J. Austral. Math. Soc. Ser. B 36(1994), 60–63

ON TWO LEMMAS OF BROWN AND SHEPP HAVING APPLICATION TO SUM SETS AND FRACTALS

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(Received 3 January 1993; revised 20 September 1993)

Abstract

An improvement is made to two results of Brown and Shepp which are useful in calculations with fractal sets.

1. Introduction

Recently there has been a resurgence the study of sum sets. They have, *inter alia*, application to fractals, which can often be attractors or Markov attractors of iterated function systems (see the seminal paper of Barnsley and Demko [1]). Measure properties of sum sets are important in the study of dynamical systems (see, for example, Newhouse [5] and Palis and Takens [6]). The calculation of associated Hausdorff dimensions and Hausdorff measures and other properties can be delicate. In [3], G. Brown and L. Shepp provided two key lemmas which have proved valuable in making available a number of simple calculations in this area.

We say two positive numbers s and t are *conjugate* if $s^{-1} + t^{-1} = 1$. By $\|f\|_p$ we denote the L^p norm of a real-valued function f. Assuming the relevant quantities exist, the results of Brown and Shepp alluded to are as follows.

(i) *Suppose* $s_1 < s_0 < s_2$ *and let* s_i *be conjugate to* t_i $(i = 0, 1, 2)$. *Then*

$$\|f\|_{s_0} \|g\|_{t_0} \le \max [\|f\|_{s_1} \|g\|_{t_1}, \|f\|_{s_2} \|g\|_{t_2}].$$

(ii) *Suppose that, for* $i = 0, 1, 2$, *we have* $s_i, t_i \ge 1$ *and* $as_i^{-1} + bt_i^{-1} = 1$ *for positive constants* a, b. *If* $s_1 \le s_0 \le s_2$, *then*

$$M_{s_0}(x, u)M_{t_0}(y, v) \le \max_{i=1,2} M_{s_i}(x, u)M_{t_i}(y, v) \tag{1}$$

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https://doi.org/10.1017/S0334270000010249 Published online by Cambridge University Press

and, if further $a : b = \log n : \log m$, *then*

$$S_{s_0}(x)S_{t_0}(y) \le \max_{i=1,2} S_{s_i}(x)S_{t_i}(y), \tag{2}$$

where the mean of order t *is*

$$M_t(x,u) = \left[\sum_{i=1}^{n} u_i x_i^t\right]^{1/t}$$

and the sum of order t *is*

$$S_t(x) = \left[\sum_{i=1}^{n} x_i^t\right]^{1/t}.$$

It is implicit in these statements that $x = (x_i)_1^n$, $u = (u_i)_1^n$, $y = (y_i)_1^m$, $v = (v_i)_1^m$ have positive entries, that $\sum u_i = 1 = \sum v_i$ and that m may differ from n. Various applications of (ii) are given by Brown [2] and Brown and Shepp [3].

2. Results

We now proceed to some useful extensions of these results.

THEOREM 1. *Suppose* $s_1 \le s_0 \le s_2$ *with* $s_i \ge a$, $t_i \ge b$ *and* $as_i^{-1} + bt_i^{-1} = 1$ $(i = 0, 1, 2)$, $a, b > 0$. *Then assuming the relevant quantities exist,*

$$\|f\|_{s_0} \|g\|_{t_0} \le \max\,[\|f\|_{s_1} \|g\|_{t_1},\ \|f\|_{s_2} \|g\|_{t_2}].$$

PROOF. As in [3] choose α_1, α_2 positive and such that $\alpha_1 + \alpha_2 = 1$ and $s_0 = \alpha_1 s_1 + \alpha_2 s_2$. By the Hölder inequality

$$\|f\|_{s_0}^{s_0} = \|f\|_{\alpha_1 s_1 + \alpha_2 s_2}^{\alpha_1 s_1 + \alpha_2 s_2} \le \|f\|_{s_1}^{\alpha_1 s_1} \|f\|_{s_2}^{\alpha_2 s_2}. \tag{3}$$

If we choose

$$\beta_i = \alpha_i \frac{s_i}{s_0}\frac{t_0}{t_i} \quad (i = 1, 2)$$

then

$$\begin{aligned} \beta_1 + \beta_2 &= \left(\alpha_1 \frac{s_1}{t_1} + \alpha_2 \frac{s_2}{t_2}\right)\frac{t_0}{s_0} \\ &= \frac{b}{s_0 - a}\left(\alpha_1 \frac{s_1 - a}{b} + \alpha_2 \frac{s_2 - a}{b}\right) \\ &= \frac{\alpha_1 s_1 + \alpha_2 s_2 - a(\alpha_1 + \alpha_2)}{b} \frac{b}{s_0 - a} \\ &= 1 \end{aligned} \tag{4}$$

https://doi.org/10.1017/S0334270000010249 Published online by Cambridge University Press

and

$$\beta_1 t_1 + \beta_2 t_2 = \frac{\alpha_1 s_1 + \alpha_2 s_2}{s_0} t_0 = t_0, \tag{5}$$

so again by Hölder's inequality

$$\|g\|_{t_0}^{t_0} = \|g\|_{\beta_1 t_1 + \beta_2 t_2}^{\beta_1 t_1 + \beta_2 t_2} \le \|g\|_{t_1}^{\beta_1 t_1} \, \|g\|_{t_2}^{\beta_2 t_2}. \tag{6}$$

Since

$$\beta_i \frac{t_i}{t_0} = \frac{\alpha_i s_i}{s_0} \quad (i = 1, 2), \tag{7}$$

(3) and (6) may be combined to provide

$$\|f\|_{s_0} \, \|g\|_{t_0} \le \left[\|f\|_{s_1} \, \|g\|_{t_1}\right]^{\alpha_1 s_1/s_0} \left[\|f\|_{s_2} \, \|g\|_{t_2}\right]^{\alpha_2 s_2/s_0}. \tag{8}$$

As

$$\frac{\alpha_1 s_1}{s_0} + \frac{\alpha_2 s_2}{s_0} = 1,$$

the right-hand side of (8) is a weighted geometric mean and the result follows.

