a combinatorial problem whose solution gives us the value of c_k in Theorem 1.10.

1.6. Notation.

We use the following standard arithmetic functions:

- $\omega(n)$, defined to equal the number of prime factors of n (not counting multiplicity);
- $\Omega(n)$, defined to equal the number of prime factors of n (counting multiplicity);
- the *Liouville function* $\lambda(n) = (-1)^{\Omega(n)}$;
- the *Möbius function* $\mu(n)$, defined to equal $\lambda(n)$ when n is square-free and 0 otherwise;
- the largest prime factor $P^+(n)$ of n and the smallest prime factor $P_-(n)$ of n (with the convention $P^-(1) = P^+(1) = 1$);
- the *Euler totient* function $\varphi(n)$, defined to equal the number $|(\mathbb{Z}/n\mathbb{Z})^\times|$ of primitive residue classes modulo n;
- the *von Mangoldt function* $\Lambda(n)$, defined to equal $\log p$ when n is a power p^j of a prime p for some $j\geqslant 1$, and equal to zero otherwise;
- the *Dickman function* $\rho(u)$, defined as the unique continuous solution to the delayed differential equation $u\rho'(u)+\rho(u-1)=0$ with the initial condition $\rho(u)=1$ for $0\leqslant u\leqslant 1$. As is well known, we have $\lim_{x\to\infty} 1/x|\{n\leqslant x : P^+(n)\leqslant x^u\}| = \rho(1/u)$; we refer to [26] for further properties of this function.

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If A is a finite set, we use $|A|$ to denote its cardinality. If A is a set of natural numbers, we define the *lower density*

$$d_-(A) := \liminf_{x\to\infty} \frac{|A \cap [1, x]|}{x}, \tag{1.15}$$

the *upper density*

$$d_+(A) := \limsup_{x\to\infty} \frac{|A \cap [1, x]|}{x}$$

and the *asymptotic density*

$$d(A) := \lim_{x\to\infty} \frac{|A \cap [1, x]|}{x}$$

(if it exists).

If A is a finite nonempty set of natural numbers and $f : A \to \mathbb{C}$ is a function, we define the average

$$\mathbb{E}_{n\in A} f(n) := \frac{\sum_{n\in A} f(n)}{\sum_{n\in A} 1}$$

and the logarithmic average

$$\mathbb{E}^{\log}_{n\in A} f(n) := \frac{\sum_{n\in A} f(n)/n}{\sum_{n\in A} 1/n}.$$

If we average over the variable p instead of n, the definitions are same, except that the summation variable is now restricted to be prime.

We utilize the Dickman function $\rho(u)$ that equals to the asymptotic density $d(\{n \in \mathbb{N} : P^+(n) \leqslant n^{1/u}\})$; see [26] for further properties of this function.

If A is a set, we use 1_A to denote the indicator function; thus, $1_A(n) = 1$ when $n \in A$ and $1_A(n) = 0$ otherwise. Similarly, if E is a statement, we let 1_E denote the indicator of E; thus, $1_E = 1$ when E is true and $1_E = 0$ when E is false.

We use $X \ll Y$, $X \gg Y$, $X = O(Y)$ to denote a bound of the form $|X| \leqslant CY$ for an absolute constant C; if we need to allow C to depend on additional parameters, we denote this by subscripts, thus for instance $X = O_k(Y)$ denotes the bound $|X| \leqslant C_k Y$ for some C_k depending on k. Given an asymptotic parameter such as x tending to infinity, we use $o(Y)$ to denote a quantity bounded in magnitude by $c(x)Y$ where $c(x)$ goes to zero as $x \to \infty$.

We use $e(x) := e^{2\pi i x}$ for the standard character. We use $n \pmod q$ for the reduction of n modulo q and $(a_1, \ldots, a_k)$ for the greatest common divisor of $a_1, \ldots, a_k$.

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2. A correspondence principle

In this section, we develop a correspondence principle for weakly stable sets, analogous to the Furstenberg correspondence principle [15], which converts problems about establishing patterns in such sets with positive lower density to problems about establishing certain patterns in measure-preserving systems. The approximately multiplicative structure of weakly stable sets will be incorporated (via the 'entropy decrement argument' [37]) to a certain prime shift in these latter patterns. This correspondence principle will then be used in later sections to establish Theorems 1.8, 1.9 and 1.10. We remark that the analogous correspondence principle with weakly stable sets replaced by bounded multiplicative functions is essentially contained in the recent work of Frantzikinakis and Host [12].

We first recall the definition of a measure-preserving system.

DEFINITION 2.1 (Measure-preserving systems). We say that a tuple $(X, \mathcal{X}, \mu, T)$ is a *measure-preserving system* if $\mathcal{X}$ is a sigma algebra on X, μ is a measure on $\mathcal{X}$ and $T : X \to X$ is measure-preserving in the sense that T is invertible with T, T^{-1} both measurable with $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{X}$. We often omit the sigma algebra $\mathcal{X}$ from the notation when it plays no specific role. We further say that $(X, \mathcal{X}, \mu, T)$ is a *separable measure-preserving system* if the sigma algebra $\mathcal{X}$ is countably generated.

The main result of this section is then as follows.

THEOREM 2.2 (Correspondence principle for weakly stable sets). *Let $A_1, \ldots, A_k \subset \mathbb{N}$ be weakly stable sets. Suppose that there is a finite index set I and for each $\alpha \in I$, one has a natural number m^α, integers $h_1^\alpha, \ldots, h_{m^\alpha}^\alpha$ and indices $c_1^\alpha, \ldots, c_{m^\alpha}^\alpha \in \{1, \ldots, k\}$ such that*

$$d_-\left(\bigcup_{\alpha\in I}\bigcap_{i=1}^{m^\alpha}(A_{c_i^\alpha} - h_i^\alpha)\right) = 0. \tag{2.1}$$

Then there exists a separable measure-preserving system $(X, \mathcal{X}, \mu, T)$ and measurable sets $B_1, \ldots, B_k \in \mathcal{X}$ such that

$$\lim_{P\to\infty}\sum_{\alpha\in I}\mathbb{E}^{\log}_{p\leqslant P}\int_X\prod_{i=1}^{m^\alpha}1_{B_{c_i^\alpha}}(T^{ph_i^\alpha}x)\,d\mu(x) = 0. \tag{2.2}$$

Furthermore, one can ensure the following additional properties:

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(i) *If for each* $j = 1, \ldots, k$, A_j *is uniformly distributed in short intervals with density* $\delta_j \in [0, 1]$, *then for every natural number* q *and* $i = 1, \ldots, k$ *one has*

$$\lim_{H\to\infty} \int_X |\mathbb{E}_{h\leqslant H} 1_{B_j}(T^{qh}x) - \delta_j| \, d\mu(x) = 0. \tag{2.3}$$

In particular (by the triangle inequality and shift invariance), each B_j *has measure* δ_j.

(ii) *If the* A_j *are disjoint up to sets of density zero, then the* B_j *are disjoint up to null sets.*

(iii) *If* $d(\bigcup_{j=1}^k A_j) = 1$, *then* $\bigcup_{j=1}^k B_j$ *has full measure.*

REMARK 2.3. For the application to Theorems 1.8 and 1.10, we are going to take I to be a singleton and $h_i^\alpha = c_i^\alpha = i$. For Theorem 1.9 in turn, we choose $I = \{1, 2, 3\}$ and $h_i^\alpha = i$, and as α ranges through I, the tuples $(c_i^\alpha)_{i\leqslant 3}$ run through solutions to $c_1^\alpha + c_2^\alpha + c_3^\alpha = c \pmod 3$.

We remark that by the ergodic theorem, the conclusion (2.3) is equivalent to $1_{B_i} - \delta_i$ being orthogonal to the *profinite factor* of X, defined as the factor generated by all the periodic functions on X (that is, functions $f : X \to \mathbb{C}$ with $f(T^k x) = f(x)$ for some natural number k and almost all $x \in X$). The presence of the dilation factor p in the shifts $T^{ph_i^\alpha}$ in (2.2) is a key feature of this principle that is not present in the classical Furstenberg correspondence principle and is introduced via the entropy decrement argument from [**37**]. We remark that the existence of the limit in (2.2) can also be derived from the general convergence results for multiple ergodic averages along the primes in [**14**, **44**], and the logarithmically averaged limit $\mathbb{E}^{\log}_{p\leqslant P}$ can then be replaced by the ordinary average $\mathbb{E}_{p\leqslant P}$. In fact, we have a useful formula for the limit; see Proposition 2.6.

We now prove the theorem. Let

$$S := \bigcup_{\alpha\in I} \bigcap_{i=1}^{m^\alpha} (A_{c_i^\alpha} - h_i^\alpha)$$

denote the set in (2.1). By hypothesis, we have $d_-(S) = 0$; thus, we can find a sequence x_l tending to infinity such that

$$\mathbb{E}_{n\leqslant x_l} 1_S(n) = o(1)$$

as $l \to \infty$. In particular, if $1 \leqslant \omega_l \leqslant x_l$ goes to infinity sufficiently slowly, one has

$$\omega_l \mathbb{E}_{n\leqslant x_l} 1_S(n) = o(1)$$

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which implies in particular that

$$\mathbb{E}^{\log}_{x_l/\omega_l \leqslant n \leqslant x_l} 1_S(n) = o(1).$$

Since

$$1_S(n) = \sum_{\alpha \in I} \prod_{i=1}^{m^\alpha} 1_{A_{c_i^\alpha}}(n + h_i^\alpha),$$

we thus have

$$\mathbb{E}^{\log}_{x_l/\omega_l \leqslant n \leqslant x_l} \prod_{i=1}^{m^\alpha} 1_{A_{c_i^\alpha}}(n + h_i^\alpha) = o(1) \tag{2.4}$$

for each $\alpha \in I$.

For each $j = 1, \ldots, k$, the set A_j is weakly stable by hypothesis. Let $B_{x,j}$ be the sets corresponding to A_j as per Definition 1.6. Then for each prime p, one has

$$\mathbb{E}_{n \leqslant x_l} 1_{p \nmid n} |1_{A_j}(n) - 1_{B_{x_l,j}}(pn)| = o(1)$$

as $l \to \infty$, which for ω_l sufficiently slowly growing depending on p implies that

$$\mathbb{E}^{\log}_{x_l/\omega_l \leqslant n \leqslant x_l} 1_{p \nmid n} |1_{A_j}(n) - 1_{B_{x_l,j}}(pn)| = o(1). \tag{2.5}$$

By a diagonalization argument, one can select ω_l so that (2.5) holds for *all* primes p (of course, the decay rate will almost certainly not be uniform in p).

Restoring the case $p|n$, we have

$$\mathbb{E}^{\log}_{x_l/\omega_l \leqslant n \leqslant x_l} |1_{A_j}(n) - 1_{B_{x_l,j}}(pn)| \ll \frac{1}{p} + o(1),$$

and hence also

$$\mathbb{E}^{\log}_{x_l/\omega_l \leqslant n \leqslant x_l} |1_{A_{c_i^\alpha}}(n + h_i^\alpha) - 1_{B_{x_l,c_i^\alpha}}(pn + ph_i^\alpha)| \ll \frac{1}{p} + o(1)$$

for all $\alpha \in I$ and $i = 1, \ldots, m^\alpha$. From this, (2.4) and the triangle inequality, we conclude that

$$\mathbb{E}^{\log}_{x_l/\omega_l \leqslant n \leqslant x_l} \prod_{i=1}^{m^\alpha} 1_{B_{x_l,c_i^\alpha}}(pn + ph_i^\alpha) \ll \frac{1}{p} + o(1)$$

for all primes p, all $\alpha \in I$ and $i = 1, \ldots, m^\alpha$, where we allow implied constants in the asymptotic notation to depend on I and the m^α. Writing this average in

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terms of pn instead of n (which only impacts the logarithmic average in n by a negligible amount, other than by now restricting n to multiples of p), we obtain

$$\mathbb{E}^{\log}_{x_l/\omega_l \leqslant n \leqslant x_l} \prod_{i=1}^{m^\alpha} 1_{B_{x_l, c_i^\alpha}}(n + ph_i^\alpha) p 1_{p|n} \ll \frac{1}{p} + o(1).$$

If we logarithmically average over primes $p \leqslant P$, we conclude from the convergence of $\sum_p 1/p^2$ and the divergence of $\sum_p 1/p$ that

$$\lim_{P\to\infty} \limsup_{l\to\infty} \mathbb{E}^{\log}_{p\leqslant P} \mathbb{E}^{\log}_{x_l/\omega_l \leqslant n \leqslant x_l} \prod_{i=1}^{m^\alpha} 1_{B_{x_l, c_i^\alpha}}(n + ph_i^\alpha) p 1_{p|n} = 0.$$

On the other hand, by the entropy decrement argument [41, Theorem 3.6], we have

$$\lim_{P\to\infty} \limsup_{l\to\infty} \mathbb{E}^{\log}_{p\leqslant P} \mathbb{E}^{\log}_{x_l/\omega_l \leqslant n \leqslant x_l} \prod_{i=1}^{m^\alpha} 1_{B_{x_l, c_i^\alpha}}(n + ph_i^\alpha)(p 1_{p|n} - 1) = 0.$$

We conclude from the triangle inequality that

$$\lim_{P\to\infty} \limsup_{l\to\infty} \mathbb{E}^{\log}_{p\leqslant P} \mathbb{E}^{\log}_{x_l/\omega_l \leqslant n \leqslant x_l} \prod_{i=1}^{m^\alpha} 1_{B_{x_l, c_i^\alpha}}(n + ph_i^\alpha) = 0$$

for all $\alpha \in I$.

Next, let $\widetilde{\lim} : \ell^\infty(\mathbb{N}) \to \mathbb{C}$ denote a generalized limit functional, that is to say a bounded linear functional on $\ell^\infty(\mathbb{N})$ that extends the limit functional on convergent sequences, and such that

$$\liminf_{l\to\infty} a_l \leqslant \widetilde{\lim}(a_l)_{l\in\mathbb{N}} \leqslant \limsup_{n\to\infty} a_l$$

for all bounded real-valued sequences a_n. The existence of such a generalized limit functional easily follows from the Hahn–Banach theorem (or from the existence of nonprincipal ultrafilters on $\mathbb{N}$). Then we have

$$\lim_{P\to\infty} \widetilde{\lim} \left(\mathbb{E}^{\log}_{p\leqslant P} \mathbb{E}^{\log}_{x_l/\omega_l \leqslant n \leqslant x_l} \prod_{i=1}^{m^\alpha} 1_{B_{x_l, c_i^\alpha}}(n + ph_i^\alpha) \right)_{l\in\mathbb{N}} = 0. \tag{2.6}$$

Let X denote the product space $(\{0, 1\}^k)^{\mathbb{Z}}$ of sequences $(x_{c,m})_{c\in\{1,\ldots,k\}, m\in\mathbb{Z}}$ of numbers $x_{c,m} \in \{0, 1\}$ with the product sigma algebra $\mathcal{X}$ (so, in particular, X is a compact Hausdorff space with separable sigma algebra $\mathcal{X}$) and the shift

$$T(x_{c,m})_{c\in\{1,\ldots,k\}, m\in\mathbb{Z}} := (x_{c,m+1})_{c\in\{1,\ldots,k\}, m\in\mathbb{Z}}.$$

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We define a probability measure μ on X by requiring that

$$\int_X \prod_{\beta\in J} 1_{x_{c_\beta,m_\beta}=1}\, d\mu(x) = \widetilde{\lim}\left(\mathbb{E}^{\log}_{x_l/\omega_l\leqslant n\leqslant x_l} \prod_{\beta\in J} 1_{B_{x_l,c_\beta}}(n+m_\beta)\right)_{l\in\mathbb{N}} \tag{2.7}$$

for any finite index set J, any $c_\beta \in \{1,\ldots,k\}$ and any integers m_β. The existence (and uniqueness) of this measure follows from the Kolmogorov extension theorem. The measure μ is a probability measure that is invariant under the shift T since the right-hand side of (2.7) remains invariant when the m_β are replaced by $m_\beta+1$. Next, we define the measurable sets B_j for $j=1,\ldots,k$ by the formula

$$B_j := \{(x_{c,m})_{c\in\{1,\ldots,k\},m\in\mathbb{Z}} \in X : x_{j,0}=1\};$$

then one can rewrite the left-hand side of (2.7) as

$$\int_X \prod_{\beta\in J} 1_{B_{c_\beta}}(T^{m_\beta}x)\, d\mu(x). \tag{2.8}$$

In particular, from (2.6), one has

$$\lim_{P\to\infty} \mathbb{E}^{\log}_{p\leqslant P} \int_X \prod_{i=1}^{m^\alpha} 1_{B_{c_i^\alpha}}(T^{ph_i^\alpha}x)\, d\mu = 0$$

for all $\alpha\in I$, which gives (2.2).

