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Symmetry Classification for Jackson Integrals Associated with Irreducible Reduced Root Systems

Dedicated to Professor Kazuhiko Aomoto on the occasion of his 60th birthday

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Abstract. We state certain product formulae for Jackson integrals associated with irreducible reduced root systems. The Jackson integral is defined here as a sum over any full-rank sublattice of the coweight lattice for the root system. In particular, a Weyl group symmetry classification of the Jackson integrals is done when they have an expression of a product of the Jacobi elliptic theta functions. Most of the product formulae investigated by Aomoto, Macdonald and Gustafson appear in the list of classifications. A new product formula for an F_4 root system is included in it.

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1. Introduction

There are a lot of generalizations of the Selberg integral

$$\int_{[0,1]^n}\prod_{i=1}^{n}x_i^{\alpha-1}(1-x_i)^{\beta-1}\prod_{1\leqslant i<j\leqslant n}|x_i-x_j|^{2\gamma}\,\mathrm{d}x_1\cdots\mathrm{d}x_n$$
$$=\prod_{j=1}^{n}\frac{\Gamma(j\gamma+1)\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+(n+j-2)\gamma)},$$

which was proved by Selberg in 1944, and they have been studied in various way. One of the extensions is the q-Selberg integral investigated by Andrews, Askey and many others [As, E, H, Kad, Kan]. In [Ao1], Aomoto extended the q-Selberg integral to a sum which has the symmetry of a Weyl group of irreducible reduced root systems. Using the Poincaré series for affine root systems, Macdonald [Ma3] showed the relation between Aomoto's sum and the q-Macdonald–Morris identity investigated by Cherednik [C1] and others. The product formula (see Proposition 4.4) investigated in [Ao1, Ito1, Ma3] was recently applied by van Diejen and Vinet [vDV] to an eigenvalue problem of the quantum Hamiltonian for the compactified trigonometric Ruijsenaars–Schneider model.

On the other hand, the formula

$$\sum_{\nu=-\infty}^{\infty}\frac{(1-aq^{2\nu})(b;q)_\nu(c;q)_\nu(d;q)_\nu(e;q)_\nu}{(1-a)(aq/b;q)_\nu(aq/c;q)_\nu(aq/d;q)_\nu(aq/e;q)_\nu}\left(\frac{a^2q}{bcde}\right)^\nu$$
$$=\frac{(q/a;q)_\infty(aq;q)_\infty(aq/bc;q)_\infty(aq/bd;q)_\infty}{(q/b;q)_\infty(q/c;q)_\infty(q/d;q)_\infty(q/e;q)_\infty}\times$$
$$\times\frac{(aq/be;q)_\infty(aq/cd;q)_\infty(aq/ce;q)_\infty(aq/de;q)_\infty(q;q)_\infty}{(aq/b;q)_\infty(aq/c;q)_\infty(aq/d;q)_\infty(aq/e;q)_\infty(a^2q/bcde;q)_\infty}$$

was proved by Bailey in 1936 and is called Bailey's very-well-poised ${}_6\psi_6$ summation formula. This formula and the q-Selberg integral can be regarded as a q-series of the hypergeometric type expressed as a product of q-gamma functions. Gustafson [Gu4] established a multidimensional generalization of a ${}_6\psi_6$ summation formula corresponding to semi-simple Lie algebras. By using Gustafson's C_n-type sum, van Diejen [vD] proved a summation formula for his BC_n-type sum, which includes Aomoto's B_n and C_n-type sums as special cases.

In this paper we define certain sums which have the symmetry of a Weyl group for the irreducible reduced root system R. We call them Jackson integrals associated with R. The main results of this paper are Theorems 4.5 and 4.10, which classify them when they are expressed as a product of the Jacobi elliptic theta functions. Aomoto's sums and Gustafson's B_n and G_2-type sums [Gu1] are included in the classification list in Theorem 4.5. One advantage of this list is to be able to find a new product formula for F_4-type [Ito3] which seems not to be known yet. The sums not appearing in it are Gustafson's A_n, C_n, D_n-type sums, and van Diejen's BC_n-type sum. But these sums, except for Gustafson's A_n-type sum, are included in the classification list for the Jackson integral associated with a nonreduced root system (BC_n-type root system). (See the list for the BC_n-type case in a sequel [Ito4] to this paper.) Thus, essentially the sum not belonging to our lists is Gustafson's A_n-type sum.

Throughout this paper, we use the notation

$$(a;q)_\infty:=\prod_{i=0}^{\infty}(1-aq^i)\quad\text{and}\quad\vartheta(\xi;q):=(\xi;q)_\infty(q/\xi;q)_\infty(q;q)_\infty.$$

2. Definition of Jackson Integral

Let R be an irreducible reduced root system, spanning a real vector space E of dimension n, and let $\langle\cdot,\cdot\rangle$ be a positive definite scalar product on E under the Weyl group W of R. We denote by R^+ the set of positive roots relative to a fixed basis $\{\alpha_1,\dots,\alpha_n\}$ of R. For each $\alpha\in R$, let $\alpha^\vee=2\alpha/\langle\alpha,\alpha\rangle$. Let P be the *coweight lattice* $\{\chi\in E;\ \langle\alpha,\chi\rangle\in\mathbf{Z}$ for any $\alpha\in R\}$ and let Q be the *coroot lattice* of R defined by