THEOREM 2. *Let* $x = (x_i)_1^n$, $u = (u_i)_1^n$, $y = (y_i)_1^m$, $v = (v_i)_1^m$ *be sequences of positive numbers and let* s_i, t_i $(i = 0, 1, 2)$ *satisfy the conditions of Theorem 1. Then*

$$S_n^{[s_0]}(x, u) S_m^{[t_0]}(y, v) \le \max_{i=1,2} S_n^{[s_i]}(x, u) S_m^{[t_i]}(y, v), \tag{9}$$

where

$$S_n^{[t]}(x, u) = \left[\sum_{i=1}^n u_i x_i^t\right]^{1/t}.$$

PROOF. We proceed as in Theorem 1 but use the discrete Hölder inequality. With α_1, α_2 as in Theorem 1 we have

$$\sum_{i=1}^n u_i x_i^{s_0} = \sum_{i=1}^n u_i x_i^{\alpha_1 s_1 + \alpha_2 s_2} \le \left[\sum_{i=1}^n u_i x_i^{s_1}\right]^{\alpha_1} \left[\sum_{i=1}^n u_i x_i^{s_2}\right]^{\alpha_2}$$

by the weighted form of the Hölder inequality [4, p. 136 Theorem 1(c)]. That is,

$$S_n^{[s_0]}(x, u)^{s_0} \le S_n^{[s_1]}(x, u)^{\alpha_1 s_1} S_n^{[s_2]}(x, u)^{\alpha_2 s_2}. \tag{10}$$

If β_1, β_2 are chosen as in Theorem 1, (4) and (5) hold as before and Hölder's inequality again gives

$$\sum_{i=1}^m v_i y_i^{t_0} = \sum_{i=1}^m v_i y_i^{\beta_1 t_1 + \beta_2 t_2} \le \left[\sum_{i=1}^m v_i y_i^{t_1}\right]^{\beta_1} \left[\sum_{i=1}^m v_i y_i^{t_2}\right]^{\beta_2},$$

so that

$$S_m^{[t_0]}(y,v)^{t_0} \le S_m^{[t_1]}(y,v)^{\beta_1 t_1} S_m^{[t_2]}(y,v)^{\beta_2 t_2}. \tag{11}$$

Combining (10) and (11) gives, *via* (7), that

$$S_n^{[s_0]}(x,u)S_m^{[t_0]}(y,v) \le \left[S_n^{[s_1]}(x,u)S_m^{[t_1]}(y,v)\right]^{\alpha_1 s_1/s_0} \left[S_n^{[s_2]}(x,u)S_m^{[t_2]}(y,v)\right]^{\alpha_2 s_2/s_0},$$

and the desired result follows as before.

REMARK 1. If $\sum_1^n u_i = 1 = \sum_1^m v_i$, we have (1) from (9) but with the wider supposition that $s_i \ge a$ and $t_i \ge b$ in place of $s_i \ge 1$, $t_i \ge 1$. Moreover, if $u_i = 1 \quad (i = 1, \dots, n)$, $v_i = 1 \quad (i = 1, \dots, m)$, then we have (2) without the requirement that $a : b = \log n : \log m$.

REMARK 2. The proof of Theorem 1 depends only on Hölder's inequality and certain convexity properties of the (real) exponents. Consequently the proof is valid in a general measure space. Indeed, f and g can even be taken from different measure spaces. From this viewpoint, Theorem 2 becomes a special case of the more general form of Theorem 1, since the expression $S_n^{[t]}(x,u)$ is then the usual norm $\|\cdot\|_t$ with respect to the obvious discrete measure.

Acknowledgement

The authors wish to thank the referee for helpful comments on an earlier version of this paper.

References

[1] M. F. Barnsley and S. Demko, "Iterated function systems and the global construction of fractals", *Proc. Roy. Soc. London* **A 399** (1985) 243—275.

[2] G. Brown, "Some inequalities that arise in measure theory", *J. Austral. Math. Soc. Ser. A* **45** (1988) 83–95.

[3] G. Brown and L. Shepp, "A convolution inequality", in *Contributions to Prob. and Stat. Essays in Honor of Ingram Olkin*, (Springer, New York, 1989) 51–57.

[4] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and their inequalities* (D. Reidel, Dordrecht, 1988).

[5] S. E. Newhouse, "The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms", *Publ. Inst. des Hautes Etudes Scient. Publ. Math.* **50** (1979) 101–151.

[6] J. Palis and F. Takens, "Cycles and measure of bifurcation sets for two-dimensional diffeomorphisms", *Invent. Math.* **82** (1985) 397–422.