Now we prove (ii). If $A_j, A_{j'}$ are disjoint up to zero density sets, then

$$\mathbb{E}^{\log}_{x_l/\omega_l\leqslant n\leqslant x_l} 1_{A_j}(n)1_{A_{j'}(n)} = o(1).$$

Repeating the previous arguments using this bound in place of (2.4), we eventually arrive at

$$\int_X 1_{B_j}(x)1_{B_{j'}}(x)\, d\mu(x) = 0,$$

and hence $B_j, B_{j'}$ are disjoint up to null sets. This gives (ii). Similarly, if $\bigcup_{j=1}^k A_j$ has density one, then

$$\mathbb{E}^{\log}_{x_l/\omega_l\leqslant n\leqslant x_l} \prod_{j=1}^k (1-1_{A_j}(n)) = o(1),$$

and then by repeating the previous arguments,

$$\int_X \prod_{j=1}^k (1-1_{B_j}(x))\, d\mu(x) = 0,$$

so that $\bigcup_{j=1}^k B_j$ has full measure. This establishes (iii).

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Now we turn to (i). Fix b, q, j, let $\varepsilon > 0$, let Q be sufficiently large (depending on b, q, ε) and then let H be sufficiently large (depending on b, q, ε, Q). Further, let p be a prime in $[\log Q, Q]$. Since A_j is uniformly distributed in short intervals with density δ_j, we then conclude (if ω_l grows slowly enough) that

$$\sup_{p\in[\log Q,Q]} \mathbb{E}^{\log}_{x_l/\omega_l\leqslant y\leqslant x_l} ||A_j \cap [y/p, y/p + qH/p] \cap (q\mathbb{Z} + b\overline{p})| - \delta_j H/p| = o(1), \tag{2.9}$$

where $\overline{p}$ denotes the inverse of p in $\mathbb{Z}/q\mathbb{Z}$ (this exists since $p \geqslant \log Q > q$ for Q large enough). Also, since A_j is weakly stable, we have

$$\mathbb{E}_{n\leqslant x/p:p\nmid n} |1_{A_j}(n) - 1_{B_{x,j}}(pn)| = o(1),$$

and hence

$$\sup_{p\in[\log Q,Q]} \mathbb{E}^{\log}_{x_l/\omega_l\leqslant y\leqslant x_l} \mathbb{E}_{y/p\leqslant n\leqslant y/p+qH/p:p\nmid n} |1_{A_j}(n) - 1_{B_{x,j}}(pn)| = o(1). \tag{2.10}$$

From (2.10), we have, in particular, that for each $p \in [\log Q, Q]$,

$$\mathbb{E}^{\log}_{x_l/\omega_l\leqslant y\leqslant x_l} \mathbb{E}_{n\in[y/p,y/p+qH/p]\cap(q\mathbb{Z}+b\overline{p}):p\nmid n} |1_{A_j}(n) - 1_{B_{x_l,j}}(pn)| = o(1),$$

and hence on removing the $p \nmid n$ constraint,

$$\mathbb{E}^{\log}_{x_l/\omega_l\leqslant y\leqslant x_l} \mathbb{E}_{n\in[y/p,y/p+qH/p]\cap(q\mathbb{Z}+b\overline{p})} |1_{A_j}(n) - 1_{B_{x_l,j}}(pn)| \ll \frac{1}{p} + o(1).$$

Meanwhile, from (2.9), one has

$$\mathbb{E}^{\log}_{x_l/\omega_l\leqslant y\leqslant x_l} |\mathbb{E}_{n\in[y/p,y/p+qH/p]\cap(q\mathbb{Z}+b\overline{p})} (1_{A_j}(n) - \delta_j)| = o(1).$$

By the triangle inequality, we conclude that

$$\mathbb{E}^{\log}_{x_l/\omega_l\leqslant y\leqslant x_l} |\mathbb{E}_{n\in[y/p,y/p+qH/p]\cap(q\mathbb{Z}+b\overline{p})} (1_{B_{x_l,j}}(pn) - \delta_j)| \ll 1/p + o(1)$$

or, equivalently,

$$\mathbb{E}^{\log}_{x_l/\omega_l\leqslant y\leqslant x_l} |\mathbb{E}_{n\in[y,y+qH]\cap(q\mathbb{Z}+b)} (1_{B_{x_l,j}}(n) - \delta_j) 1_{p|n}| \ll \frac{1}{p}\left(\frac{1}{p} + o(1)\right).$$

We can estimate $1/p + o(1)$ by $O(\varepsilon)$ for Q sufficiently large. We then sum in p and use the triangle inequality to conclude that

$$\mathbb{E}^{\log}_{x_l/\omega_l\leqslant y\leqslant x_l} \left| \mathbb{E}_{n\in[y,y+qH]\cap(q\mathbb{Z}+b)} (1_{B_{x_l,j}}(n) - \delta_j) \sum_{\log Q\leqslant p\leqslant Q} 1_{p|n} \right| \ll \varepsilon \log\log Q$$

for Q sufficiently large. On the other hand, from the Turan–Kubilius inequality (or a direct second moment calculation), we have

$$\mathbb{E}_{n\in[y,y+qH]\cap(q\mathbb{Z}+b)}\left|\sum_{\log Q\leqslant p\leqslant Q} 1_{p|n} - \log\log Q\right|^2 \ll \varepsilon^2(\log\log Q)^2,$$

and hence by Cauchy–Schwarz,

$$\mathbb{E}_{n\in[y,y+qH]\cap(q\mathbb{Z}+b)}|1_{B_{x_l,j}}(n)-\delta_j|\left|\sum_{\log Q\leqslant p\leqslant Q} 1_{p|n} - \log\log Q\right| \ll \varepsilon\log\log Q.$$

From the triangle inequality, we thus have

$$\mathbb{E}^{\log}_{x_l/\omega_l\leqslant y\leqslant x_l}|\mathbb{E}_{n\in[y,y+qH]\cap(q\mathbb{Z}+b)}(1_{B_{x_l,j}}(n)-\delta_j)| \ll \varepsilon.$$

This implies that

$$\limsup_{l\to\infty}\mathbb{E}^{\log}_{x_l/\omega_l\leqslant n\leqslant x_l:n=b\ (q)}|\mathbb{E}_{h\leqslant H}(1_{B_{x_l,j}}(n+qh)-\delta_j)| \ll \varepsilon;$$

averaging in b, this implies

$$\limsup_{l\to\infty}\mathbb{E}^{\log}_{x_l/\omega_l\leqslant n\leqslant x_l}|\mathbb{E}_{h\leqslant H}(1_{B_{x_l,j}}(n+qh)-\delta_j)| \ll \varepsilon,$$

and thus,

$$\lim_{H\to\infty}\limsup_{l\to\infty}\mathbb{E}^{\log}_{x_l/\omega_l\leqslant n\leqslant x_l}|\mathbb{E}_{h\leqslant H}(1_{B_{x_l,j}}(n+qh)-\delta_j)|^2 = 0.$$

Using (2.7) and (2.8) and expanding the square, we conclude that

$$\lim_{H\to\infty}\int_X|\mathbb{E}_{h\leqslant H}(1_{B_j}(T^{qh}x)-\delta_j)|^2\,d\mu(x)=0,$$

and (2.3) follows from the Cauchy–Schwarz inequality. This completes the proof of Theorem 2.2.

In view of this correspondence principle (taken in the contrapositive), Theorems 1.8, 1.9 and 1.10 are immediate consequences of the following ergodic-theoretic counterparts (specialized to the case when $F_j = 1_{B_j}$ are indicator functions).

THEOREM 2.4 (Main theorem, ergodic version). *Let $F_1, \ldots, F_k : X \to [0, 1]$ be measurable functions on a measure-preserving system $(X, \mathcal{X}, \mu, T)$, and let $\delta_1, \ldots, \delta_k \in (0, 1]$ be such that*

$$\lim_{H\to\infty}\int_X|\mathbb{E}_{h\leqslant H}F_j(T^{qh}x)-\delta_j|\,d\mu(x)=0 \tag{2.11}$$

for all $q \geqslant 1$ and $j = 1, \ldots, k$.

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(i) *($k = 3$, large density) If $k = 3$ and $\delta_1 + \delta_2 + \delta_3 > 1$, then*

$$\limsup_{P \to \infty} \mathbb{E}^{\log}_{p \leqslant P} \int_X F_1(T^p x) F_2(T^{2p} x) F_3(T^{3p} x)\, d\mu(x) > 0.$$

(ii) *($k = 3$, critical density, first part) If $k = 3$ and $\delta_1 + \delta_2 + \delta_3 = 1$, then*

$$\limsup_{P \to \infty} \sum_{\substack{c_1, c_2, c_3 \in \{0,1,2\} \\ c_1 + c_2 + c_3 \equiv c \pmod 3}} \mathbb{E}^{\log}_{p \leqslant P} \int_X F_{c_1}(T^p x) F_{c_2}(T^{2p} x) F_{c_3}(T^{3p} x)\, d\mu(x) > 0$$

for all $c = 0, 1, 2$.

(iii) *($k = 3$, critical density, second part) If $k = 3$, $\delta_1 + \delta_2 + \delta_3 = 1$, $\delta_1 \neq \delta_3$ and $F_1 + F_2 + F_3 = 1$ almost everywhere, then*

$$\limsup_{P \to \infty} \mathbb{E}^{\log}_{p \leqslant P} \int_X F_1(T^p x) F_2(T^{2p} x) F_3(T^{3p} x)\, d\mu(x) > 0.$$

(iv) *($k > 3$) If $k > 3$ and $\delta_1, \ldots, \delta_k > c_k$ (where c_k is as in Theorem 1.10), then*

$$\limsup_{P \to \infty} \mathbb{E}^{\log}_{p \leqslant P} \int_X F_1(T^p x) \ldots F_k(T^{kp} x)\, d\mu(x) > 0.$$

REMARK 2.5. The condition $\delta_1 \neq \delta_3$ in part (iii) is necessary. To see this, let $X = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, equipped with its Haar measure and the measure-preserving map $T(x, n) = (x + \alpha, n + 1)$ for α irrational. In addition, for $0 < \delta_2 < 1$, define the intervals

$$I_1 = \left[\frac{\delta_2}{2}, \frac{1}{2}\right], \quad I_2 = \left[0, \frac{\delta_2}{2}\right) \cup \left[\frac{1}{2}, \frac{1 + \delta_2}{2}\right], \quad I_3 = \left[\frac{1 + \delta_2}{2}, 1\right)$$

and the functions $F_i(x, n) = 1_{I_i}(x + (n\%2)/4)$, where $n\%2$ equals 0 when n is even and 1 when n is odd. We then have $F_1 + F_2 + F_3 \equiv 1$. By Weyl's equidistribution theorem, condition (2.11) is satisfied for $j = 1, 2, 3$ with densities $(1 - \delta_2)/2, \delta_2, (1 - \delta_2)/2$, respectively. However, for any p, we have $F_1(T^p x) F_2(T^{2p} x) F_3(T^{3p} x) = 0$ since for any $x, y \in \mathbb{R}/\mathbb{Z}$, we cannot simultaneously have $x + y \in I_1$, $x + 2y \in I_2 \pm 1/4$, $x + 3y \in I_3$.

Analogously, if we define the sets of *integers*

$$A_i = \{n \equiv 0 \pmod 2 : \alpha n \in I_i \bmod 1\} \cup \{n \equiv 1 \pmod 2 : \alpha n - 1/4 \in I_i \bmod 1\}$$

for $i = 1, 2, 3$, then A_1, A_2, A_3 are uniformly distributed in short intervals with densities $(1 - \delta_2)/2, \delta_2, (1 - \delta_2)/2$, respectively, but $n + d \in A_1$, $n + 2d \in A_2$, $n + 3d \in A_3$ for d odd never happens.

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To prove this theorem, we will use the following explicit formula for the limit of multiple ergodic averages along primes, which is essentially implicit in [14].

PROPOSITION 2.6 (Limit formula). *Let $F_1, \dots, F_k \in L^\infty(X)$ be bounded measurable functions on a measure-preserving system $(X, \mathcal{X}, \mu, T)$. Then*

$$\lim_{P\to\infty} \mathbb{E}^{\log}_{p\leqslant P} \int_X F_1(T^p x) \dots F_k(T^{kp}x)\, d\mu(x)$$
$$= \lim_{w\to\infty} \lim_{P\to\infty} \mathbb{E}^{\log}_{d\leqslant P:(d,W)=1} \int_X F_1(T^d x) \dots F_k(T^{kd}x)\, d\mu(x),$$

where $W := \prod_{p\leqslant w} p$.

We remark that the convergence of the inner limit on the right-hand side was first established by Host and Kra [27]; see also [45] for an alternate proof.

Proof. To abbreviate notation, we write $A(d) := \int_X F_1(T^d x) \dots F_k(T^{kd}x)\, d\mu(x)$. It suffices to show that

$$\lim_{w\to\infty} \limsup_{P\to\infty} |\mathbb{E}^{\log}_{p\leqslant P} A(p) - \mathbb{E}^{\log}_{d\leqslant P:(d,W)=1} A(d)| = 0.$$

By summation by parts, it will suffice to show that

$$\lim_{w\to\infty} \limsup_{P\to\infty} |\mathbb{E}_{p\leqslant P} A(p) - \mathbb{E}_{d\leqslant P:(d,W)=1} A(d)| = 0,$$

and by dyadic decomposition, it then suffices to show that

$$\lim_{w\to\infty} \limsup_{P\to\infty} |\mathbb{E}_{P\leqslant p\leqslant 2P} A(p) - \mathbb{E}_{P\leqslant d\leqslant 2P:(d,W)=1} A(d)| = 0.$$

Equivalently, we need to show that

$$\mathbb{E}_{P\leqslant p\leqslant 2P} A(p) = \mathbb{E}_{P\leqslant d\leqslant 2P:(d,W)=1} A(d) + o(1)$$

as $P \to \infty$, if $w = w(P)$ goes to infinity sufficiently slowly as $P \to \infty$. By splitting into residue classes modulo W, it suffices to show that

$$\mathbb{E}_{P\leqslant p\leqslant 2P:p=b \pmod W} A(p) = \mathbb{E}_{P\leqslant d\leqslant 2P:d=b \pmod W} A(d) + o(1)$$

uniformly for all $1 \leqslant b < W$ coprime to W.