$Q = \mathbf{Z}\alpha_1^\vee + \cdots + \mathbf{Z}\alpha_n^\vee \subset P$. Let L be any sublattice of P of rank n. We assume L is W-stable, i.e. $L = wL$ for $w \in W$. The scalar product $\langle \cdot, \cdot \rangle$ is uniquely extended linearly to $E_{\mathbf{C}} = E \otimes_{\mathbf{R}} \mathbf{C} \simeq \mathbf{C}^n$. Let q be a real number such that $0 < q < 1$. For $x \in E_{\mathbf{C}}$, we define

$$\begin{aligned}
&\Phi_R(b_1, \cdots, b_s, c_1, \cdots, c_l; x)\\
&\quad = \Phi_R(\{b_i\}, \{c_j\}; x)\\
&\quad := \prod_{i=1}^{s} \prod_{\substack{\alpha>0\\ \alpha:\,\text{short}}} q^{(1/2-b_i)\langle\alpha,x\rangle} \frac{(q^{1-b_i+\langle\alpha,x\rangle}; q)_\infty}{(q^{b_i+\langle\alpha,x\rangle}; q)_\infty} \times\\
&\qquad \times \prod_{j=1}^{l} \prod_{\substack{\alpha>0\\ \alpha:\,\text{long}}} q^{(1/2-c_j)\langle\alpha,x\rangle} \frac{(q^{1-c_j+\langle\alpha,x\rangle}; q)_\infty}{(q^{c_j+\langle\alpha,x\rangle}; q)_\infty},
\end{aligned}$$

where $s, l \in \mathbf{Z}_{\geqslant 0}$, $b_i, c_j \in \mathbf{C}$ and $\alpha > 0$ means $\alpha \in R^+$. We denote by $\Delta_R(x)$ the Weyl denominator as

$$\Delta_R(x) := \prod_{\alpha>0} (q^{\langle\alpha,x\rangle/2} - q^{-\langle\alpha,x\rangle/2}).$$

For $w \in W$, we define $wF(x) := F(w^{-1}x)$ for a function $F(x)$ of $x \in E_{\mathbf{C}}$. The function $\Phi_R(\{b_i\}, \{c_j\}; x)$ is *quasi-symmetric* with respect to W:

$$w\Phi_R(\{b_i\}, \{c_j\}; x) = U_w(x) \quad \Phi_R(\{b_i\}, \{c_j\}; x), \quad w \in W, \tag{1}$$

where $U_w(x)$ is a *pseudo-constant*, i.e. an invariant under the shift $x \to x + \chi$ for $\chi \in P$, as

$$\begin{aligned}
U_w(x) = &\prod_{i=1}^{s} \prod_{\substack{\alpha>0\\ w^{-1}\alpha<0\\ \alpha:\,\text{short}}} q^{(2b_i-1)\langle\alpha,x\rangle} \frac{\vartheta(q^{b_i+\langle\alpha,x\rangle}; q)}{\vartheta(q^{1-b_i+\langle\alpha,x\rangle}; q)} \times\\
&\times \prod_{j=1}^{l} \prod_{\substack{\alpha>0\\ w^{-1}\alpha<0\\ \alpha:\,\text{long}}} q^{(2c_j-1)\langle\alpha,x\rangle} \frac{\vartheta(q^{c_j+\langle\alpha,x\rangle}; q)}{\vartheta(q^{1-c_j+\langle\alpha,x\rangle}; q)}.
\end{aligned}$$

The Weyl denominator $\Delta_R(x)$ changes by the action of W as

$$w\Delta_R(x) = \operatorname{sgn} w \, \Delta_R(x). \tag{2}$$

For $z \in E_{\mathbf{C}}$, we define the *Jackson integral associated with* R as

$$J_R(\{b_i\}, \{c_j\}; L; z) := \sum_{\chi \in L} \Phi_R(\{b_i\}, \{c_j\}; z + \chi) \, \Delta_R(z + \chi). \tag{3}$$

By definition, the Jackson integral $J_R(\{b_i\}, \{c_j\}; L; z)$ is obviously invariant under the shift $z \to z + \chi$ for $\chi \in L$:

$$J_R(\{b_i\}, \{c_j\}; L; z + \chi) = J_R(\{b_i\}, \{c_j\}; L; z). \tag{4}$$

Remark. If all roots in R have the same length, we regard the roots as all short, so that

$$\Phi_R(\{b_i\}; x) = \prod_{i=1}^{s} \prod_{\alpha>0} q^{(1/2-b_i)\langle\alpha,x\rangle} \frac{(q^{1-b_i+\langle\alpha,x\rangle}; q)_\infty}{(q^{b_i+\langle\alpha,x\rangle}; q)_\infty}.$$

LEMMA 2.1. *The following holds for* $w \in W$:

$$wJ_R(\{b_i\}, \{c_j\}; L; z) = \mathrm{sgn}w \quad U_w(z) \quad J_R(\{b_i\}, \{c_j\}; L; z).$$

Proof. From Definition (3), we have

$$wJ_R(\{b_i\}, \{c_j\}; L; z) = \sum_{\chi\in L} \Phi_R(\{b_i\}, \{c_j\}; w^{-1}z + \chi)\ \Delta_R(w^{-1}z + \chi), \tag{5}$$

and, since L is W-stable, we have

$$\begin{aligned} &wJ_R(\{b_i\}, \{c_j\}; L; z) \\ &\quad = \sum_{\chi\in L} w\Phi_R(\{b_i\}, \{c_j\}; z + w\chi)\ w\Delta_R(z + w\chi) \\ &\quad = \sum_{\chi\in L} w\Phi_R(\{b_i\}, \{c_j\}; z + \chi)\ w\Delta_R(z + \chi). \end{aligned}$$

Hence, from (1) and (2), we have Lemma 2.1. □

3. Examples

In this section, in our setting, we state some sums which are already known. Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be the standard basis of $\mathbf{R}^n$ satisfying $\langle\varepsilon_i, \varepsilon_j\rangle = \delta_{ij}$ for all $i, j = 1, \dots, n$, where δ_{ij} is the Kronecker delta.

3.1. A_n-type

Basis: $\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_n = \varepsilon_n - \varepsilon_{n+1}$,
Positive roots: $\varepsilon_i - \varepsilon_j = \sum_{i \leqslant k<j} \alpha_k \quad (1 \leqslant i < j \leqslant n+1)$,
Coweight lattice: $P = \mathbf{Z}\chi_1 + \mathbf{Z}\chi_2 + \mathbf{Z}\chi_3 + \cdots + \mathbf{Z}\chi_n, \quad \langle\alpha_i, \chi_j\rangle = \delta_{ij}$.
Coroot lattice: $Q = \mathbf{Z}\alpha_1 + \mathbf{Z}\alpha_2 + \mathbf{Z}\alpha_3 + \cdots + \mathbf{Z}\alpha_n$,

$$\left(\langle\alpha_i, \alpha_j\rangle\right)_{i,j=1}^{n} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

