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A DEGREE ONE BORSUK-ULAM THEOREM

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We generalise the Borsuk-Ulam theorem for maps $M^n \to \mathbb{R}^n$.

Everyone knows the Borsuk-Ulam theorem as a simple application of some of the first ideas one encounters in algebraic topology.

THEOREM 0.1. (Borsuk-Ulam) *Let $f : S^n \to \mathbb{R}^n$ be any continuous map. Then there are antipodal points in S^n which are mapped to the same point under f.*

The purpose of this brief note is to observe that there is an easy generalisation of this theorem for maps $f : M^n \to \mathbb{R}^n$ where M^n is a closed n-manifold.

THEOREM 0.2. *Let M be a closed n-manifold. Let $f : M \to \mathbb{R}^n$ be any continuous map and $g : M \to S^n$ a degree one map. Then there are points $p, q \in M$ such that $f(p) = f(q)$ and $g(p) = -g(q)$.*

PROOF: We wiggle g to be smooth and generic. By compactness of the space of antipodal points in S^n, it suffices to prove the theorem in this case, since then we can extract a subsequence of pairs of points in M with the desired properties for a sequence of degree one smooth maps $g_i : M \to S^n$ approximating g.

We define the following spaces

$$\widehat{M} \subset M \times M - \Delta = \big\{(p,q) : g(p) = -g(q)\big\}$$

$$S \subset S^n \times S^n - \Delta = \big\{(p,q) : p = -q\big\}$$

Observe that S is homeomorphic to S^n. There is an induced map $\widehat{g} : \widehat{M} \to S$ given by $\widehat{g} : (p,q) \to \big(g(p), g(q)\big)$. Since g was degree one, one easily observes that there are an odd number of points in the generic fibre of $\widehat{g}$ so that there is some connected component of $\widehat{M}$ for which the restricted map $\widehat{g}$ has odd degree. Moreover, the $\mathbb{Z}/2\mathbb{Z}$ action on $\widehat{M}$ and S given by interchanging the co-ordinates commutes with $\widehat{g}$, so there is an induced map on the quotients. We define $N = \widehat{M}/\!\sim$ and call the quotient map $h : N \to \mathbb{R}P^n$.

Assume on the contrary that points in M mapping to antipodal points in S^n map to distinct points in $\mathbb{R}^n$. Then there is a map

$$\widehat{f} : \widehat{M} \to S^{n-1}$$

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defined by

$$\hat{f} : (p,q) \to \frac{f(p) - f(q)}{\|f(p) - f(q)\|}.$$

It is obvious that this descends to a map $j : N \to \mathbb{R}P^{n-1}$ where $\mathbb{R}P^{n-1}$ is obtained from S^n by quotienting out by the antipodal map.

In the sequel, we consider homology and cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients. For simplicity of notation, we omit the coefficients.

Since the degree of h is odd, h^* pulls back the generator $[\mathbb{R}P^n]$ of $H^n(\mathbb{R}P^n)$ to the generator $[N]$ of $H^n(N)$. Furthermore, if α generates $H^1(\mathbb{R}P^n)$ then $h^*\alpha \in H^1(N)$ is an element whose nth power is $[N]$. Moreover by construction for every cycle $C \in H_1(N)$ we have $h_*C \neq 0$ in $H_1(\mathbb{R}P^n)$ if and only if $j_*C \neq 0$ in $H_1(\mathbb{R}P^{n-1})$, since these are exactly the C which do not lift to $\widehat{M}$.

It follows that if β denotes the generator of $H^1(\mathbb{R}P^{n-1})$ then $j^*\beta(C) = h^*\alpha(C)$ for all C, and therefore $j^*\beta = h^*\alpha$ so that the nth power of $j^*\beta$ is nontrivial. But $(j^*\beta)^n = j^*(\beta^n)$ which is trivial, giving us a contradiction. ∎

REMARK 0.1. Notice that the proof works in exactly the same way if $g : M \to S^n$ is a map of odd degree.

The following corollary led the author to observe the theorem above:

COROLLARY 0.3. *Let $M^n \subset \mathbb{R}^{n+1}$ be an embedded submanifold bounding a closed region which contains a ball of diameter t. Let $f : M^n \to \mathbb{R}^n$ be a continuous map. Then there are points in M at distance at least t apart from each other which have the same image under f.*

PROOF: Let g be the map which is radial projection of M onto the boundary of the ball of diameter t. ∎

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