Using the von Mangoldt function Λ and the prime number theorem in arithmetic progressions, we can write the left-hand side as

$$\mathbb{E}_{P\leqslant d\leqslant 2P:d=b \pmod W} \frac{\phi(W)}{W} \Lambda(d) A(d),$$

so it suffices to show that

$$\mathbb{E}_{P\leqslant d\leqslant 2P:d=b \pmod W}\left(\frac{\phi(W)}{W}\Lambda(d)-1\right)A(d)=o(1)$$

or, equivalently, that

$$\mathbb{E}_{P/W\leqslant d\leqslant 2P/W}(\Lambda_{b,W}(d)-1)\int_X F_1(T^{Wd+b}x)\cdots F_k(T^{Wkd+kb}x)\,d\mu(x)=o(1),$$

where $\Lambda_{b,W}(d):=(\phi(W)/W)\Lambda(Wd+b)$. Replacing x by $T^n x$ for $n\leqslant P$ and averaging, it suffices to show that

$$\mathbb{E}_{P/W\leqslant d\leqslant 2P/W}\mathbb{E}_{n\leqslant P}\int_X(\Lambda_{b,W}(d)-1)F_1(T^{n+Wd+b}x)\cdots F_k(T^{n+Wkd+kb}x)\,d\mu(x)=o(1)$$

uniformly in b. By the generalized von Neumann theorem in the form of [40, Lemma 5.2], this will follow from the claim

$$\|\Lambda_{b,W}(d)-1\|_{U^k[2P/W]}=o(1),$$

where the Gowers norm U^k is defined, for instance, in [18]. But this follows from [18, Theorem 7.2] (combined with the main results of [19, 20]). □

It will thus suffice to prove the following slightly stronger version of Theorem 2.4.

THEOREM 2.7 (Main theorem, ergodic version, II). *Let $F_1,\ldots,F_k: X\to[0,1]$ be measurable functions on a measure-preserving system $(X,\mathcal{X},\mu,T)$, and let $\delta_1,\ldots,\delta_k\in(0,1]$ be such that*

$$\lim_{H\to\infty}\int_X|\mathbb{E}_{h\leqslant H}F_j(T^{qh}x)-\delta_j|\,d\mu(x)=0 \tag{2.12}$$

for all $q\geqslant 1$ and $j=1,\ldots,k$. We allow implied constants to depend on $k,\delta_1,\ldots,\delta_k$. Let W be a natural number.

(i) *($k=3$, large density) If $k=3$ and $\delta_1+\delta_2+\delta_3>1$, then*

$$\limsup_{P\to\infty}\mathbb{E}_{d\leqslant P:(d,W)=1}\int_X F_1(T^d x)F_2(T^{2d}x)F_3(T^{3d}x)\,d\mu(x)\gg 1.$$

(ii) *($k=3$, critical density, first part) If $k=3$ and $\delta_1+\delta_2+\delta_3=1$, then*

$$\limsup_{P\to\infty}\sum_{\substack{c_1,c_2,c_3\in\{0,1,2\}\\ c_1+c_2+c_3\equiv c \pmod 3}}\mathbb{E}_{d\leqslant P:(d,W)=1}\int_X F_{c_1}(T^d x)F_{c_2}(T^{2d}x)F_{c_3}(T^{3d}x)\,d\mu(x)\gg 1 \tag{2.13}$$

for all $c=0,1,2$.

(iii) *($k = 3$, critical density, second part) If $k = 3$, $\delta_1 + \delta_2 + \delta_3 = 1$, $\delta_1 \neq \delta_3$ and $F_1 + F_2 + F_3 = 1$ almost everywhere, then*

$$\limsup_{P\to\infty} \mathbb{E}_{d\leqslant P:(d,W)=1} \int_X F_1(T^d x) F_2(T^{2d} x) F_3(T^{3d} x)\, d\mu(x) \gg 1.$$

(iv) *($k > 3$) If $k > 3$ and $\delta_1, \ldots, \delta_k > c_k$ (where c_k is as in Theorem 1.10), then*

$$\limsup_{P\to\infty} \mathbb{E}_{d\leqslant P:(d,W)=1} \int_X F_1(T^d x) \cdots F_k(T^{kd} x)\, d\mu(x) \gg 1. \tag{2.14}$$

A key point here is that the lower bound is independent of W (and of the system X).

3. The main theorems for $k = 3$

In this section, we prove parts (i)–(iii) of Theorem 2.7. We begin with some standard reductions.

3.1. Reduction to the case of X being ergodic. We claim that to prove any part of Theorem 2.7, it suffices to do so in the case when the measure-preserving system X is ergodic (that is to say, all T-invariant subsets of X have measure zero or full measure). For the sake of discussion, we only present this in the case (ii) as the other cases are similar. Let X be a separable measure-preserving system that is not necessarily ergodic. Applying the ergodic decomposition (see, for example, [**17**, Theorem 3.42]), one can obtain a disintegration

$$\mu = \int_Y \mu_y\, d\nu(y), \tag{3.1}$$

where (Y, ν) is the T-invariant factor of (X, μ), and for ν-almost every y, the (X, T, μ_y) are ergodic measure-preserving systems. Assume that Theorem 2.7(ii) is established whenever X is ergodic. By dominated convergence, (3.1) and (2.12), we have

$$\int_Y \lim_{H\to\infty} \int_X |\mathbb{E}_{h\in[H]} F_c(T^{qh} x) - \delta_c|\, d\mu_y(x)\, d\nu(y)$$
$$= \lim_{H\to\infty} \int_Y \int_X |\mathbb{E}_{h\in[H]} F_c(T^{qh} x) - \delta_c|\, d\mu_y(x)\, d\nu(y) = 0.$$

Thus, for any $c = 1, 2, 3$, $q \geqslant 1$ and ν-almost every y, we have that $\mathbb{E}_{h\in[H]} F_c(T^{qh} x)$ converges in $L^1(X, \mu_y)$ norm to δ_c as $H \to \infty$. Applying

Theorem 2.7(ii) in the ergodic case, we conclude that for every W and $c \in \mathbb{Z}/3\mathbb{Z}$, one has

$$\liminf_{P\to\infty} \sum_{\substack{c_1,c_2,c_3\in\{1,2,3\} \\ c_1+c_2+c_3=c \pmod 3}} \mathbb{E}_{P\leqslant r\leqslant 2P:(r,W)=1} \int_X F_{c_1}(T^r x) F_{c_2}(T^{2r} x) F_{c_3}(T^{3r} x)\, d\mu_y(x) \gg 1$$

for ν-almost every y. Integrating in y and applying Fatou's lemma, this implies that

$$\liminf_{P\to\infty} \sum_{\substack{c_1,c_2,c_3\in\{1,2,3\} \\ c_1+c_2+c_3=c \pmod 3}} \mathbb{E}_{P\leqslant r\leqslant 2P:(r,W)=1} \int_X F_{c_1}(T^r x) F_{c_2}(T^{2r} x) F_{c_3}(T^{3r} x)\, d\mu(x) \gg 1,$$

giving Theorem 2.7(ii) in the general case. (Though it is not strictly necessary, one could use the results of [**15**] (see also [**27**]) to upgrade the limit inferior here to a limit.) A similar argument works for all other components of Theorem 2.7.

3.2. Reduction to the case of X being a Kronecker system. Next we make a reduction of parts (i)–(iii) of Theorem 2.7 to the case when X is a *Kronecker system*, by which we mean that X is a compact separable abelian group with shift T given by a translation $T : x \mapsto x + \alpha$; the argument here relies crucially on the fact that $k = 3$, and it does not extend to part (iv). Again, we only detail this reduction for the case (ii). If (X, T, μ) is an ergodic separable measure-preserving system, then (as is well known, see, for example, [**16**]) we can form the *Kronecker factor* (Z^1, S, ν), which is a Kronecker system together with a factor map $\pi : X \mapsto Z^1$ that pushes forward μ to ν and intertwines T and S (with the measurable functions on Z^1 pulling back to the functions on X generated by the eigenfunctions of T). Furthermore, any average of the form

$$\lim_{P\to\infty} \mathbb{E}_{r\in[P]} \int_X G_1(T^{ar}x) G_2(T^{br}x) G_3(T^{cr}x)\, d\mu$$

for distinct integers a, b, c will vanish whenever at least one of the functions $G_1, G_2, G_3 \in L^\infty(X)$ is orthogonal to the Kronecker factor in the sense that the conditional expectation $\mathbb{E}(G_i|Z^1)$ vanishes for some i. As such, we see (as in [**16**]) that the Kronecker factor is *characteristic* for the average in (2.2), in the sense that one can replace each of the functions F_c by the conditional expectation $\mathbb{E}(F_c|Z^1)$ without affecting the average. The Kronecker factor is also characteristic for the ergodic averages in (2.12). Finally, as the functions F_1, F_2, F_3 take values in $[0, 1]$ and sum to 1, the same is true for $\mathbb{E}(F_1|Z^1), \mathbb{E}(F_2|Z^1), \mathbb{E}(F_3|Z^1)$. As such, we see that to prove Theorem 2.7(ii) for the functions F_1, F_2, F_3, it suffices to do so for $\mathbb{E}(F_1|Z^1), \mathbb{E}(F_2|Z^1), \mathbb{E}(F_3|Z^1)$. Thus, Theorem 2.7(ii) for general ergodic systems will follow from the case of Kronecker systems. Similarly for parts (i) or (iii) of this theorem.

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3.3. Reduction to the case of X being a Kronecker system corresponding to a Lie group. We make a further reduction of parts (i)–(iii) of Theorem 2.7 to the case when the Kronecker system is a compact abelian *Lie* group. Again, we only discuss the case (ii). It is easy to see that a general Kronecker system X is expressible as the inverse limit of Kronecker systems X_n that are compact abelian Lie groups (see also [27] for the generalization of this claim to higher step). Suppose that Theorem 2.7(ii) has been proven for Kronecker systems that are compact abelian Lie groups. If F_1, F_2, F_3, X are as in that theorem, then $\mathbb{E}(F_c|X_n)$ will converge in $L^1(X,\mu)$ norm to F_c for $c \in \mathbb{Z}/3\mathbb{Z}$. Applying conditional expectations to (2.12) and using the dominated convergence theorem, we see that this hypothesis continues to hold if each function F_c is replaced with $\mathbb{E}(F_c|Z_n)$. Thus, by hypothesis, we see that for any $c \in \mathbb{Z}/3\mathbb{Z}$, we have

$$\liminf_{P\to\infty} \sum_{\substack{c_1,c_2,c_3\in\{1,2,3\} \\ c_1+c_2+c_3=c \pmod 3}} \mathbb{E}_{P\leqslant r\leqslant 2P:(r,W)=1} \int_X \mathbb{E}(F_{c_1}|Z_n)(T^r x)\mathbb{E}(F_{c_2}|Z_n)(T^{2r} x) \cdot \mathbb{E}(F_{c_3}|Z_n)(T^{3r} x)\, d\mu(x) \gg 1,$$

with the implied constants uniform in n. Taking limits in n, we obtain Theorem 2.7(ii) for arbitrary Kronecker systems. Similarly for Theorem 2.7(i) or Theorem 2.7(iii).

3.4. Main argument. We continue the proof of Theorem 2.7(ii). Henceforth, X is a Kronecker system that is a compact abelian Lie group. As the translation map T is ergodic, the system X must (up to isomorphism) take the form $X = G \times \mathbb{Z}/M\mathbb{Z}$ for some *connected* compact abelian Lie group (that is, a torus) and some $M \geqslant 1$, with shift given by $T(x,a) := (x+\alpha, a+1)$ for some $\alpha \in G$, such that the translation $x \mapsto x+\alpha$ is ergodic on G (and, hence, totally ergodic since G is connected and so the Pontryagin dual $\hat{G}$ is torsion-free). Applying the hypothesis (2.12) with $q = M$, we conclude, in particular, that

$$\lim_{H\to\infty} \int_G |\mathbb{E}_{h\in[H]} F_c(x + M\alpha h, a) - \delta_c|\, d\mu_G(x) = 0$$

for all $c = 1, 2, 3$ and $a \in \mathbb{Z}/M\mathbb{Z}$, where μ_G is the Haar probability measure on G. By the ergodic theorem and total ergodicity of the shift $x \mapsto x+\alpha$, we thus have

$$\int_G F_c(x,a)\, d\mu_G(x) = \delta_c \tag{3.2}$$

for all $c = 1, 2, 3$ and $a \in \mathbb{Z}/M\mathbb{Z}$.