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Sum	Type
$J_{A_n}(b_1; P; z)$ $J_{A_n}(b_1; Q; z)$	Aomoto's A_n-type [Ito1, p. 132]

3.2. B_n-type

Basis: $\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n,\ \alpha_n = \varepsilon_n,$
Positive short roots: ε_i $(1 \leqslant i \leqslant n)$,
Positive long roots: $\varepsilon_i \pm \varepsilon_j$ $(1 \leqslant i < j \leqslant n)$,
Coweight lattice:

$$\begin{aligned} P &= \mathbf{Z}\varepsilon_1 + \mathbf{Z}(\varepsilon_1 + \varepsilon_2) + \cdots + \mathbf{Z}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n) \\ &= \mathbf{Z}\varepsilon_1 + \mathbf{Z}\varepsilon_2 + \mathbf{Z}\varepsilon_3 + \cdots + \mathbf{Z}\varepsilon_n. \end{aligned}$$

Sum	Type
$J_{B_n}(b_1, c_1; P; z)$	Aomoto's B_n-type [Ao1, Ito1]
$J_{B_n}(\{b_i\}_{i=1}^{2n-1}; P; z)$	Gustafson's B_n-type [Gu1]

3.3. C_n-type

Basis: $\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n,\ \alpha_n = 2\varepsilon_n,$
Positive short roots: $\varepsilon_i \pm \varepsilon_j$ $(1 \leqslant i < j \leqslant n)$,
Positive long roots: $2\varepsilon_i$ $(1 \leqslant i \leqslant n)$,
Coweight lattice:

$$\begin{aligned} P &= \mathbf{Z}\varepsilon_1 + \mathbf{Z}(\varepsilon_1 + \varepsilon_2) + \cdots + \mathbf{Z}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{n-1}) + \\ &\qquad + \mathbf{Z}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \cdots + \varepsilon_n)/2 \\ &= \mathbf{Z}\varepsilon_1 + \mathbf{Z}\varepsilon_2 + \cdots + \mathbf{Z}\varepsilon_{n-1} + \mathbf{Z}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \cdots + \varepsilon_n)/2, \end{aligned}$$

Coroot lattice:

$$\begin{aligned} Q &= \mathbf{Z}(\varepsilon_1 - \varepsilon_2) + \mathbf{Z}(\varepsilon_2 - \varepsilon_3) + \cdots + \mathbf{Z}(\varepsilon_{n-1} - \varepsilon_n) + \mathbf{Z}\varepsilon_n \\ &= \mathbf{Z}\varepsilon_1 + \mathbf{Z}\varepsilon_2 + \mathbf{Z}\varepsilon_3 + \cdots + \mathbf{Z}\varepsilon_n. \end{aligned}$$

Sum	Type
$J_{C_n}(b_1, c_1; P; z)$ $J_{C_n}(b_1, c_1; Q; z)$	Aomoto's C_n-type [Ao1, p. 122 (3.5)]

3.4. D_n-type

Basis: $\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots\ \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n,\ \alpha_n = \varepsilon_{n-1} + \varepsilon_n,$
Positive roots: $\varepsilon_i \pm \varepsilon_j$ $(1 \leqslant i < j \leqslant n)$,

Coweight lattice:

$$\begin{aligned}P &= \mathbf{Z}\varepsilon_1 + \mathbf{Z}(\varepsilon_1+\varepsilon_2) + \cdots + \mathbf{Z}(\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_{n-2}) + \\ &\quad + \mathbf{Z}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\cdots+\varepsilon_{n-1}-\varepsilon_n)/2 + \\ &\quad + \mathbf{Z}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\cdots+\varepsilon_{n-1}+\varepsilon_n)/2 \\ &= \mathbf{Z}\varepsilon_1 + \mathbf{Z}\varepsilon_2 + \cdots + \mathbf{Z}\varepsilon_{n-1} + \mathbf{Z}(\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_n)/2,\end{aligned}$$

Coroot lattice:

$$Q = \mathbf{Z}(\varepsilon_1-\varepsilon_2) + \mathbf{Z}(\varepsilon_2-\varepsilon_3) + \cdots + \mathbf{Z}(\varepsilon_{n-1}-\varepsilon_n) + \mathbf{Z}(\varepsilon_{n-1}+\varepsilon_n).$$

Sum	Type
$J_{D_n}(b_1; P; z)$	Aomoto's D_n-type [Ao1, p. 122 (3.6)]
$J_{D_n}(b_1; Q; z)$	

3.5 G_2-type

Basis: $\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$,
Positive short roots: $\alpha_1,\ \alpha_1+\alpha_2,\ 2\alpha_1+\alpha_2$,
Positive long roots: $\alpha_2,\ 3\alpha_1+\alpha_2,\ 3\alpha_1+2\alpha_2$,
Coweight lattice: $P = Q = \mathbf{Z}\chi_1 + \mathbf{Z}\chi_2,\quad \langle\alpha_i, \chi_j\rangle = \delta_{ij}$.

Sum	Type
$J_{G_2}(b_1, c_1; P; z)$	Amomoto's G_2-type [Ito1, p.152]
$J_{G_2}(\{b_i\}_{i=1}^4; P; z)$	Gustafson's G_2-type [Gu1, p. 103 (8.12), Ito2]

3.6 F_4-type

Since the root systems F_4 and $F_4^\vee$ are isomorphic with orthogonal transformation [Ga1, p. 806], we take a basis of $F_4^\vee$.
Basis: $\alpha_1 = \varepsilon_2 - \varepsilon_3,\ \alpha_2 = \varepsilon_3 - \varepsilon_4,\ \alpha_3 = 2\varepsilon_4,\ \alpha_4 = \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4$,
Positive short roots: $\varepsilon_i \pm \varepsilon_j\ (1 \leqslant i < j \leqslant 4)$,
Positive long roots: $2\varepsilon_i\ (1 \leqslant i \leqslant 4),\ \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4$,
Coweight lattice:

$$\begin{aligned}P = Q &= \mathbf{Z}(\varepsilon_2-\varepsilon_3) + \mathbf{Z}(\varepsilon_3-\varepsilon_4) + \mathbf{Z}\varepsilon_4 + \mathbf{Z}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)/2 \\ &= \mathbf{Z}\varepsilon_1 + \mathbf{Z}\varepsilon_2 + \mathbf{Z}\varepsilon_3 + \mathbf{Z}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)/2.\end{aligned}$$