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Next, we expand the left-hand side of (2.13) as

$$\liminf_{P\to\infty} \sum_{\substack{c_1,c_2,c_3\in\{1,2,3\}\\ c_1+c_2+c_3=c \pmod 3}} \mathbb{E}_{P\leqslant r\leqslant 2P:(r,W)=1}\mathbb{E}_{a\in\mathbb{Z}/M\mathbb{Z}} \int_G F_{c_1}(x+r\alpha, a+r) \cdot F_{c_2}(x+2r\alpha, a+2r) F_{c_3}(x+3r\alpha, a+3r)\, d\mu_G(x).$$

We split r into residue classes modulo MW to write this as

$$\liminf_{P\to\infty} \sum_{\substack{c_1,c_2,c_3\in\{1,2,3\}\\ c_1+c_2+c_3=c \pmod 3}} \mathbb{E}_{b\in[MW]:(b,W)=1}\mathbb{E}_{P\leqslant r\leqslant 2P:r=b \pmod{MW}}\mathbb{E}_{a\in\mathbb{Z}/M\mathbb{Z}} \int_G F_{c_1}(x+r\alpha, a+b) F_{c_2}(x+2r\alpha, a+2b) F_{c_3}(x+3r\alpha, a+3b)\, d\mu_G(x). \tag{3.3}$$

A standard calculation (see [16, Theorem 2.1]) shows that

$$\lim_{P\to\infty} \mathbb{E}_{P\leqslant r\leqslant 2P:r=b\ (MW)} \int_G F_{c_1}(x+r\alpha, a+b) F_{c_2}(x+2r\alpha, a+2b) \cdot F_{c_3}(x+3r\alpha, a+3b)\, d\mu_G(x)$$
$$= \int_G\int_G F_{c_1}(x+y, a+b) F_{c_2}(x+2y, a+2b) F_{c_3}(x+3y, a+3b)\, d\mu_G(x)\, d\mu_G(y),$$

and so the expression in (3.3) can be simplified to

$$\sum_{\substack{c_1,c_2,c_3\in\{1,2,3\}\\ c_1+c_2+c_3=c \pmod 3}} \mathbb{E}_{b\in[MW]:(b,W)=1}\mathbb{E}_{a\in\mathbb{Z}/M\mathbb{Z}} A_{c_1,c_2,c_3}(a+b, a+2b, a+3b),$$

where

$$A_{c_1,c_2,c_3}(a_1,a_2,a_3) := \int_G\int_G F_{c_1}(x+y, a_1) F_{c_2}(x+2y, a_2) F_{c_3}(x+3y, a_3)\, d\mu_G(x)\, d\mu_G(y).$$

The condition $(b, W) = 1$ clearly implies $(r, M, W) = 1$ for any $r \in \mathbb{Z}/M\mathbb{Z}$ with $b = r\ (M)$. Conversely, if $(r, M, W) = 1$, then from the Chinese remainder theorem, we see that there are precisely $(M, W)\phi(W)/\phi((M, W))$ values of $b \in [MW]$ with $(b, W) = 1$ and $b = r(M)$. Thus, the above expression can also be written as

$$\sum_{\substack{c_1,c_2,c_3\in\{1,2,3\}\\ c_1+c_2+c_3=c \pmod 3}} \mathbb{E}_{a,r\in\mathbb{Z}/M\mathbb{Z}:(r,M,W)=1} A_{c_1,c_2,c_3}(a+r, a+2r, a+3r).$$

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Thus, to prove Theorem 2.7(ii), we can assume for the sake of contradiction that

$$\mathbb{E}_{a,r\in\mathbb{Z}/M\mathbb{Z}:(r,M,W)=1} A_{c_1,c_2,c_3}(a+r,a+2r,a+3r) \leqslant \varepsilon \tag{3.4}$$

for all $c_1, c_2, c_3 \in \mathbb{Z}/3\mathbb{Z}$ satisfying $c_1 + c_2 + c_3 = c$ and some sufficiently small $\varepsilon > 0$ depending on $\delta_1, \delta_2, \delta_3$. Similarly, to prove Theorem 2.7(i) or Theorem 2.7(iii), we may assume for the sake of contradiction that

$$\mathbb{E}_{a,r\in\mathbb{Z}/M\mathbb{Z}:(r,M,W)=1} A_{1,2,3}(a+r,a+2r,a+3r) \leqslant \varepsilon. \tag{3.5}$$

We can now easily dispose of the case (i) by using the following inequality of 'Pollard-type' [34].

Lemma 3.1 (Pollard-type inequality). *Let G be a torus of any dimension equipped with its Haar measure μ_G and let $F_1, F_2, F_3 : G \to [0,1]$ be measurable functions. Set $\delta_i := \int_G F_i(x)\,d\mu(x)$ for $i = 1, 2, 3$ and write $\delta := \min(\delta_1, \delta_2, \delta_3)$. Then, for any distinct integers m_1, m_2, m_3, one has*

$$\int_G \int_G F_1(x+m_1y)F_2(x+m_2y)F_3(x+m_3y)\,d\mu_G(x)\,d\mu_G(y) \geqslant \frac{1}{4}\max(\delta_1+\delta_2+\delta_3-1,0)^2$$

if $\delta_1 + \delta_2 + \delta_3 \leqslant 1 + 2\delta$, and

$$\int_G \int_G F_1(x+m_1y)F_2(x+m_2y)F_3(x+m_3y) \geqslant \delta(\delta_1+\delta_2+\delta_3-1-\delta)$$

if $\delta_1 + \delta_2 + \delta_3 > 1 + 2\delta$.

Proof. We can replace the functions F_i with indicator functions by the following lifting trick: if we define the subsets A_i of the torus $\tilde{G} := G \times (\mathbb{R}/\mathbb{Z})^3$ for $i = 1, 2, 3$ by the formula

$$A_i := \{(x, t_1, t_2, t_3) \in \tilde{G} : t_i \in [0, F_i(x)]\},$$

then we see that $\delta_i = \mu_{\tilde{G}}(A_i)$ and

$$\begin{aligned}\int_G \int_G & F_1(x+m_1y)F_2(x+m_2y)F_3(x+m_3y)\,d\mu_G(x)\,d\mu_G(y)\\ &= \int_{\tilde{G}}\int_{\tilde{G}} 1_{A_1}(\tilde{x}+m_1\tilde{y})1_{A_2}(\tilde{x}+m_2\tilde{y})1_{A_3}(\tilde{x}+m_3\tilde{y})\,d\mu_{\tilde{G}}(\tilde{x})\,d\mu_{\tilde{G}}(\tilde{y}).\end{aligned} \tag{3.6}$$

Observe that as $(\tilde{x}, \tilde{y})$ ranges in $\tilde{G} \times \tilde{G}$, the triple $(\tilde{x}+m_1\tilde{y}, \tilde{x}+m_2\tilde{y}, \tilde{x}+m_3\tilde{y})$ ranges surjectively in the torus

$$\{(z_1, z_2, z_3) \in \tilde{G}^3 : (m_3-m_2)z_1 + (m_1-m_3)z_2 + (m_2-m_1)z_3 = 0\},$$

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and furthermore that the Haar probability measure on $\tilde{G} \times \tilde{G}$ pushes forward to Haar probability measure on this torus. Thus, we can write the expression (3.6) as a convolution

$$1_{(m_3-m_2)^{-1}A_1} * 1_{(m_1-m_3)^{-1}A_2} * 1_{(m_2-m_1)^{-1}A_3}(0),$$

where $(m_3-m_2)^{-1}A_1 := \{\tilde{x} \in \tilde{G} : (m_3-m_2)\tilde{x} \in A_1\}$ has the same measure δ_1 as A_1 (because the pushforward of Haar probability measure on $\tilde{G}$ by $\tilde{x} \mapsto (m_3 - m_2)\tilde{x}$ is also Haar probability measure), and similarly for $(m_1 - m_3)^{-1}A_2$ and $(m_2 - m_1)^{-1}A_3$. By inner regularity, we may assume that A_1, A_2, A_3 are all compact. The claim now follows from [**36**, Corollary 3] (see also [**39**, Theorem 1.1] for a closely related inequality). □

For any choice of a, r, we see from (3.2), Lemma 3.1 and the hypothesis $\delta_1 + \delta_2 + \delta_3 > 1$ of (i) that

$$A_{1,2,3}(a + r, a + 2r, a + 3r) \gg 1.$$

Averaging over a, r, we contradict (3.5) if ε is small enough.

It remains to handle the critical cases (ii), (iii). For this, we use the following inverse theorem for Lemma 3.1 that is deduced from the recent results in [**39**].

THEOREM 3.2 (Inverse theorem). *Let $\delta_1, \delta_2, \delta_3 > 0$ be real numbers with $\delta_1+\delta_2+\delta_3 = 1$. Let $\kappa > 0$, and suppose that $\varepsilon > 0$ is sufficiently small depending on κ. Let G be a torus with Haar probability measure $d\mu_G$, and let $g_1, g_2, g_3 : G \to [0, 1]$ be such that*

$$\delta_i - \varepsilon^{1/2} \leqslant \int_G g_i(x_0)\, d\mu_G(x_0) \leqslant \delta_i + \varepsilon^{1/2} \tag{3.7}$$

for $i = 1, 2, 3$ and such that

$$\int_G \int_G g_1(x_0 + y_0)g_2(x_0 + 2y_0)g_3(x_0 + 3y_0)\, d\mu_G(x_0)\, d\mu_G(y_0) \leqslant \varepsilon^{1/2}. \tag{3.8}$$

Then there exists a nonzero element ϕ of the Pontryagin dual group $\hat{G}$ (thus, $\phi : G \to \mathbb{R}/\mathbb{Z}$ is a continuous homomorphism that is not identically zero) and arcs I_1, I_2, I_3 in $\mathbb{R}/\mathbb{Z}$ of lengths exactly $\delta_1, \delta_2, \delta_3$, such that

$$g_1 \approx_\kappa 1_{\phi^{-1}(I_1)}$$
$$g_2 \approx_\kappa 1_{(2\phi)^{-1}(I_2)}$$
$$g_3 \approx_\kappa 1_{\phi^{-1}(I_3)},$$

where $2\phi : G \to \mathbb{R}/\mathbb{Z}$ is the map $(2\phi)(x_0) := 2(\phi(x_0))$ and $g \approx_\kappa h$ denotes the estimate $\|g - h\|_{L^1(G,d\mu_G)} \ll \kappa$.

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Proof. Introduce the sets $E_1, E_3 \subset G$ by the formulae

$$E_i := \{x_0 \in G : g_i(x_0) \geqslant \varepsilon^{1/8}\}$$

for $i = 1, 3$. On the one hand, we have the pointwise bound

$$1_{E_i} \geqslant g_i - \varepsilon^{1/8},$$

and hence from (3.7),

$$\mu_G(E_i) \geqslant \delta_i - O(\varepsilon^{1/8}) \tag{3.9}$$

for $i = 1, 3$. On the other hand, from the pointwise bound

$$1_{E_i} \leqslant \varepsilon^{-1/8} g_i$$

and (3.8), we have

$$\int_G \int_G 1_{E_1}(x_0 + y_0) g_2(x_0 + 2y_0) 1_{E_3}(x_0 + 3y_0)\, d\mu_G(x_0)\, d\mu_G(y_0) \ll \varepsilon^{1/4}$$

or equivalently (writing $x_0 + 2y_0 = z_0$)

$$\int_G g_2(z_0) 1_{E_1} * 1_{E_3}(2z_0)\, d\mu_G(z_0) \ll \varepsilon^{1/4}.$$

In particular, if we let F denote the set of points $x_0 \in G$ such that $1_{E_1} * 1_{E_3}(x_0) \geqslant \varepsilon^{1/8}$, then

$$\int_G g_2(z_0) 1_F(2z_0)\, d\mu_G(z_0) \ll \varepsilon^{1/8}. \tag{3.10}$$

Applying [**39**, Corollary 1.2] and (3.9), we have

$$\mu_G(F) \geqslant \mu_G(E_1) + \mu_G(E_3) - O(\varepsilon^{1/16}) \geqslant \delta_1 + \delta_3 - O(\varepsilon^{1/16}). \tag{3.11}$$

If one sets $F' := \{z_0 : 2z_0 \in F\}$, then (as G is a torus) F' has the same measure as F; thus,

$$\mu_G(F') \geqslant \mu_G(E_1) + \mu_G(E_3) - O(\varepsilon^{1/16}) \geqslant \delta_1 + \delta_3 - O(\varepsilon^{1/16}). \tag{3.12}$$

In particular, since $\delta_1 + \delta_2 + \delta_3 = 1$, we obtain

$$\int_{G \setminus F'} g_2(z_0)\, d\mu_G(z_0) \leqslant 1 - \mu_G(F') \leqslant \delta_2 + O(\varepsilon^{1/16}).$$

On the other hand, from (3.10), one has

$$\int_{F'} g_2(z_0)\, d\mu_G(z_0) \ll \varepsilon^{1/8}. \tag{3.13}$$

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From (3.7) and (3.13), we get

$$\int_{G\setminus F'} g_2(z_0)\,d\mu_G(z_0) = \delta_2 + O(\varepsilon^{1/16}). \tag{3.14}$$

From

$$\delta_2 - O(\varepsilon^{1/2}) \leqslant \int_G g_2(z_0)\,d\mu_G(z_0) \leqslant \int_{F'} g_2(z_0)\,d\mu_G(z_0) + \mu_G(G \setminus F')$$

and (3.12), (3.13), we get

$$\mu_G(F') = \delta_1 + \delta_3 - O(\varepsilon^{1/16}). \tag{3.15}$$

Finally, from (3.9) and (3.12), we get

$$\mu_G(E_1) = \delta_1 + O(\varepsilon^{1/16}), \quad \mu_G(E_3) = \delta_3 + O(\varepsilon^{1/16}). \tag{3.16}$$

By (3.15), we have $\mu_G(G \setminus F') = \delta_2 + O(\varepsilon^{1/16})$, which, together with (3.13) and (3.14), implies that

$$\|g_2 - 1_{F'}\|_{L^1(G)} \ll \varepsilon^{1/16}.$$

From (3.16), we have for $i = 1, 3$ that

$$\int_{E_i} g_i(x_0)\,d\mu_G(x_0) \leqslant \delta_i + O(\varepsilon^{1/16});$$

but by definition of E_i, we have

$$\int_{G\setminus E_i} g_i(x_0)\,d\mu_G(x_0) \ll \varepsilon^{1/8}.$$

Now, by (3.7) actually

$$\int_{E_i} g_i(x_0)\,d\mu_G(x_0) = \delta_i + O(\varepsilon^{1/16}).$$

Comparing the two previous formulae with (3.16), we conclude that

$$\|g_i - 1_{E_i}\|_{L^1(G)} \ll \varepsilon^{1/16}. \tag{3.17}$$

As F has the same measure as F', we see from (3.15) and (3.16) that

$$\mu_G(F) = \mu_G(E_1) + \mu_G(E_2) - O(\varepsilon^{1/16}).$$

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Applying [39, Theorem 1.5], there exists a nontrivial element $\phi \in \hat{G}$ and arcs $I_1, I_3 \subset \mathbb{R}/\mathbb{Z}$ such that

$$\mu_G(E_1 \triangle \phi^{-1}(I_1)), \mu_G(E_3 \triangle \phi^{-1}(I_3)) \leqslant \kappa^2, \tag{3.18}$$

where $\triangle$ denotes symmetric difference. (See also [5, 21] for closely related results.) (Note from the connectedness of G that $\phi(G)$ must be all of $\mathbb{R}/\mathbb{Z}$, and hence ϕ pushes forward μ_G to Haar probability measure on $\mathbb{R}/\mathbb{Z}$. The same claim then holds for 2ϕ.) Moreover, from (3.16), we see that necessarily $\mu_G(I_i) = \delta_i + O(\varepsilon^{1/16})$, and since ε is small enough in terms of κ, we may in fact add or remove a segment from I_i so that its length becomes exactly δ_i while keeping (3.18) true (with possibly $2\kappa^2$ in place of κ^2).

Combining (3.18) with (3.17), we see that

$$\|g_i - 1_{\phi^{-1}(I_i)}\|_{L^1(G)} \ll \kappa^2$$

for $i = 1, 3$. From (3.8), we conclude that

$$\int_G \int_G 1_{\phi^{-1}(I_1)}(x_0) g_2(x_0 + y_0) 1_{\phi^{-1}(I_3)}(x_0 + 2y_0)\, d\mu_G(x_0)\, d\mu_G(y_0) \ll \kappa^2$$

or equivalently

$$\int_G g_2(z_0) 1_{\phi^{-1}(I_1)} * 1_{\phi^{-1}(I_3)}(2z_0)\, d\mu_G(z_0) \ll \kappa^2.$$

Let J be the interval $I_1 + I_3$, shrunk on both sides by κ. Then J is an arc of length $\delta_1 + \delta_3 - 2\kappa$ and

$$1_{\phi^{-1}(I_1)} * 1_{\phi^{-1}(I_3)}(x_0) \gg \kappa$$

for $x_0 \in \phi^{-1}(J)$. We conclude that

$$\int_G g_2(z_0) 1_{\phi^{-1}(J)}(2z_0)\, d\mu_G(z_0) \ll \kappa.$$

If we let I_2 denote the complement of $I_1 + I_3$, then I_2 is an arc of length δ_2 that differs from the complement of J by two arcs of total length κ, and thus

$$\int_{G \backslash (2\phi)^{-1}(I_2)} g_2(z_0)\, d\mu_G(z_0) \ll \kappa.$$

Also

$$\int_{(2\phi)^{-1}(I_2)} g_2(z_0)\, d\mu_G(z_0) \leqslant \mu_G(\phi^{-1}(I_2)) = \delta_2.$$

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Combining this with (3.7), we see that

$$\int_{(2\phi)^{-1}(I_2)} g_2(x_0)\, d\mu_G(x_0) = \delta_2 + O(\kappa),$$

so

$$\|g_2 - 1_{(2\phi)^{-1}(I_2)}\|_{L^1(G)} \ll \kappa,$$

and the claim follows. □

Let $\kappa > 0$ be a small absolute constant to be chosen later, and suppose $\varepsilon > 0$ is sufficiently small depending on κ. Suppose first that (3.4) holds for some $c \in \mathbb{Z}/3\mathbb{Z}$. By Markov's inequality, this implies that for $1 - O(\varepsilon^{1/2})$ of the pairs of $(a, r) \in \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$ with $(r, M, W) = 1$, and any $c_1, c_2, c_3 \in \{1, 2, 3\}$ with $c_1 + c_2 + c_3 = c \pmod 3$, one has

$$A_{c_1,c_2,c_3}(a + r, a + 2r, a + 3r) \ll \varepsilon^{1/2}.$$

Applying Theorem 3.2, we conclude that for such pairs (a, r), there exists a nontrivial element $\phi_{a,r;c_1,c_2,c_3} \in \hat{G}$ and arcs $I_{a,r;c_1,c_2,c_3,i} \subset \mathbb{R}/\mathbb{Z}$ for $i = 1, 2, 3$ and any $c_1, c_2, c_3 \in \{1, 2, 3\}$ with $c_1 + c_2 + c_3 = c \pmod 3$, one has

$$\begin{aligned} F_{c_1}(\cdot, a + r) &\approx_\kappa 1_{\phi^{-1}_{a,r;c_1,c_2,c_3}(I_{a,r;c_1,c_2,c_3,1})} \\ F_{c_2}(\cdot, a + 2r) &\approx_\kappa 1_{(2\phi_{a,r;c_1,c_2,c_3})^{-1}(I_{a,r;c_1,c_2,c_3,2})} \\ F_{c_3}(\cdot, a + 3r) &\approx_\kappa 1_{\phi^{-1}_{a,r;c_1,c_2,c_3}(I_{a,r;c_1,c_2,c_3,3})}. \end{aligned}$$

From (3.2), we see that the arc $I_{a,r;c_1,c_2,c_3,i}$ has length $\delta_i + O(\kappa)$ for $i = 1, 2, 3$.

Now we start removing the dependence of $\phi_{a,r;c_1,c_2,c_3}$ on the various parameters a, r, c_1, c_2, c_3. The key lemma is the following.

LEMMA 3.3. *Let $0 < \sigma < 1/2$, and suppose that $\delta > 0$ is sufficiently small depending on σ. Let $\phi_1, \phi_2 \in \hat{G}$ be nontrivial and let $I_1, I_2 \subset \mathbb{R}/\mathbb{Z}$ be arcs of length between σ and $1 - \sigma$. Suppose that $1_{\phi_1^{-1}(I_1)} \approx_\delta 1_{\phi_2^{-1}(I_2)}$. Then we have $\phi_2 = \pm\phi_1$.*

Proof. By hypothesis, we have

$$\int_G 1_{\phi_1^{-1}(I_1)} 1_{\phi_2^{-1}(I_2)}\, d\mu_G = \mu_G(\phi_1^{-1}(I_1)) + O(\delta) = m(I_1) + O(\delta) \tag{3.19}$$

and similarly for ϕ_2 and I_2, where m denotes the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$. In particular, $m(I_2) = m(I_1) + O(\delta)$. By Fourier inversion, the left-hand side of (3.19)

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is equal to

$$\sum_{\substack{n,m\in\mathbb{Z}\\ n\phi_1+m\phi_2=0}} \check{1}_{I_1}(n)\check{1}_{I_2}(m),$$

where

$$\check{1}_{I_1}(n) := \int_{\mathbb{R}/\mathbb{Z}} 1_{I_1}(\alpha)e(-n\alpha)\,d\alpha$$

and similarly for $\check{1}_{I_2}(m)$. On the other hand, as G is connected, the Pontryagin dual $\hat{G}$ is torsion-free, so for each n, there is at most one m such that $n\phi_1 + m\phi_2 = 0$ and vice versa. If ϕ_1 is not an integer multiple of ϕ_2, then we may omit the $n = 1$ terms and conclude from Cauchy–Schwarz that

$$\Big(\sum_{n\in\mathbb{Z}\backslash\{1\}} |\check{1}_{I_1}(n)|^2\Big)^{1/2}\Big(\sum_{m\in\mathbb{Z}} |\check{1}_{I_2}(m)|^2\Big)^{1/2} \geqslant m(I_1) + O(\delta).$$

On the other hand, from the Plancherel identity, one has

$$\sum_{m\in\mathbb{Z}} |\check{1}_{I_2}(m)|^2 = m(I_2)$$

and (by explicit computation of $\check{1}_{I_1}(1)$)

$$\sum_{n\in\mathbb{Z}\backslash\{1\}} |\check{1}_{I_1}(n)|^2 = m(I_1) - |\check{1}_{I_1}(1)|^2 \leqslant m(I_1) - c_\sigma$$

for some quantity $c_\sigma > 0$ depending only on σ. For δ small enough, this leads to a contradiction. Thus, ϕ_1 is an integer multiple of ϕ_2, and, similarly, ϕ_2 is an integer multiple of ϕ_1; thus, $\phi_2 = \pm\phi_1$ as claimed. □

From this lemma, we see that for each $c_1 \in \mathbb{Z}/3\mathbb{Z}$ and $a \in \mathbb{Z}/M\mathbb{Z}$, there is at most one nontrivial $\phi_{a;c_1} \in \hat{G}$ up to sign such that $F_{c_1}(\cdot, a) \approx_\kappa 1_{\phi_{a,c}^{-1}(I_{c_1})}$ for some arc I_{c_1} of length δ_{c_1}. Select such a $\phi_{a;c_1}$ for each a, c_1 (or select ϕ arbitrarily if no such I_{c_1} exists). Then for $1 - O(\varepsilon^{1/2})$ of the pairs of (a, r) with $(r, M, W) = 1$, and any c_1, c_2, c_3 with $c_1 + c_2 + c_3 = c$, we have

$$\begin{aligned}\phi_{a,r;c_1,c_2,c_3} &= \pm\phi_{a+r;c_1}\\ 2\phi_{a,r;c_1,c_2,c_3} &= \pm\phi_{a+2r;c_2}\\ \phi_{a,r;c_1,c_2,c_3} &= \pm\phi_{a+3r;c_3}.\end{aligned}$$

In particular, for such a pair (a, r), we have

$$\phi_{a+r;c_1} = \pm\phi_{a+3r;c_3}$$

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for any $c_1, c_3 \in \{1, 2, 3\}$ (choosing c_2 to be congruent to $c - c_1 - c_3$ modulo 3), which implies, in particular, that $\phi_{a+r;c_1}$ does not depend on c_1 up to sign. Thus, we can actually find a nontrivial $\phi_a \in \hat{G}$ for all $a \in \mathbb{Z}/M\mathbb{Z}$, such that one has

$$\phi_{a+r} = \pm\phi_{a+3r}; \quad 2\phi_{a+r} = \pm\phi_{a+2r}$$

for $1 - O(\varepsilon^{1/2})$ of the pairs of (a, r) with $(r, M, W) = 1$. Replacing (a, r) by $(a - r, r)$ and $(a + 2r, -r)$, we also see that for $1 - O(\varepsilon^{1/2})$ of such pairs, we simultaneously have

$$\phi_a = \pm\phi_{a+2r}; \quad 2\phi_a = \pm\phi_{a+r}$$

and

$$\phi_{a+r} = \pm\phi_{a-r}; \quad 2\phi_{a+r} = \pm\phi_a,$$

which implies, in particular, that

$$4\phi_a = \pm\phi_a.$$

But this is impossible since $\hat{G}$ is torsion-free and ϕ_a is nontrivial. This proves Theorem 2.7(ii).

Now we turn to Theorem 2.7(iii). With κ and ε as above, we now assume instead that $\delta_1 \neq \delta_3$ and that (3.5) holds. Again using Markov's inequality followed by Theorem 3.2, we now conclude that for $1 - O(\varepsilon^{1/2})$ of the pairs of $(a, r) \in \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$ with $(r, M, W) = 1$, one has a nontrivial element $\phi_{a,r} \in \hat{G}$ and arcs $I_{a,r;i} \subset \mathbb{R}/\mathbb{Z}$ for $i = 1, 2, 3$ such that

$$\begin{aligned} F_1(\cdot, a + r) &\approx_\kappa 1_{\phi_{a,r}^{-1}(I_{a,r;1})} \\ F_2(\cdot, a + 2r) &\approx_\kappa 1_{(2\phi_{a,r})^{-1}(I_{a,r;2})} \\ F_3(\cdot, a + 3r) &\approx_\kappa 1_{\phi_{a,r}^{-1}(I_{a,r;3})}. \end{aligned}$$

From (3.2), we see that the arc $I_{a,r;i}$ has length $\delta_i + O(\varepsilon^{1/2})$ for $i = 1, 2, 3$.

Applying Lemma 3.3, we conclude that one can find nontrivial characters $\phi_a^{(i)} \in \hat{G}$ for $a \in \mathbb{Z}/M\mathbb{Z}$, $i = 1, 2, 3$ such that for $1 - O(\varepsilon^{1/2})$ of the pairs of $(a, r) \in \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$ with $(r, M, W) = 1$, we have

$$\phi_{a,r} = \pm\phi_{a+r}^{(1)} \tag{3.20}$$

$$2\phi_{a,r} = \pm\phi_{a+2r}^{(2)} \tag{3.21}$$

$$\phi_{a,r} = \pm\phi_{a+3r}^{(3)}, \tag{3.22}$$

so in particular

$$\phi_{a+2r}^{(2)} = \pm 2\phi_{a+r}^{(1)}.$$

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Replacing a by $a-r$, we conclude that for $1-O(\varepsilon^{1/2})$ of the above pairs (a,r), we have

$$\phi^{(2)}_{a+r} = \pm 2\phi^{(1)}_a.$$

This implies that for $1-O(\varepsilon^{1/2})$ of the triples $(a,r,r') \in \mathbb{Z}/M\mathbb{Z}\times\mathbb{Z}/M\mathbb{Z}\times\mathbb{Z}/M\mathbb{Z}$ with $(r,M,W),(r',M,W)=1$, we have

$$\phi^{(2)}_{a+r} = \pm 2\phi^{(1)}_a$$

and

$$\phi^{(2)}_{a+r} = \pm 2\phi^{(1)}_{a+r-r'},$$

which implies (by the torsion-free nature of $\hat{G}$) that

$$\phi^{(1)}_{a+r-r'} = \pm \phi^{(1)}_a.$$

Iterating this two more times, we see that for $1-O(\varepsilon^{1/2})$ of the septuples $(a,(r_i)_{i=1}^6) \in (\mathbb{Z}/M\mathbb{Z})^7$ with $(r_i,M,W)=1$ for $1\leqslant i\leqslant 6$, one has

$$\phi^{(1)}_{a+r_1-r_2+r_3-r_4+r_5-r_6} = \pm \phi^{(1)}_a.$$

For any $h\in\mathbb{Z}/M\mathbb{Z}$, the number of sextuples $(r_i)_{i=1}^6 \in (\mathbb{Z}/M\mathbb{Z})^6$ with $r_1-r_2+r_3-r_4+r_5-r_6=2h$ and $(r_i,M,W)=1$ for $1\leqslant i\leqslant 6$ can be computed using the Chinese remainder theorem to be comparable (up to absolute constants) to the quantity $1/M((\phi((M,W))/(M,W))M)^6$. (The factor of 2 here is needed to avoid the parity obstruction that $r_1-r_2+r_3-r_4$ is necessarily even if (M,W) is even.) On the other hand, the number of representations of $r_1-r_2+r_3-r_4+r_5-r_6=2h$, where $r_i\in(\mathbb{Z}/M\mathbb{Z})$ and $(r_i,M,W)=1$, and r_1 (say) belongs to an exceptional set of size $O(\varepsilon^{1/2}|\{r\in\mathbb{Z}/M\mathbb{Z} : (r,M,W)=1\}|)$ is bounded by $\ll \varepsilon^{1/2}1/M((\phi((M,W))/(M,W))M)^6$. (The validity of this bound follows from the fact that the number of representations $-r_2+r_3-r_4+r_5-r_6=h'$ with $r_i\in\mathbb{Z}/M\mathbb{Z}$ and $(r_i,M,W)=1$ is uniformly $\ll 1/M((\phi((M,W))/(M,W))M)^5$.)

From this and a double counting argument, we see that for $1-O(\varepsilon^{1/2})$ of the pairs $(a,h)\in(\mathbb{Z}/M\mathbb{Z})^2$, we have

$$\phi^{(1)}_{a+2h} = \pm \phi^{(1)}_a.$$

We conclude that there exists a nonzero element ϕ of $\hat{G}$ such that

$$\phi^{(1)}_{2a} = \pm\phi$$

for $1-O(\varepsilon^{1/2})$ of $a\in\mathbb{Z}/M\mathbb{Z}$. Inserting this back into (3.20), (3.21) and (3.22) and double counting, we conclude that

$$\phi^{(2)}_{2a} = \pm 2\phi$$

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and

$$\phi^{(3)}_{2a} = \pm\phi$$

for $1 - O(\varepsilon^{1/2})$ of $a \in \mathbb{Z}/M\mathbb{Z}$. This implies that for $1 - O(\varepsilon^{1/2})$ of $a \in \mathbb{Z}/M\mathbb{Z}$, we can find arcs $I_{a;1}, I_{a;2}, I_{a;3}$ in $\mathbb{R}/\mathbb{Z}$ such that

$$\begin{aligned} F_1(\cdot, a) &\approx_\kappa 1_{\phi^{-1}(I_{a;1})} \\ F_2(\cdot, a) &\approx_\kappa 1_{(2\phi)^{-1}(I_{a;2})} \\ F_3(\cdot, a) &\approx_\kappa 1_{\phi^{-1}(I_{a;3})}. \end{aligned}$$

Fix such an a. From (3.2), we see that each arc $I_{a;i}$ has length $\delta_i + O(\kappa)$ for $i = 1, 2, 3$. As $F_1 + F_2 + F_3 = 1$, we have

$$1 \approx_\kappa 1_{\phi^{-1}(I_{a;1})} + 1_{(2\phi)^{-1}(I_{a;2})} + 1_{\phi^{-1}(I_{a;3})}.$$

Since ϕ pushes forward μ_G to Haar measure m on $\mathbb{R}/\mathbb{Z}$, we conclude that

$$\int_{\mathbb{R}/\mathbb{Z}} |1_{I_{a;1}}(\theta) + 1_{I_{a;2}}(2\theta) + 1_{I_{a;3}}(\theta) - 1|\, dm(\theta) \ll \delta,$$

which implies that the set $I_{a;1} \cup I_{a;3}$ differs by at most $O(\delta)$ in measure from the set $\{\theta \in \mathbb{R}/\mathbb{Z} : 2\theta \notin I_{a;2}\}$. But since $I_{a;2}$ is an arc length $\delta_2 + O(\kappa)$, the set $\{\theta \in \mathbb{R}/\mathbb{Z} : 2\theta \notin I_{a;2}\}$ is the union of two arcs of length $(1 - \delta_2)/2 + O(\kappa)$, separated from each other by distance $\delta_2/2 + O(\kappa)$. Since $\delta_1 \neq \delta_3$ and $\delta_1 + \delta_2 + \delta_3 = 1$, δ_1 and δ_3 are both distinct from $(1 - \delta_2)/2$. As $I_{a;1}$ and $I_{a;3}$ are arcs of length $\delta_1 + O(\kappa)$ and $\delta_3 + O(\kappa)$, this leads to a contradiction for κ small enough. This proves Theorem 2.7(iii).