Sum	Type
$J_{F_4}(b_1, c_1; P; z)$	Aomoto's F_4-type [Ao1]
$J_{F_4}(\{b_i\}_{i=1}^3; P; z)$	[Ito3]

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4. Product Formula

In this section, we discuss the sum $J_R(\{b_i\},\{c_j\};L;z)$ which can be expressed as a product of the *Jacobi elliptic theta function* $\vartheta(\xi;q)$. The theta function $\vartheta(\xi;q)$ has a quasi-periodic property such as $\vartheta(q\xi;q)=-\vartheta(\xi;q)/\xi$. By using this property, for $\chi\in L$, we have

$$\begin{aligned}&\vartheta(q^{c+\langle\alpha,z+\chi\rangle};q)\\&\quad=(-1)^{\langle\alpha,\chi\rangle}q^{(\frac{1}{2}-c)\langle\alpha,\chi\rangle-\frac{1}{2}\langle\alpha,\chi\rangle^2-\langle\alpha,z\rangle\langle\alpha,\chi\rangle}\vartheta(q^{c+\langle\alpha,z\rangle};q),\end{aligned} \tag{6}$$

which is used in the succeeding discussion.

LEMMA 4.1. *For $\alpha\in R^+$, if $\langle\alpha,z\rangle=0$, then $J_R(\{b_i\},\{c_j\};L;z+\chi)=0$ for all $\chi\in L$.*

Proof. First we consider $J_R(\{b_i\},\{c_j\};L;z)=0$ if $\langle\alpha,z\rangle=0$. We denote by w_α the orthogonal reflection with respect to the hyperplane H_α perpendicular to $\alpha\in R$, i.e.

$$w_\alpha(x):=x-2\frac{\langle\alpha,x\rangle}{\langle\alpha,\alpha\rangle}\alpha\quad\text{for } x\in E_{\mathbf{C}},\quad\text{and}\quad H_\alpha:=\{x\in E_{\mathbf{C}}|\langle\alpha,x\rangle=0\}.$$

We denote by S_α the set $\{\beta\in R|\beta>0 \text{ and } w_\alpha(\beta)<0\}$. For simplicity, we abbreviate $J_R(\{b_i\},\{c_j\};L;z)$ and $\Phi_R(\{b_i\},\{c_j\};x)$ by $J_R(L;z)$ and $\Phi_R(x)$ respectively. If $\langle\alpha,z\rangle=0$, we have

$$q^{(2c-1)\langle\alpha,z\rangle}\frac{\vartheta(q^{c+\langle\alpha,z\rangle};q)}{\vartheta(q^{1-c+\langle\alpha,z\rangle};q)}=1 \tag{7}$$

because of the property $\vartheta(\xi;q)=\vartheta(q/\xi;q)$. Since $z\in H_\alpha$, it follows that

$$\langle w_\alpha(\beta),z\rangle=\langle\beta,z\rangle\quad\text{for}\quad\beta\in R, \tag{8}$$

so that, for $\beta\in R$, we have

$$q^{(2c-1)\langle\beta,z\rangle}\frac{\vartheta(q^{c+\langle\beta,z\rangle};q)}{\vartheta(q^{1-c+\langle\beta,z\rangle};q)}q^{(2c-1)\langle-w_\alpha(\beta),z\rangle}\frac{\vartheta(q^{c+\langle-w_\alpha(\beta),z\rangle};q)}{\vartheta(q^{1-c+\langle-w_\alpha(\beta),z\rangle};q)}=1, \tag{9}$$

because $\vartheta(\xi;q)=\vartheta(q/\xi;q)$. If $\beta\in S_\alpha$, then $-w_\alpha(\beta)\in S_\alpha$. When $\langle\alpha,z\rangle=0$, using the relations (7) and (9) for the definition of $U_{w_\alpha}(z)$, we have

$$U_{w_\alpha}(z)=1. \tag{10}$$

By using (10), Lemma 2.1, and $\operatorname{sgn} w_\alpha=-1$, we have

$$w_\alpha J_R(L;z)=-J_R(L;z). \tag{11}$$

On the other hand, for $\beta\in R$, by Equation (8), we have

$$\langle\beta,w_\alpha z+\chi\rangle=\langle w_\alpha(\beta),z\rangle+\langle\beta,\chi\rangle=\langle\beta,z\rangle+\langle\beta,\chi\rangle=\langle\beta,z+\chi\rangle,$$

so that

$$\Phi_R(w_\alpha z+\chi)=\Phi_R(z+\chi)\quad\text{and}\quad\Delta_R(w_\alpha z+\chi)=\Delta_R(z+\chi). \tag{12}$$

By Equations (12) and (5), we have

$$w_\alpha J_R(L;z) = J_R(L;z). \tag{13}$$

Hence, from (11) and (13), it follows that

$$J_R(L;z) = 0.$$

By using (4), it is clear that $J_R(L; z+\chi) = 0$ for $\chi \in L$. □

LEMMA 4.2. *Assume that* $L = P$ *or* Q. *For* $\alpha \in R^+$, *if* $\langle \alpha, z\rangle = 0$, *then* $J_R(\{b_i\},\{c_j\}; L; z+\chi) = 0$ *for all* $\chi \in P$.