4. The main theorem for $k \geqslant 4$.

We now prove Theorem 2.7(iv). By reducing the functions F_i by an appropriate scalar multiple, we may assume that $\delta_1 = \cdots = \delta_k = \delta$ for some $\delta > c_k$. From (2.12) and the triangle inequality, we have

$$\lim_{P\to\infty} \mathbb{E}_{d \leqslant P:(d,W)=1} \int_X F_i(T^{id}x)\, d\mu(x) = \delta$$

for any $1 \leqslant i \leqslant k$, and also

$$\lim_{P\to\infty} \mathbb{E}_{d \leqslant P:(d,W)=1} \int_X F_i(T^{id}x) F_{i'}(T^{i'd}x)\, d\mu(x) = \delta^2$$

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for any $1 \leqslant i < i' \leqslant k$ (this can be seen by first changing variables from x to $T^{id}x$, pulling the d sum inside the integral and using (2.12) and the triangle inequality). This implies that

$$\lim_{P\to\infty} \mathbb{E}_{d\leqslant P:(d,W)=1} \int_X (1 - F_i(T^{id}x))\, d\mu(x) = 1 - \delta \tag{4.1}$$

and

$$\lim_{P\to\infty} \mathbb{E}_{d\leqslant P:(d,W)=1} \int_X (1 - F_i(T^{id}x))(1 - F_{i'}(T^{i'd}x))\, d\mu(x) = (1 - \delta)^2. \tag{4.2}$$

Now, we bound triple correlations.

LEMMA 4.1. *For $1 \leqslant i < i' < i'' \leqslant k$, one has*

$$\lim_{P\to\infty} \mathbb{E}_{d\leqslant P:(d,W)=1} \int_X (1-F_i(T^{id}x))(1-F_{i'}(T^{i'd}x))(1-F_{i''}(T^{i''d}x))\, d\mu(x) \leqslant \frac{3}{4}(1-\delta)^2.$$

Proof. By inclusion–exclusion, it suffices to show that

$$\lim_{P\to\infty} \mathbb{E}_{d\leqslant P:(d,W)=1} \int_X (1-F_i(T^{id}x))(1-F_{i'}(T^{i'd}x))F_{i''}(T^{i''d}x)\, d\mu(x) \geqslant \frac{1}{4}(1-\delta)^2.$$

By repeating the arguments of the previous section, to prove this, it suffices to do so when $X = G \times \mathbb{Z}/M\mathbb{Z}$ with G a torus with shift $T(x, a) = (x + \alpha, a + 1)$. (Here, it is essential that there are only three factors in the average considered here so that the average is of 'complexity one' and can thus be controlled by the Kronecker factor. The same is not true for the original average (2.14), but we will not need to directly pass to characteristic factors for that average.) It then suffices to establish the lower bound

$$\int_G \int_G (1-F_{i,a+ir}(x+iy))(1-F_{i',a+i'r}(x+i'y))F_{i'',a+i''r}(x+i''y)\, d\mu_G(x)d\mu_G(y) \geqslant \frac{1}{4}(1-\delta)^2$$

for all $a, r \in \mathbb{Z}/M\mathbb{Z}$, where the $F_{i,a} : G \to [0, 1]$ are measurable functions of mean δ. But this follows from Lemma 3.1 (noting that $\delta > c_k > 1/2$ and hence $(1 - \delta) + (1 - \delta) + \delta - 1 \leqslant 1 + 2\min(\delta, 1 - \delta)$). □

Let $\tilde{X}$ be the space $X \times [0, 1]^k$ with the product measure $d\mu dt_1 \ldots dt_k$, and for each $1 \leqslant i \leqslant k$, let $E_i \subset \tilde{X}$ denote the set

$$E_i := \{(x, t_1, \ldots, t_k) \in \tilde{X} : t_i > F_i(x)\},$$

then from the above lemma, we have

$$\lim_{P\to\infty} \mathbb{E}_{d\leqslant P:(d,W)=1} \int_{\tilde{X}} 1_{E_i}(T^{id}x, t)1_{E_{i'}}(T^{i'd}x, t)1_{E_{i''}}(T^{i''d}x, t)\, d\mu(x)dt \leqslant \frac{3}{4}(1 - \delta)^2.$$

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Hence, if $N(d,x,t)$ denotes the counting function

$$N(d,x,t) := \sum_{i=1}^{k} 1_{E_i}(T^{id}x,t),$$

then on summing the preceding assertion in i, i', i'', we obtain

$$\lim_{P\to\infty} \mathbb{E}_{d\leqslant P:(d,W)=1} \int_{\tilde{X}} \binom{N(d,x,t)}{3}\, d\mu(x)\, dt \leqslant \binom{k}{3}\frac{3}{4}(1-\delta)^2.$$

Applying similar arguments to (4.1) and (4.2), we obtain

$$\lim_{P\to\infty} \mathbb{E}_{d\leqslant P:(d,W)=1} \int_{\tilde{X}} \binom{N(d,x,t)}{2}\, d\mu(x)\, dt = \binom{k}{2}(1-\delta)^2$$

and

$$\lim_{P\to\infty} \mathbb{E}_{d\leqslant P:(d,W)=1} \int_{\tilde{X}} \binom{N(d,x,t)}{1}\, d\mu(x)\, dt = \binom{k}{1}(1-\delta).$$

On the other hand, if (2.14) fails, then

$$\lim_{P\to\infty} \mathbb{E}_{d\leqslant P:(d,W)=1} \int_{\tilde{X}} 1_{N(d,x,t)=0}\, dt \ll \varepsilon$$

for any given ε.

Next, note that for any integer $1 \leqslant a \leqslant k$, we have the inequality

$$(N(d,x,t)-1)(N(d,x,t)-a)(N(d,x,t)-a+1)) \geqslant 0$$

whenever $N(d,x,t) \neq 0$ since $N(d,x,t)$ is then an integer from 1 to k. On using the identities

$$x^3 = 6\binom{x}{3} + 6\binom{x}{2} + \binom{x}{1},$$
$$x^2 = 2\binom{x}{2} + \binom{x}{1},$$

this gives

$$6\binom{N(d,x,t)}{3} + (6-4a)\binom{N(d,x,t)}{2} + (a^2-a)\binom{N(d,x,t)}{1} - a(a-1) \geqslant 0.$$

(More generally, one has $x^n = \sum_{k=1}^{n} k!S(n,k)\binom{x}{k}$, where $S(n,k)$ are the Stirling numbers of the second kind.) Averaging in d, x, t and using the previous estimates, we conclude that

$$\left(\frac{9}{2}\binom{k}{3} + (6-4a)\binom{k}{2}\right)(1-\delta)^2 + (a^2-a)k(1-\delta) - a(a-1) \geqslant -O_k(\varepsilon). \quad (4.3)$$

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We take $a = a_k = \lceil (3k+2)/4 \rceil$ here (this turns out to be the optimal choice). Then this becomes exactly the same quadratic equation as in the definition of c_k in Theorem 1.10, which gives the desired contradiction if ε is small enough. This concludes the proof of Theorem 2.7(iv).

5. Obstructions for higher values of k

In this section, we give some limitations as to how much the value $c_k = 1 - 1/(k - \frac{4}{3} + o(1))$ appearing in Theorem 2.7(iv) may be lowered for large values of k.

LEMMA 5.1. *Let $m \geqslant 3$ be a natural number. Then there exist shifts $a_1, \dots, a_{m^2} \in \mathbb{R}/\mathbb{Z}$ such that the 'strips'*

$$S_i := \left\{ (x, y) \in (\mathbb{R}/\mathbb{Z})^2 : x + iy \in a_i + \left[0, \frac{2}{m}\right] \bmod 1 \right\}$$

for $i = 1, \dots, m^2$ cover the entire torus $(\mathbb{R}/\mathbb{Z})^2$.

Proof. We set $a_1 = \cdots = a_m = 0$. Then for any $y \in [1/m, 2/m]$, the strips $S_1, \dots, S_m$ intersect the circle $\{(x, y) : x \in \mathbb{R}/\mathbb{Z}\}$ in arcs $\{(x, y) : x \in [0, 2/m] - iy\}$. These m arcs have length $2/m$, with consecutive arcs intersecting in an arc of length at most $1/m$. The union of these m arcs is then an arc of length at least 1 and thus covers the whole circle. Thus, we have the inclusion

$$(\mathbb{R}/\mathbb{Z}) \times \left[\frac{1}{m}, \frac{2}{m}\right] \subset S_1 \cup \cdots \cup S_m.$$

By applying a 'Galilean transformation', we conclude that for any $1 \leqslant j < m$, if we define $a_{jm+i} := (j/m)i$ for $i = 1, \dots, m$, then for $y \in j/m + [1/m, 2/m] = [(j+1)/m, (j+2)/m]$, the strips $S_{jm+1}, \dots, S_{jm+m}$ intersect the circle $\{(x, y) : x \in \mathbb{R}/\mathbb{Z}\}$ in overlapping arcs of total length at least 1 so that

$$(\mathbb{R}/\mathbb{Z}) \times \left[\frac{j+1}{m}, \frac{j+2}{m}\right] \subset S_{jm+1} \cup \cdots \cup S_{jm+m}.$$

Taking the union over all $j = 0, \dots, m-1$, we obtain the claim. □

COROLLARY 5.2. *Let $k \geqslant 9$. In Theorem 2.7, one cannot replace c_k with any quantity lower than $1 - 2/\lfloor\sqrt{k}\rfloor$. (For $3 \leqslant k < 9$, this conclusion is vacuously true.)*

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Proof. Set $m := \lfloor\sqrt{k}\rfloor$ so that $m \geqslant 3$ and $m^2 \leqslant k$. Let $a_1, \ldots, a_{m^2} \in \mathbb{R}/\mathbb{Z}$ be as in the preceding lemma, set a_i arbitrarily for $m^2 < i \leqslant k$ and let I_i be the complement of $a_i + [0, 2/m]$ mod 1 in $\mathbb{R}/\mathbb{Z}$ for $i = 1, \ldots, k$. By the above lemma, we have

$$\prod_{i=1}^{k} 1_{I_i}(x + iy) = 0$$

for all $x, y \in \mathbb{R}/\mathbb{Z}$. If we then set X to be the unit circle $\mathbb{R}/\mathbb{Z}$ with Haar measure and an irrational shift $T : x \mapsto x+\alpha$ for some irrational $\alpha \in \mathbb{R}/\mathbb{Z}$, and set $F_i := 1_{I_i}$, we obtain the claim (with $\delta_i = 1 - 2/m = 1 - 2/\lfloor\sqrt{k}\rfloor$ for $i = 1, \ldots, k$). □

Clearly, any quantitative improvement in the covering construction in Lemma 5.1 would lead to a stronger lower bound on the optimal value of c_k in Corollary 5.2. We do not know, however, whether the optimal value behaves like $1 - 1/k$, like $1 - 1/\sqrt{k}$ or has some intermediate behaviour.

6. Proofs of the applications

Proof of Theorem 1.2. Consider the sets $Q_{\alpha,\beta} = \{n \in \mathbb{N} : n^\alpha < P^+(n) < n^\beta\}$ with $0 \leqslant \alpha < \beta \leqslant 1$. These sets are always stable since for any prime p, we have $1_{Q_{\alpha,\beta}}(pn) = 1_{Q_{\alpha,\beta}}(n) + O(1_{P^+(n)\in[n^\alpha,(pn)^\alpha]\cup[n^\beta,(pn)^\beta]}) + O(1_{p>n^\alpha})$ and after taking expectations over $n \leqslant x$, the $O(\cdot)$ term becomes negligible (as follows, for instance, from the continuity of the Dickman function). Also, $Q_{\alpha,\beta}$ is uniformly distributed in short intervals with density $\rho(1/\beta) - \rho(1/\alpha)$ since for $x/\log x \leqslant n \leqslant x$, we have

$$1_{Q_{\alpha,\beta}}(n) = 1_{P^+(n)\leqslant x^\beta} - 1_{P^+(n)\leqslant x^\alpha} + O(1_{P^+(n)\in[(x/\log x)^\alpha, x^\alpha]\cup[(x/\log x)^\beta, x^\beta]}),$$

and the $O(\cdot)$ term is negligible, whereas $1_{P^+(n)\leqslant x^\alpha}$ is a real-valued multiplicative function, so by [42, Lemma 3.4], we have

$$\int_0^x |\mathbb{E}_{y\leqslant n\leqslant y+H, n=b \pmod q} 1_{P^+(n)\leqslant x^\alpha} - \rho(1/\alpha)|\, dy = o(1),$$

and the same holds with α replaced by β. Now, since $d(Q_{0,\alpha}) + d(Q_{\alpha,\beta}) + d(Q_{\beta,1}) = 1$, $d(Q_{0,\alpha} \cup Q_{\alpha,\beta} \cup Q_{\beta,1}) = 1$ and

$$d(Q_{0,\alpha}) = \rho(1/\alpha) \neq 1 - \rho(1/\beta) = d(Q_{\beta,1})$$

by hypothesis, we conclude from Theorem 1.9 that

$$d_-((Q_{0,\alpha} - 1) \cap (Q_{\alpha,\beta} - 2) \cap (Q_{\beta,1} - 3)) > 0$$

whenever $\rho(1/\alpha) \neq 1 - \rho(1/\beta)$, and the positivity of the first density in Theorem 1.2 follows. The positivity of the second density is proven completely symmetrically. □

Proof of Theorem 1.12. We know from the proof of Theorem 1.2 that $\{n \in \mathbb{N} : P^+(n) > n^\gamma\}$ is a stable set that is uniformly distributed in short intervals with density $1-\rho(1/\gamma)$. Thus, as long as $3(1-\rho(1/\gamma)) > 1$, we can apply Theorem 1.8 to obtain the desired conclusion. But $3(1-\rho(1/\gamma)) > 1$ holds exactly when $\gamma < e^{-1/3}$, as wanted. □

Proof of Theorem 1.13. Employing Theorem 1.10, we only need to show that if c_k are as in that theorem, then $1 - \rho(1/\gamma_k) > c_k$ for $k = 4, 5$, and this is true by a numerical computation. □