Proof. If $L = P$, it is straightforward from Lemma 4.1. We assume that $L = Q$. From Lemma 2.1, it follows that

$$\begin{aligned} &w_\alpha J_R(Q; z+\chi) \\ &\quad = -U_{w_\alpha}(z+\chi) J_R(Q; z+\chi) = -U_{w_\alpha}(z) J_R(Q; z+\chi). \end{aligned} \tag{14}$$

If $\langle \alpha, z\rangle = 0$, by using (10) and (14), we have

$$w_\alpha J_R(Q; z+\chi) = -J_R(Q; z+\chi). \tag{15}$$

On the other hand, since $w_\alpha(\chi) \in \chi + Q$ for $\alpha \in R$, we have

$$J_R(Q; z + w_\alpha(\chi)) = J_R(Q; z+\chi) \quad \text{for all} \quad \chi \in P,$$

so that,

$$w_\alpha J_R(Q; z+\chi) = J_R(Q; w_\alpha z + w_\alpha(\chi)) = J_R(Q; w_\alpha z + \chi). \tag{16}$$

If $\langle \alpha, z\rangle = 0$, from (12) and (16), we have

$$w_\alpha J_R(Q; z+\chi) = J_R(Q; z+\chi). \tag{17}$$

Hence, from (15) and (17), it follows that $J_R(Q; z+\chi) = 0$. □

PROPOSITION 4.3. *For* $L = P$ *or* Q, *the sum* $J_R(\{b_i\},\{c_j\}; L; z)$ *is expressed as*

$$f(z) \prod_{\substack{\alpha>0 \\ \alpha:\,\text{hort}}} \frac{q^{\left(s-1/2-\sum_{i=1}^{s} b_i\right)\langle\alpha,z\rangle} \vartheta(q^{\langle\alpha,z\rangle}; q)}{\prod_{i=1}^{s} \vartheta(q^{b_i+\langle\alpha,z\rangle}; q)} \times$$
$$\times \prod_{\substack{\alpha>0 \\ \alpha:\,\text{long}}} \frac{q^{\left(l-1/2-\sum_{j=1}^{l} c_j\right)\langle\alpha,z\rangle} \vartheta(q^{\langle\alpha,z\rangle}; q)}{\prod_{j=1}^{l} \vartheta(q^{c_j+\langle\alpha,z\rangle}; q)}$$

where $f(z)$ *is a holomorphic function of* $z \in E_{\mathbf{C}}$.

Proof. We set $\zeta^\alpha := q^{\langle\alpha,z\rangle}$. By definition, $J_R(\{b_i\},\{c_j\}; L; z)$ can be regarded as a function of $\zeta = (\zeta^{\alpha_1}, \ldots, \zeta^{\alpha_n}) \in (\mathbf{C}^*)^n$, and we denote by $J_R(\zeta)$ the sum

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$J_R(\{b_i\}, \{c_j\}; L; z)$. Since $J_R(\zeta)$ has poles lying in the set

$$\left\{\zeta \in (\mathbf{C}^*)^n \; ; \; \prod_{i=1}^{s} \prod_{\substack{\alpha>0\\ \alpha:\,\text{short}}} \vartheta(q^{b_i}\zeta^\alpha; q) \prod_{j=1}^{l} \prod_{\substack{\alpha>0\\ \alpha:\,\text{long}}} \vartheta(q^{c_j}\zeta^\alpha; q) = 0\right\},$$

the sum $J_R(\zeta)$ is written as

$$g(\zeta)\frac{\prod\limits_{\substack{\alpha>0\\ \alpha:\,\text{short}}} \zeta^{\alpha\left(s-1/2-\sum_{i=1}^{s} b_i\right)} \cdot \prod\limits_{\substack{\alpha>0\\ \alpha:\,\text{long}}} \zeta^{\alpha\left(l-1/2-\sum_{j=1}^{l} c_j\right)}}{\prod\limits_{i=1}^{s} \prod\limits_{\substack{\alpha>0\\ \alpha:\,\text{short}}} \vartheta(q^{b_i}\zeta^\alpha; q) \cdot \prod\limits_{j=1}^{l} \prod\limits_{\substack{\alpha>0\\ \alpha:\,\text{long}}} \vartheta(q^{c_j}\zeta^\alpha; q)},$$

where $g(\zeta)$ is a holomorphic function of $\zeta \in (\mathbf{C}^*)^n$. By Lemma 4.2, we have $J_R(\zeta) = 0$ if $\zeta^\alpha = 1, q^{\pm 1}, q^{\pm 2}, \ldots$, so that $g(\zeta) = 0$ if $\zeta^\alpha = 1, q^{\pm 1}, q^{\pm 2}, \ldots$. Therefore the function $g(\zeta)$ is divided out by the product $\prod_{\alpha>0} \vartheta(\zeta^\alpha; q)$. □

PROPOSITION 4.4 (Aomoto). *For $L = P$ or Q, the sum $J_R(b_1, c_1; L; z)$ is expressed as*

$$C_R(b_1, c_1; L) \prod_{\substack{\alpha>0\\ \alpha:\,\text{short}}} \frac{q^{-b_1\langle\alpha,z\rangle}\vartheta(q^{\langle\alpha,z\rangle}; q)}{\vartheta(q^{b_1+\langle\alpha,z\rangle}; q)} \prod_{\substack{\alpha>0\\ \alpha:\,\text{long}}} \frac{q^{-c_1\langle\alpha,z\rangle}\vartheta(q^{\langle\alpha,z\rangle}; q)}{\vartheta(q^{c_1+\langle\alpha,z\rangle}; q)}$$

where $C_R(b_1, c_1; L)$ is a constant not depending on $z \in E_{\mathbf{C}}$.

Proof. See [Ao1]. □

Remark 4.4.1. In [Ito1, Ma3], an explicit form of the constant $C_R(b_1, c_1; L)$ was obtained when $L = P$ or Q and it is written as a product of q-gamma functions. The constant $C_R(b_1, c_1; Q)$ is expressed as

$$C_R(b_1, c_1; Q) = \prod_{\substack{\alpha>0\\ \alpha:\,\text{short}}} \frac{(q^{1-\langle\rho_k,\alpha^\vee\rangle-b_1}; q)_\infty (q^{\delta_\alpha-\langle\rho_k,\alpha^\vee\rangle+b_1}; q)_\infty}{(q^{1-\langle\rho_k,\alpha^\vee\rangle}; q)_\infty (q^{-\langle\rho_k,\alpha^\vee\rangle}; q)_\infty} \times$$
$$\times \prod_{\substack{\alpha>0\\ \alpha:\,\text{long}}} \frac{(q^{1-\langle\rho_k,\alpha^\vee\rangle-c_1}; q)_\infty (q^{\delta_\alpha-\langle\rho_k,\alpha^\vee\rangle+c_1}; q)_\infty}{(q^{1-\langle\rho_k,\alpha^\vee\rangle}; q)_\infty (q^{-\langle\rho_k,\alpha^\vee\rangle}; q)_\infty},$$

where

$$2\rho_k := b_1 \sum_{\substack{\alpha>0\\ \alpha:\,\text{short}}} \alpha + c_1 \sum_{\substack{\alpha>0\\ \alpha:\,\text{long}}} \alpha,$$