Proof of Theorem 1.1. By applying Theorem 1.2 for any α, β satisfying $\rho(1/\alpha) \neq 1 - \rho(1/\beta)$, we already know that

$$\begin{aligned} &d\_(n \in \mathbb{N} : P^+(n+1) < P^+(n+2) < P^+(n+3)) > 0, \\ &d\_(n \in \mathbb{N} : P^+(n+1) > P^+(n+2) > P^+(n+3)) > 0. \end{aligned} \tag{6.1}$$

We prove the positivity of the first density in Theorem 1.1; the second one is proven completely symmetrically. We follow the strategy of [31, Corollary 2.8]. Suppose, for a contradiction, that we had

$$\lim_{l\to\infty} \mathbb{E}_{n\leqslant x_l} 1_{P^+(n+1)<P^+(n+2)<P^+(n+3)>P^+(n+4)} = 0$$

for some sequence $(x_l)_{l\in\mathbb{N}}$ tending to infinity. Let

$$\mathcal{S} := \{n \in \mathbb{N} : P^+(n+1) < P^+(n+2) < P^+(n+3)\}.$$

Then as $l \to \infty$, we have

$$\mathbb{E}_{n\leqslant x_l} 1_{n\in\mathcal{S}, n+1\notin\mathcal{S}} = o(1).$$

Iterating this, for any $H \in \mathbb{N}$, we see that for almost all $n \leqslant x_l$, we have

$$\mathbb{E}_{n\leqslant x_l} 1_{n\in\mathcal{S}} 1_{n+1\notin\mathcal{S} \text{ or } n+2\notin\mathcal{S} \text{ or} \dots \text{or } n+H\notin\mathcal{S}} = o(1).$$

In particular, this yields

$$\mathbb{E}_{n\leqslant x_l} 1_{n\notin\mathcal{S}} + \mathbb{E}_{n\leqslant x_l} 1_{n+1,\dots,n+H\in\mathcal{S}} \geqslant 1 - o(1)$$

as $l \to \infty$. By (6.1), we must then have

$$\mathbb{E}_{n\leqslant x_l} 1_{n+1,\dots,n+H\in\mathcal{S}} \geqslant c - o(1)$$

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for some $c > 0$ independent of H and for all large enough l. However, for any $\varepsilon > 0$, we have

$$(\mathcal{S} - 1) \cap \cdots \cap (\mathcal{S} - H) \subset \{n \in \mathbb{N} : P^+(n+1) \leqslant n^\varepsilon\} \cup \{n \in \mathbb{N} : P^+(n+h) > n^\varepsilon \text{ for all } 2 \leqslant h \leqslant H\}. \tag{6.2}$$

The density of the first set on the right-hand side of (6.2) over $n \leqslant x_l$ is $\rho(1/\varepsilon) + o(1)$, whereas by the Matomäki–Radziwiłł theorem, the density of the second set is $\leqslant \varepsilon + o(1)$ as soon as H is large enough in terms of ε. Thus,

$$\mathbb{E}_{n \leqslant x_l} 1_{n,n+1,\ldots,n+H \in \mathcal{S}} \leqslant \rho(1/\varepsilon) + \varepsilon + o(1)$$

for all large enough H, and letting $\varepsilon \to 0$, we get the desired contradiction. □

Proof of Theorem 1.3. We prove the theorem for $\omega(n)$; the case of $\Omega(n)$ is similar (and in fact slightly simpler). We first note that the sets $A_a := \{n \in \mathbb{N} : \omega(n) \equiv a \pmod 3\}$ are weakly stable; indeed, for any prime $p \nmid n$, we have

$$1_{A_a}(n) = 1_{A_{a+1}}(pn).$$

Also, we can represent $1_A(n)$ as a linear combination of 1-bounded multiplicative functions by the Fourier expansion

$$1_A(n) = \frac{1}{3} \sum_{j=0}^{2} \zeta^{-aj} \zeta^{\omega(n)j}, \tag{6.3}$$

where $\zeta := e(\frac{1}{3})$. The constant function $1/3$ is certainly uniformly distributed in short intervals with density $1/3$. The multiplicative function $n \mapsto \zeta^{\omega(n)}$ is uniformly distributed in short intervals with density 0, thanks to [30, Theorem A.1], since

$$\inf_{|t| \leqslant x} \sum_{p \leqslant x} \frac{1 - \mathrm{Re}(\zeta^{\omega(p)} \overline{\chi(p) p^{it}})}{p} \gg_\chi \log\log x \tag{6.4}$$

for every Dirichlet character χ by the Vinogradov–Korobov zero-free region for Dirichlet L-functions. Thus, A_a itself is uniformly distributed in short intervals with density $1/3$.

Our objective is to show that $(A_{a_1} - 1) \cap (A_{a_2} - 2) \cap (A_{a_3} - 3)$ has positive lower density for any $a_1, a_2, a_3 \in \mathbb{Z}/3\mathbb{Z}$. By modifying the first part of the proof of Theorem 2.2, it suffices to show that for every function $1 \leqslant \omega(X) \leqslant X$ tending to infinity, we have

$$\mathbb{E}^{\log}_{x/\omega(x) \leqslant n \leqslant x} 1_{A_{a_1}}(n+1) 1_{A_{a_2}}(n+2) 1_{A_{a_3}}(n+3) \gg 1. \tag{6.5}$$

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The left-hand side of (6.5) can be expanded using (6.3) as

$$\frac{1}{27}\sum_{j_1,j_2,j_3\in\{0,1,2\}}\zeta^{-(a_1j_1+a_2j_2+a_3j_3)}C_{j_1,j_2,j_3}, \tag{6.6}$$

where

$$C_{j_1,j_2,j_3} := \mathbb{E}^{\log}_{x/\omega(x)\leqslant n\leqslant x}\zeta^{j_1\omega(n+1)+j_2\omega(n+2)+j_3\omega(n+3)}.$$

Clearly $C_{0,0,0} = 1$. From [37, Theorem 1.3] and (6.4), we also have $C_{j_1,j_2,j_3} = o(1)$ when one or two of the j_1, j_2, j_3 vanish. Finally, from the weak form of the logarithmic Elliott conjecture from [41, Corollary 1.6] combined with (6.4), we also see that $C_{j_1,j_2,j_3} = o(1)$ whenever $j_1+j_2+j_3 \neq 0 \pmod 3$. Finally, we have $C_{2,2,2} = \overline{C_{1,1,1}}$. Putting all this together, we can write the left-hand side of (6.5) as

$$\tfrac{1}{27}(1+2\mathrm{Re}(\zeta^{-a_1-a_2-a_3}C_{1,1,1}))+o(1). \tag{6.7}$$

For $c \in \{0, 1, 2\}$, let

$$\delta_c := \mathbb{E}^{\log}_{x/\omega(x)\leqslant n\leqslant x}1_{\omega(n+1)+\omega(n+2)+\omega(n+3)\equiv c \pmod 3}.$$

Then (6.7) can be rewritten as

$$\tfrac{1}{27}(1+2\mathrm{Re}(\zeta^{-a}\delta_0+\zeta^{-a+1}\delta_1+\zeta^{-a+2}\delta_2)+o(1)), \tag{6.8}$$

where $a := a_1+a_2+a_3 \pmod 3$. Since $\mathrm{Re}(\zeta^{-a}) = 1$ if $a \equiv 0 \pmod 3$ and $\mathrm{Re}(\zeta^{-a}) = -1/2$ otherwise, using $\delta_0+\delta_1+\delta_2 = 1$, we can rewrite this as

$$\tfrac{1}{27}(3\delta_{3-a}+o(1)).$$

Thus, in order to show that (6.8) is $\gg 1$, what remains to be shown is that δ_0, δ_1, $\delta_2 \gg 1$. But since the sets A_i are weakly stable and uniformly distributed with densities $1/3$ each, by Theorem 1.9, we have

$$d_-\Bigg(\bigcup_{\substack{c_1,c_2,c_3\in\{0,1,2\}\\ c_1+c_2+c_3=c \pmod 3}}(A_{c_1}-1)\cap(A_{c_2}-2)\cap(A_{c_3}-3)\Bigg) > 0,$$

or in other words,

$$d_-(\{n\in\mathbb{N} : \omega(n+1)+\omega(n+2)+\omega(n+3)\equiv c \pmod 3\}) > 0$$

for every $c \in \mathbb{Z}/3\mathbb{Z}$, which, by partial summation, implies $\delta_c > 0$ for each c. The proof is now complete. □

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7. Sign patterns of the Liouville function

Before proving Theorem 1.14, we present a few lemmas. In what follows, $\omega(x) \leqslant x$ will be an arbitrary function tending to infinity. By modifying the first part of the proof of Theorem 2.2, it suffices to show that

$$\limsup_{x\to\infty} \mathbb{E}^{\log}_{x/\omega(x)\leqslant n\leqslant x} 1_{\lambda(n+1)=\varepsilon_1} \cdots 1_{\lambda(n+5)=\varepsilon_5} > 0$$

for at least 24 choices of $(\varepsilon_1, \ldots, \varepsilon_5) \in \{-1, +1\}^5$ and that the patterns listed in Theorem 1.14 are among these 24 patterns.

LEMMA 7.1. *Let $k \geqslant 1$ and let $h_1, \ldots, h_k \in \mathbb{N}$. Let $1 \leqslant \omega(X) \leqslant X$ be any function tending to infinity. Extend the Liouville function arbitrarily to negative integers. Then we have*

$$\mathbb{E}^{\log}_{x/\omega(x)\leqslant n\leqslant x} \lambda(n+h_1)\cdots\lambda(n+h_k) = \mathbb{E}^{\log}_{x/\omega(x)\leqslant n\leqslant x} \lambda(n-h_1)\cdots\lambda(n-h_k) + o(1).$$

Proof. This is a direct corollary of the 'isotopy formula' [**41**, Theorem 1.2(iii)]. □

LEMMA 7.2. *Let $k \geqslant 1$ be an integer and let $1 \leqslant \omega(X) \leqslant X$ be any function tending to infinity. Then we have*

$$\limsup_{x\to\infty} |\mathbb{E}^{\log}_{x/\omega(x)\leqslant n\leqslant x} \lambda(n+1)\cdots\lambda(n+k)| \leqslant \tfrac{1}{2}.$$

Proof. This is a simple generalization of [**41**, Proposition 7.1]. By the triangle inequality, we have

$$\begin{aligned}
&|\mathbb{E}^{\log}_{x/\omega(x)\leqslant n\leqslant x} \lambda(n+1)\cdots\lambda(n+k) + \lambda(n+2)\cdots\lambda(n+k+1)| \\
&\quad \leqslant \mathbb{E}^{\log}_{x/\omega(x)\leqslant n\leqslant x} |\lambda(n+1)\cdots\lambda(n+k) + \lambda(n+2)\cdots\lambda(n+k+1)| \\
&\quad = \mathbb{E}^{\log}_{x/\omega(x)\leqslant n\leqslant x} |\lambda(n+1) + \lambda(n+k+1)|.
\end{aligned}$$

Here, the first expression is equal to $2|\mathbb{E}^{\log}_{x/\omega\leqslant n\leqslant x} \lambda(n+1)\cdots\lambda(n+k)| + o(1)$ by the shift invariance of logarithmic averages. But since $(\lambda(n+1), \lambda(n+k+1))$ takes each sign pattern in $\{-1, +1\}^2$ with density $1/4 + o(1)$ with respect to the density $\mathbb{E}^{\log}_{x/\omega\leqslant n\leqslant x}$, by [**37**, Theorem 1.2], we get

$$2|\mathbb{E}^{\log}_{x/\omega(x)\leqslant n\leqslant x} \lambda(n+1)\cdots\lambda(n+k)| \leqslant \tfrac{1}{2}|1+1| + \tfrac{1}{2}|1-1| + o(1) = 1 + o(1),$$

as required. □

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LEMMA 7.3. *We have*

$$\limsup_{x\to\infty} |\mathbb{E}^{\log}_{x/\omega(x)\leqslant n\leqslant x}\lambda(n+1)\lambda(n+2)\lambda(n+4)\lambda(n+5)| < 1.$$

Proof. Suppose the contrary. Then there exists a sign $\varepsilon_0 \in \{-1, +1\}$ and an infinite sequence $x_l \to \infty$ such that

$$\mathbb{E}^{\log}_{x_l/\omega(x_l)\leqslant n\leqslant x}\lambda(n+1)\lambda(n+2)\lambda(n+4)\lambda(n+5) = \varepsilon_0 + o(1).$$

Consequently, we have

$$\mathbb{E}^{\log}_{x_l/\omega(x_l)\leqslant n\leqslant x} 1_{\lambda(n+1)\lambda(n+2)\lambda(n+4)\lambda(n+5)=\varepsilon_0} = 1 + o(1). \tag{7.1}$$

Shifting by one, we also have

$$\mathbb{E}^{\log}_{x_l/\omega(x_l)\leqslant n\leqslant x} 1_{\lambda(n+2)\lambda(n+3)\lambda(n+5)\lambda(n+6)=\varepsilon_0} = 1 + o(1).$$

Putting the last two equations together, we obtain

$$\mathbb{E}^{\log}_{x_l/\omega(x_l)\leqslant n\leqslant x} 1_{\lambda(n+1)\lambda(n+3)\lambda(n+4)\lambda(n+6)=1} = 1 + o(1).$$

Shifting by one again, we have

$$\mathbb{E}^{\log}_{x_l/\omega(x_l)\leqslant n\leqslant x} 1_{\lambda(n+2)\lambda(n+4)\lambda(n+5)\lambda(n+7)=1} = 1 + o(1). \tag{7.2}$$

Finally, putting (7.1) and (7.2) together yields

$$\mathbb{E}^{\log}_{x_l/\omega(x_l)\leqslant n\leqslant x} 1_{\lambda(n+1)\lambda(n+7)=\varepsilon_0} = 1 + o(1),$$

and therefore,

$$\mathbb{E}^{\log}_{x/\omega(x)\leqslant n\leqslant x}\lambda(n+1)\lambda(n+7) = \varepsilon_0 + o(1).$$

This, however, is in contradiction with the two-point logarithmic Chowla conjecture [**40**, Theorem 1.2]. □

Proof of Theorem 1.14. Let us define

$$C_A := \widetilde{\lim}\left(\mathbb{E}^{\log}_{x_l/\omega(x_l)\leqslant n\leqslant x_l} \prod_{j\in A}\lambda(n+j)\right)_{\ell\in\mathbb{N}},$$

where $\widetilde{\lim}$ is any generalized limit functional. Using the identity $1_{\lambda(n)=\varepsilon} = (1+\varepsilon\lambda(n))/2$ for $\varepsilon \in \{-1, +1\}$ and expanding, we have

$$32\,\widetilde{\lim}(\mathbb{E}^{\log}_{x_l/\omega(x_l)\leqslant n\leqslant x_l} 1_{\lambda(n+1)=\varepsilon_1}\cdots 1_{\lambda(n+5)=\varepsilon_5})_{\ell\in\mathbb{N}} = 1 + \sum_{\substack{A\subset[5]\\ A\neq\emptyset}} C_A \prod_{j\in A}\varepsilon_j. \tag{7.3}$$

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It suffices to show that there are at least 24 sign patterns $(\varepsilon_1, \ldots, \varepsilon_5)$ for which (7.3) is > 0, regardless of which generalized limit $\widetilde{\lim}$ we choose, including the six explicit patterns listed in the theorem and their reversals. (It is this part of the argument that results in us obtaining a positive *upper* density result rather than a positive *lower* density result. Indeed, we show that for every generalized limit $\widetilde{\lim}$, there are at least 24 sign patterns $(\varepsilon_1, \ldots, \varepsilon_5)$ for which (7.3) is > 0, but, theoretically, the choice of these 24 sign patterns could depend on the choice of $\widetilde{\lim}$, thus leading only to a lim sup result. However, for each of the explicit patterns listed in Theorem 1.14, we do obtain a lower density result by showing that (7.3) is always > 0 for these sign patterns.)