$\delta_\alpha = 1$ if $\langle\rho_k, \alpha^\vee\rangle = b_1$ or c_1, and $\delta_\alpha = 0$ otherwise. And the constant $C_R(b_1, c_1; P)$ is expressed as

$$C_R(b_1, c_1; P) = |P/Q| \quad C_R(b_1, c_1; Q),$$

where $|P/Q|$ is the order of the fundamental group P/Q of R, so that

$$\begin{array}{c||c|c|c|c|c} R & A_n & B_n, C_n, E_7 & D_n & E_6 & G_2, F_4, E_8 \\ \hline |P/Q| & n+1 & 2 & 4 & 3 & 1 \end{array}. \tag{18}$$

First, we consider the case where the holomorphic function $f(z)$ in Proposition 4.3 is a constant not depending on z. As we see in Proposition 4.4, when $(s, l) = (1, 1)$, the function $f(z)$ is obviously a constant. In the following theorem, we see other possible (s, l) for $J_R(\{b_i\}, \{c_j\}; L; z)$ when the function $f(z)$ is a constant:

THEOREM 4.5. *For $L = P$ or Q, the sum $J_R(\{b_i\}, \{c_j\}; L; z)$ is expressed as*

$$C_R(\{b_i\}, \{c_j\}; L) \prod_{\substack{\alpha>0 \\ \alpha:\text{short}}} \frac{q^{\left(s-1/2-\sum_{i=1}^{s} b_i\right)\langle\alpha,z\rangle} \vartheta(q^{\langle\alpha,z\rangle}; q)}{\prod_{i=1}^{s} \vartheta(q^{b_i+\langle\alpha,z\rangle}; q)} \times$$
$$\times \prod_{\substack{\alpha>0 \\ \alpha:\text{long}}} \frac{q^{\left(l-1/2-\sum_{j=1}^{l} c_j\right)\langle\alpha,z\rangle} \vartheta(q^{\langle\alpha,z\rangle}; q)}{\prod_{j=1}^{l} \vartheta(q^{c_j+\langle\alpha,z\rangle}; q)},$$

where $C_R(\{b_i\}, \{c_j\}; L)$ is a constant not depending on $z \in E_{\mathbf{C}}$, if and only if

$s = 1$ for A_n, D_n, E_6, E_7 and E_8-type,
$(s, l) = (1, 1)$ *or* $(2n-1, 0)$ *for B_n-type,*
$(s, l) = (1, 1)$ *or* $(0, (n+1)/2)$ *for C_n-type if n is odd,*
$(s, l) = (1, 1)$ *or* $(4, 0)$ *for G_2-type,*
$(s, l) = (1, 1)$ *or* $(3, 0)$ *for F_4-type.*

Remark 4.5.1. The cases $(s, l) = (2n-1, 0)$ for B_n-type and $(s, l) = (4, 0)$ for G_2-type were investigated by Gustafson and explicit forms of the constants $C_R(\{b_i\}; P)$ of them are known (see [Gu1, Ito2]). For the case $(s, l) = (3, 0)$ for F_4-type and its constant $C_{F_4}(\{b_i\}; P)$, see [Ito2, Ito3].

Before proving Theorem 4.5, we show two lemmas. We define positive definite integral symmetric matrices $A_L = (a_{ij})_{i,j=1}^n$ and $B_L = (b_{ij})_{i,j=1}^n$ as

$$a_{ij} = \sum_{\substack{\alpha>0 \\ \alpha:\text{short}}} \langle\alpha, \chi_i\rangle\langle\alpha, \chi_j\rangle, \qquad b_{ij} = \sum_{\substack{\alpha>0 \\ \alpha:\text{long}}} \langle\alpha, \chi_i\rangle\langle\alpha, \chi_j\rangle$$

for a basis $\{\chi_1, \dots, \chi_n\}$ of a sublattice L.

LEMMA 4.6. *If $R = B_n, C_n, G_2$ and F_4, for any sublattice L, the relation between A_L and B_L are the following:*

$$\begin{aligned} B_L &= (2n-2)A_L \quad \text{for } B_n,\\ 2A_L &= (n-1)B_L \quad \text{for } C_n,\\ B_L &= 3A_L \quad \text{for } G_2,\\ B_L &= 2A_L \quad \text{for } F_4. \end{aligned}$$

Proof. We denote by $\{\chi'_1, \ldots, \chi'_n\}$ a basis of the coweight lattice P. For a basis $\{\chi_1, \ldots, \chi_n\}$ of a sublattice $L \subset P$, we write $(\chi_1, \ldots, \chi_n) = (\chi'_1, \ldots, \chi'_n)C$, where C is a matrix such as $|\det C| \geqslant 1$. By definition of A_L and B_L, we have $A_L = CA_P{}^tC$ and $B_L = CB_P{}^tC$. Therefore, for each R, it is enough to check the relation in Lemma 4.6 for an L which is easy to calculate. We can easily check it and this is left to the reader. □

LEMMA 4.7. *Let L be a sublattice of the coweight lattice P of any irreducible root system R. For any R, we have that* $\det A_L > 1$ *except for the case $L = P$ for B_n, and* $\det A_P = 1$ *only for B_n.*

Proof. Since $\det A_L = (\det C)^2 \det A_P \geqslant \det A_P$, it is enough to show that $\det A_P > 1$ except for B_n and this is left to the reader. □

Proof of Theorem 4.5. By the q-periodicity (4) of $J_R(\{b_i\}, \{c_j\}; L; z)$ and (6), for $\chi \in L$, the function $f(z)$ in Proposition 4.3 satisfies

$$f\left(z + \frac{2\pi\sqrt{-1}}{\log q}\chi\right) = f(z) \tag{19}$$

and

$$f(z + \chi) = V_\chi(z) f(z), \tag{20}$$

where

$$\begin{aligned} V_\chi(z) = &\prod_{\substack{\alpha>0\\ \alpha:\,\text{short}}} (-1)^{(s-1)\langle\alpha,\chi\rangle} q^{(s-1)(-\frac{1}{2}\langle\alpha,\chi\rangle^2 - \langle\alpha,z\rangle\langle\alpha,\chi\rangle)} \times \\ &\times \prod_{\substack{\alpha>0\\ \alpha:\,\text{long}}} (-1)^{(l-1)\langle\alpha,\chi\rangle} q^{(l-1)(-\frac{1}{2}\langle\alpha,\chi\rangle^2 - \langle\alpha,z\rangle\langle\alpha,\chi\rangle)}. \end{aligned} \tag{21}$$

We denote by $M = \left(m_{ij}\right)_{i,j=1}^n$ the positive definite integral symmetric matrix such that

$$m_{ij} = (s-1) \sum_{\substack{\alpha>0\\ \alpha:\,\text{short}}} \langle\alpha, \chi_i\rangle\langle\alpha, \chi_j\rangle + (l-1) \sum_{\substack{\alpha>0\\ \alpha:\,\text{long}}} \langle\alpha, \chi_i\rangle\langle\alpha, \chi_j\rangle. \tag{22}$$

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If $f(x)$ is a constant, $V_\chi(z) = 1$. By (21), for a basis $\{\chi_1, \ldots, \chi_n\}$ of L, we have

$$(s-1) \sum_{\substack{\alpha>0 \\ \alpha:\,\text{short}}} \langle \alpha, \chi_i \rangle + (l-1) \sum_{\substack{\alpha>0 \\ \alpha:\,\text{long}}} \langle \alpha, \chi_i \rangle \equiv 0 \pmod 2 \tag{23}$$

for $i = 1, 2, \ldots, n$, and

$$M = (s-1)A_L + (l-1)B_L = 0. \tag{24}$$

By Lemma 4.6 and Equation (24), we have (s, l) as in Theorem 4.5. Equation (23) is valid for such (s, l). □

PROPOSITION 4.8. *Assume that* (s, l) *satisfies the condition in Theorem 4.5. The following relation holds for* $L = P$ *or* Q*:*

$$J_R((\{b_i\}, \{c_j\}; P; z) = |P/Q| J_R(\{b_i\}, \{c_j\}; Q; z),$$

in particular,

$$C_R(\{b_i\}, \{c_j\}; P) = |P/Q| C_R(\{b_i\}, \{c_j\}; Q),$$

where $|P/Q|$ *is the order of the fundamental group* P/Q *as in* (18).

Proof. For $\lambda \in P$, we set $\lambda + Q := \{\lambda + \chi; \chi \in Q\}$. Let m be the order of the fundamental group P/Q. Then, there exist $\lambda_1, \ldots, \lambda_m \in P$ such that

$$P = \bigcup_{k=1}^{m} (\lambda_k + Q), \quad \lambda_i + Q \neq \lambda_j + Q \quad \text{if } i \neq j. \tag{25}$$

By the definition (3) of $J_R(\{b_i\}, \{c_j\}; L; z)$ and (25), we have

$$J_R(\{b_i\}, \{c_j\}; P; z) = \sum_{k=1}^{m} J_R(\{b_i\}, \{c_j\}; Q; z + \lambda_k).$$

From the theta product expression of $J_R(\{b_i\}, \{c_j\}; Q; z)$ in Theorem 4.5, it follows that

$$J_R(\{b_i\}, \{c_j\}; Q; z + \lambda_k) = J_R(\{b_i\}, \{c_j\}; Q; z).$$

Thus, we have

$$J_R(\{b_i\}, \{c_j\}; P; z) = m J_R(\{b_i\}, \{c_j\}; Q; z).$$

This concludes the proof. □

According to Aomoto [Lemma 2.1 in Ao2], the number $\kappa := \det M$ is the dimension of the space of holomorphic functions satisfying (19) and (20), and the function in this space is described as a linear combination of theta functions of number κ. Next we consider the problem of finding (s, l) such that the holomorphic function $f(z)$ in Proposition 4.3 satisfies the condition $\kappa = 1$. From Lemmas 4.6 and

4.7 and the definition (22) of M, we have $\kappa = 1$ only if $L = P$ and $(s, l) = (2, 1)$ or $(2n, 0)$ for B_n. The sum $J_{B_n}(\{b_i\}_{i=1}^{s}, \{c_j\}_{j=1}^{l}; P; z)$ of $(s, l) = (2, 1)$ or $(2n, 0)$ is, indeed, realized as a special case of the following theta product formulae (26) and (27) investigated by Gustafson and van Diejen. We set

$$\Phi_G(b_1, \dots, b_{2n+2}; x) := \prod_{i=1}^{2n+2} \prod_{k=1}^{n} q^{(1/2-b_i)\langle\varepsilon_k, x\rangle} \frac{(q^{1-b_i+\langle\varepsilon_k, x\rangle}; q)_\infty}{(q^{b_i+\langle\varepsilon_k, x\rangle}; q)_\infty},$$

$$\begin{aligned}\Phi_D(b_1, b_2, b_3, b_4, c_1; x) &:= \prod_{i=1}^{4} \prod_{k=1}^{n} q^{(1/2-b_i)\langle\varepsilon_k, x\rangle} \frac{(q^{1-b_i+\langle\varepsilon_k, x\rangle}; q)_\infty}{(q^{b_i+\langle\varepsilon_k, x\rangle}; q)_\infty} \times \\ &\times \prod_{1 \leqslant \mu < \nu \leqslant n} q^{(1-2c_1)\langle\varepsilon_\mu, x\rangle} \frac{(q^{1-c_1+\langle\varepsilon_\mu-\varepsilon_\nu, x\rangle}; q)_\infty}{(q^{c_1+\langle\varepsilon_\mu-\varepsilon_\nu, x\rangle}; q)_\infty} \frac{(q^{1-c_1+\langle\varepsilon_\mu+\varepsilon_\nu, x\rangle}; q)_\infty}{(q^{c_1+\langle\varepsilon_\mu+\varepsilon_\nu, x\rangle}; q)_\infty},\end{aligned}$$

$$\begin{aligned}\Delta_{C_n}(x) &= \prod_{k=1}^{n} (q^{\langle\varepsilon_k, x\rangle} - q^{-\langle\varepsilon_k, x\rangle}) \times \\ &\times \prod_{1 \leqslant \mu < \nu \leqslant n} (q^{\langle\varepsilon_\mu-\varepsilon_\nu, x\rangle/2} - q^{-\langle\varepsilon_\mu-\varepsilon_\nu, x\rangle/2})(q^{\langle\varepsilon_\mu+\varepsilon_\nu, x\rangle/2} - q^{-\langle\varepsilon_\mu+\varepsilon_\nu, x\rangle/2})\end{aligned}$$

and

$$Q = \mathbf{Z}\varepsilon_1 + \mathbf{Z}\varepsilon_2 + \mathbf{Z}\varepsilon_3 + \cdots + \mathbf{Z}\varepsilon_n, \quad \langle\varepsilon_i, \varepsilon_j\rangle = \delta_{ij}.$$