By the odd order logarithmic Chowla conjecture [**41**], all the odd order correlations are 0, and by the two-point logarithmic Chowla conjecture [**37**, Theorem 1.2], all the two-point correlations are 0 as well. Thus, if we denote the average on the left-hand side of (7.3) by $\mathbb{P}_{\varepsilon_1,\ldots,\varepsilon_5}$, then

$$32\mathbb{P}_{\varepsilon_1,\ldots,\varepsilon_5} = 1 + \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5(\varepsilon_1 C_{[5]\setminus\{1\}} + \cdots + \varepsilon_5 C_{[5]\setminus\{5\}}).$$

If we denote $C_{[5]\setminus\{1\}} := a$, then by shift invariance also $C_{[5]\setminus\{5\}} = a$. Furthermore, by Lemma 7.1, if $C_{[5]\setminus\{2\}} = b$, then $C_{[5]\setminus\{4\}} = b$. Finally, denote $C_{[5]\setminus\{3\}} = c$. We conclude that

$$32\mathbb{P}_{\varepsilon_1,\ldots,\varepsilon_5} = 1 + \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5((\varepsilon_1 + \varepsilon_5)a + (\varepsilon_2 + \varepsilon_4)b + \varepsilon_3 c). \tag{7.4}$$

Next, we split into several cases.

Case $a = b = 0$. When this holds, by Lemma 7.3, we have

$$32\mathbb{P}_{\varepsilon_1,\ldots,\varepsilon_5} \geqslant 1 - |c| > 0$$

for each of the 32 patterns.

Case $c \neq 0$, exactly one of $a, b \neq 0$. Suppose that $a \neq 0$, $b = 0$; the other case is symmetric. Then

$$32\mathbb{P}_{\varepsilon_1,\ldots,\varepsilon_5} = 1 + \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5((\varepsilon_1 + \varepsilon_5)a + \varepsilon_3 c).$$

Since $|a| \leqslant \frac{1}{2}$ and $|c| < 1$ by Lemmas 7.2 and 7.3, respectively, the only way that the probability can be zero is if $\varepsilon_1 = \varepsilon_5$, $\varepsilon_1 \mathrm{sgn}(a) = \varepsilon_3 \mathrm{sgn}(c)$ and $\varepsilon_1\varepsilon_2\varepsilon_4\varepsilon_5 \mathrm{sgn}(c) = -1$. This happens for $32 \cdot \frac{1}{2^3} = 4$ sign patterns, so there are $32 - 4 = 28$ sign patterns having positive probability.

Case $c = 0$, exactly one of $a, b \neq 0$. Suppose that $a \neq 0$, $b = 0$; the opposite case is symmetric. Then

$$32\mathbb{P}_{\varepsilon_1,\ldots,\varepsilon_5} = 1 + (\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 + \varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5)a,$$

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and the only way this can be zero is if $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = \varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5 = -\mathrm{sgn}(a)$, which happens for exactly $32 \cdot \frac{1}{2^2} = 8$ patterns. Thus, there are $32 - 8 = 24$ patterns with positive density.

Case $c = 0$, both $a, b \neq 0$. Then we have

$$32\mathbb{P}_{\varepsilon_1,\ldots,\varepsilon_5} = 1 + \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5((\varepsilon_1 + \varepsilon_5)a + (\varepsilon_2 + \varepsilon_4)b).$$

Now, consider ε_i satisfying

$$\begin{aligned}\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5 &= +1,\\ \varepsilon_1 = \varepsilon_5 &= -\mathrm{sgn}(a)\\ \varepsilon_2 = \varepsilon_4 &= -\mathrm{sgn}(b),\end{aligned}$$

which can always be found. The resulting probability is nonnegative, so

$$1 - 2|a| - 2|b| \geqslant 0,$$

so $|a| + |b| \leqslant \frac{1}{2}$. Therefore, since $a, b \neq 0$, the only way that $\mathbb{P}_{\varepsilon_1,\ldots,\varepsilon_5} = 0$ can happen is if $\varepsilon_1 = \varepsilon_5$, $\varepsilon_2 = \varepsilon_4$ and $\varepsilon_1\mathrm{sgn}(a) = \varepsilon_2\mathrm{sgn}(b)$. This happens for $32 \cdot \frac{1}{2^3} = 4$ patterns, so there must be at least $32 - 4 = 28$ patterns for which the probability is positive.

Case $a, b, c \neq 0$. Now suppose that $\mathbb{P}_{\varepsilon_1,\ldots,\varepsilon_5} = 0$ and consider the transformations of $(\varepsilon_1, \ldots, \varepsilon_5)$ given by

$$\begin{aligned}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) &\mapsto (-\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, -\varepsilon_5)\\ (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) &\mapsto (\varepsilon_1, -\varepsilon_2, \varepsilon_3, -\varepsilon_4, \varepsilon_5)\\ (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) &\mapsto (-\varepsilon_1, -\varepsilon_2, \varepsilon_3, -\varepsilon_4, -\varepsilon_5).\end{aligned}$$

Since $a \neq 0$, $b \neq 0$, each of the first two transformations changes the probability in (7.4), in particular, making it nonzero. The third transformation also changes the probability in (7.4), unless $(\varepsilon_1 + \varepsilon_5)a + (\varepsilon_2 + \varepsilon_4)b = 0$, in which case $32\mathbb{P}_{\varepsilon_1,\ldots,\varepsilon_5} = 1 + \varepsilon_3 c > 0$, contrary to our assumption. Thus, the patterns $(\varepsilon_1, \ldots, \varepsilon_5)$ can be grouped into groups of four where each group is closed under the above three transformations and has at most one pattern with zero probability. Hence, there are at least $32 - \frac{32}{4} = 24$ patterns having nonzero probability.

Since the above considerations exhaust all cases, we have now shown that there are at least 24 sign patterns of length 5 having positive upper density. We still need to show that the specific patterns mentioned in Theorem 1.14 are among the patterns having positive upper density. The existence of the patterns having exactly one plus or exactly one minus follows directly from the proof strategy of [**31**, Corollary 2.8] together with the fact that each length 4 pattern occurs in

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the Liouville function with positive lower density. When it comes to the remaining patterns, consider $(+1, +1, \pm 1, -1, -1)$: the others are similar. This pattern has probability

$$32\mathbb{P}_{\varepsilon_1,\dots,\varepsilon_5} = 1 \pm c > 0$$

by Lemma 7.3. This completes the proof. □

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Conflict of interest declaration

None.

References

[1] A. Balog, 'Problem in 'Tagunsbericht 41/1982'', *Math. Forschungsinstitut Oberwolfach*. 1982. p. 29.
[2] A. Balog, 'On triplets with descending largest prime factors', *Studia Sci. Math. Hungar.* **38** (2001), 45–50.
[3] A. Balog and T. D. Wooley, 'On strings of consecutive integers with no large prime factors', *J. Austral. Math. Soc. Ser. A* **64**(2) (1998), 266–276.
[4] S. Chowla, *The Riemann Hypothesis and Hilbert's Tenth Problem*, Mathematics and Its Applications, 4 (Gordon and Breach Science Publishers, New York–London–Paris, 1965).
[5] M. Christ and M. Iliopoulou, 'Inequalities of Riesz–Sobolev type for compact connected Abelian groups', Preprint, 2018.
[6] J.-M. De Koninck and N. Doyon, 'On the distance between smooth numbers', *Integers* **11**(A25) (2011), 22.
[7] R. de la Bretèche, C. Pomerance and G. Tenenbaum, 'Products of ratios of consecutive integers', *Ramanujan J.* **9**(1–2) (2005), 131–138.
[8] K. Dickman, 'On the frequency of numbers containing prime factors of a certain relative magnitude', *Ark. Mat.* **22A**(10) (1930), 1–14.
[9] P. D. T. A. Elliott, 'On the correlation of multiplicative and the sum of additive arithmetic functions', *Mem. Amer. Math. Soc.* **112**(538) (1994), viii+88.

https://doi.org/10.1017/fms.2019.28 Published online by Cambridge University Press

[10] P. Erdős, 'Some unconventional problems in number theory', in *Journées Arithmétiques de Luminy (Colloq. Internat. CNRS, Centre Univ. Luminy, Luminy, 1978)*, Astérisque, 61 (Soc. Math. France, Paris, 1979), 73–82.

[11] P. Erdős and C. Pomerance, 'On the largest prime factors of n and $n+1$', *Aequationes Math.* **17**(2–3) (1978), 311–321.

[12] N. Frantzikinakis and B. Host, 'Furstenberg systems of bounded multiplicative functions and applications', *Int. Math. Res. Not. IMRN* to appear.

[13] N. Frantzikinakis and B. Host, 'The logarithmic Sarnak conjecture for ergodic weights', *Ann. of Math. (2)* **187**(3) (2018), 869–931.

[14] N. Frantzikinakis, B. Host and B. Kra, 'Multiple recurrence and convergence for sequences related to the prime numbers', *J. Reine Angew. Math.* **611** (2007), 131–144.

[15] H. Furstenberg, 'Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions', *J. Anal. Math.* **31** (1977), 204–256.

[16] H. Furstenberg and B. Weiss, 'A mean ergodic theorem for $(1/N)\sum_{n=1}^{N} f(T^n x)g(T^{n^2} x)$', in *Convergence in Ergodic Theory and Probability (Columbus, OH, 1993)*, Ohio State Univ. Math. Res. Inst. Publ., 5 (de Gruyter, Berlin, 1996), 193–227.

[17] E. Glasner, *Ergodic Theory via Joinings*, Mathematical Surveys and Monographs, 101 (American Mathematical Society, Providence, RI, 2003).

[18] B. Green and T. Tao, 'Linear equations in primes', *Ann. of Math. (2)* **171**(3) (2010), 1753–1850.

[19] B. Green and T. Tao, 'The Möbius function is strongly orthogonal to nilsequences', *Ann. of Math. (2)* **175**(2) (2012), 541–566.

[20] B. Green, T. Tao and T. Ziegler, 'An inverse theorem for the Gowers $U^{s+1}[N]$-norm', *Ann. of Math. (2)* **176**(2) (2012), 1231–1372.

[21] J. T. Griesmer, 'Semicontinuity of structure for small sumsets in compact abelian groups', Preprint, 2018.

[22] A. Hildebrand, 'On a conjecture of Balog', *Proc. Amer. Math. Soc.* **95**(4) (1985), 517–523.

[23] A. Hildebrand, 'On consecutive values of the Liouville function', *Enseign. Math. (2)* **32**(3–4) (1986), 219–226.

[24] A. Hildebrand, 'On integer sets containing strings of consecutive integers', *Mathematika* **36**(1) (1989), 60–70.

[25] A. Hildebrand, 'Multiplicative properties of consecutive integers', in *Analytic Number Theory (Kyoto, 1996)*, London Math. Soc. Lecture Note Ser., 247 (Cambridge Univ. Press, Cambridge, 1997), 103–117.

[26] A. Hildebrand and G. Tenenbaum, 'Integers without large prime factors', *J. Théor. Nombres Bordeaux* **5**(2) (1993), 411–484.

[27] B. Host and B. Kra, 'Nonconventional ergodic averages and nilmanifolds', *Ann. of Math. (2)* **161**(1) (2005), 397–488.

[28] O. Klurman and A. P. Mangerel, 'Effective asymptotic formulae for multilinear averages of multiplicative functions'. Preprint, August 2017.

[29] K. Matomäki and M. Radziwiłł, 'Multiplicative functions in short intervals', *Ann. of Math. (2)* **183**(3) (2016), 1015–1056.

[30] K. Matomäki, M. Radziwiłł and T. Tao, 'An averaged form of Chowla's conjecture', *Algebra Number Theory* **9**(9) (2015), 2167–2196.

[31] K. Matomäki, M. Radziwiłł and T. Tao, 'Sign patterns of the Liouville and Möbius functions', *Forum Math. Sigma* **4**(e14) (2016), 44.

[32] K. Matomäki, M. Radziwiłł and T. Tao, 'Fourier uniformity of bounded multiplicative functions in short intervals on average', Preprint, 2018.

https://doi.org/10.1017/fms.2019.28 Published online by Cambridge University Press

[33] R. McNamara, 'Sarnak's conjecture for sequences of almost quadratic word growth', Preprint, 2019.
[34] J. M. Pollard, 'A generalisation of the theorem of Cauchy and Davenport', *J. Lond. Math. Soc. (2)* **8** (1974), 460–462.
[35] V. T. Sós, 'Turbulent years: Erdős in his correspondence with Turán from 1934 to 1940', in *Paul Erdős and His Mathematics, I (Budapest, 1999)*, Bolyai Soc. Math. Stud., 11 (János Bolyai Math. Soc., Budapest, 2002), 85–146.
[36] T. Tao, 'A variant of Kemperman's theorem', https://terrytao.wordpress.com/2011/12/26/.
[37] T. Tao, 'The logarithmically averaged Chowla and Elliott conjectures for two-point correlations', *Forum Math. Pi* **4**(e8) (2016), 36.
[38] T. Tao, 'Equivalence of the logarithmically averaged Chowla and Sarnak conjectures', in *Number Theory—Diophantine Problems, Uniform Distribution and Applications* (Springer, Cham, 2017).
[39] T. Tao, 'An inverse theorem for an inequality of Kneser', *Proc. Steklov Inst. Math.* **303**(1) (2018), 193–219. Published in Russian in Tr. Mat. Inst. Steklova **303** (2018) 209–238.
[40] T. Tao and J. Teräväinen, 'Odd order cases of the logarithmically averaged chowla conjecture', *J. Théor. Nombres Bordeaux* **30**(3) (2018), 997–1015.
[41] T. Tao and J. Teräväinen, 'The structure of logarithmically averaged correlations of multiplicative functions, with applications to the Chowla and Elliott conjectures', *Duke Math. J.* **168**(11) (2019), 1977–2027.
[42] J. Teräväinen, 'On binary correlations of multiplicative functions', *Forum Math. Sigma* **6**(e10) (2018), 41.
[43] Z. Wang, 'Sur les plus grands facteurs premiers d'entiers consécutifs', *Mathematika* **64**(2) (2018), 343–379.
[44] T. D. Wooley and T. D. Ziegler, 'Multiple recurrence and convergence along the primes', *Amer. J. Math.* **134**(6) (2012), 1705–1732.
[45] T. Ziegler, 'Universal characteristic factors and Furstenberg averages', *J. Amer. Math. Soc.* **20**(1) (2007), 53–97.