We define two types of Jackson integrals as follows:

$$J_G(b_1, \cdots, b_{2n+2}; z) := \sum_{\chi \in Q} \Phi_G(b_1, \cdots, b_{2n+2}; z + \chi) \cdot \Delta_{C_n}(z + \chi)$$

(Gustafson's C_n-type sum),

$$J_D(b_1, b_2, b_3, b_4, c_1; z) := \sum_{\chi \in Q} \Phi_D(b_1, b_2, b_3, b_4, c_1; z + \chi) \cdot \Delta_{C_n}(z + \chi)$$

(van Diejen's BC_n-type sum).

LEMMA 4.9 (Gustafson, van Diejen). *The sum* $J_G(b_1, \dots, b_{2n+2}; z)$ *and* $J_D(b_1, b_2, b_3, b_4, c_1; z)$ *are expressed as*

$$\begin{aligned}C_G(b_1, \dots, b_{2n+2}) &\prod_{k=1}^{n} \frac{q^{\left(n-\sum_{i=1}^{2n+2} b_i\right)\langle\varepsilon_k, z\rangle} \vartheta(q^{\langle 2\varepsilon_k, z\rangle}; q)}{\prod_{i=1}^{2n+2} \vartheta(q^{b_i+\langle\varepsilon_k, z\rangle}; q)} \times \\ &\times \prod_{1 \leqslant \mu < \nu \leqslant n} q^{-\langle\varepsilon_\mu, z\rangle} \vartheta(q^{\langle\varepsilon_\mu-\varepsilon_\nu, z\rangle}; q) \vartheta(q^{\langle\varepsilon_\mu+\varepsilon_\nu, z\rangle}; q) \qquad (26)\end{aligned}$$

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and

$$C_D(b_1,b_2,b_3,b_4,c_1)\prod_{k=1}^{n}\frac{q^{(1-b_1-b_2-b_3-b_4)\langle\varepsilon_k,z\rangle}\vartheta(q^{\langle 2\varepsilon_k,z\rangle};q)}{\prod_{i=1}^{4}\vartheta(q^{b_i+\langle\varepsilon_k,z\rangle};q)}\times$$
$$\times\prod_{1\leqslant\mu<\nu\leqslant n}\frac{q^{-2c_1\langle\varepsilon_\mu,z\rangle}\vartheta(q^{\langle\varepsilon_\mu-\varepsilon_\nu,z\rangle};q)\vartheta(q^{\langle\varepsilon_\mu+\varepsilon_\nu,z\rangle};q)}{\vartheta(q^{c_1+\langle\varepsilon_\mu-\varepsilon_\nu,z\rangle};q)\vartheta(q^{c_1+\langle\varepsilon_\mu+\varepsilon_\nu,z\rangle};q)} \tag{27}$$

respectively, where $C_G(b_1,\dots,b_{2n+2})$ and $C_D(b_1,b_2,b_3,b_4,c_1)$ are some constants not depending on z.

Proof. See [p. 96 (7.8) in Gu1] and [vD]. □

Remark 4.9.1. In [Gu1, vD, Ito4], the constants $C_G(b_1,\cdots,b_{2n+2})$ and $C_D(b_1,b_2,b_3,b_4,c_1)$ are expressed as a product of q-gamma functions.

Remark 4.9.2. Gustafson's D_n-type sum [p. 197 in Gu4] is deduced from Gustafson's C_n-type sum by setting $q^{b_{2n-1}}=1$, $q^{b_{2n}}=q^{\frac{1}{2}}$, $q^{b_{2n+1}}=-q^{\frac{1}{2}}$, $q^{b_{2n+2}}=-1$.

Remark 4.9.3. Gustafson's C_n-type sum and van Diejen's BC_n-type sum are both included in the classification list of BC_n-type Jackson integral (see [Ito4]).

We have theta product expressions of $J_{B_n}(\{b_i\}_{i=1}^{s},\{c_j\}_{j=1}^{l};P;z)$ of $(s,l)=(2,1)$ and $(2n,0)$ by setting

$$q^{b_3}=-q^{1/2},\qquad q^{b_4}=-1\quad\text{for } J_D(b_1,b_2,b_3,b_4,c_1;z)$$

and

$$q^{b_{2n+1}}=-q^{1/2},\qquad q^{b_{2n+2}}=-1\quad\text{for } J_G(b_1,\cdots,b_{2n+2};z),$$

respectively. Thus, we can conclude the discussion on the case $\kappa=1$ as follows:

THEOREM 4.10. *It follows that $\kappa=1$ only if $L=P$ and $(s,l)=(2,1)$ or $(2n,0)$ for B_n. The sum $J_{B_n}(\{b_i\}_{i=1}^{s},\{c_j\}_{j=1}^{l};P;z)$ where $(s,l)=(2,1)$ or $(2n,0)$ is expressed as a product of elliptic theta functions.*

Remark 4.10.1. The sum $J_{B_n}(\{b_i\}_{i=1}^{s},\{c_j\}_{j=1}^{l};P;z)$ of $(s,l)=(1,1)$ and $(2n-1,0)$ for B_n-type in Theorem 4.5 are deduced from that of $(s,l)=(2,1)$ and $(2n,0)$ for B_n-type by taking $b_2=1/2$ and $b_{2n}=1/2$ respectively.

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