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DIFFERENTIAL GAME WITH SWITCHING CONTROLS ON HILBERT SPACE

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Abstract

We study differential game problems in which the players can select different maximal monotone operators for the governing evolution system. Setting up our problem on a real Hilbert space, we show that the Elliott-Kalton upper and lower value of the game are viscosity solution of some Hamilton-Jacobi-Isaacs equations. Uniqueness is obtained by assuming condition analogous to the classical Isaacs condition, and thus the existence of value of the game follows.

1. Introduction and definitions

In various control problems of distributed parameter system, the system can be modelled by an evolution equation on a Hilbert space

$$\frac{d}{ds}X(s) + AX(s) = f(s, X(s), q(s)).$$

Here q is the control function, A is a maximal monotone operator on a Hilbert space and f is the nonhomogeneous term. In some practical applications like population control problem [13] and gain adaptive direct strain feedback control problems [8], the operator A may also depend on the control function (see also [1]). In other words, the control function can control the monotone operator of the system and it is desirable to study problems of this nature. In this paper, we are interested in control problems in which the control function can select a different A out of a given class of A's and select different f's and different cost functionals out of a given class of of f's and cost functionals. We will study such control problems in a more general setup – a differential game framework. The benefit of working on a differential game setup is to allow more flexibility in the application of our results. For instance, if a

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control problem has an uncontrollable disturbance, then we could regard this random disturbance as an opposition force that always act against the control function and our framework applies. Indeed, nonlinear H^∞ control theory can be put into a differential game framework by similar argument (see for instance [11]). Readers may consult [6, 7] and [9] for more background on differential games.

We should point out that the differential game that we are studying in this paper has no switching cost. One of the motivations for studying this no-switching-cost problem comes from "chattering control" problems [see 20], in which the optimal policy is sought for by switching the control variables indefinitely rapidly, so as to simulate the continuous actions of the true control functions. As the original problem has no switching cost whatsoever, we put no switching cost into our setup. Another important application of our results is to problems for which a Bang-Bang principle is known.

Readers should also note that J. Yong has studied differential games with positive switching costs extensively in a series of papers [14–17]. His motivation is very different from ours. He starts from a concrete practical problem that has a switching cost nature and has studied it directly. Ours, as we have pointed out, stems from an optimal problem that has no switching nature and purposely replaces the continuous action of the control function by "chattering control", and results in switching the control variables indefinitely rapidly. However, since one of the main results in Yong's papers as well as this paper is to show that the value of the differential game is unique, it is desirable to see if Yong's uniqueness result can be used to deduce ours by taking the switching costs to zero. It is our feeling that the uniqueness results of Yong and the present study supplement each others rather than one deduced from the other. As Yong pointed out in [16] that his uniqueness result holds for the positive switching cost case without any Isaacs-type condition, he also pointed out that as the switching costs go to zero, his uniqueness result will not hold unless some restrictions are put on the switching time of the control function. This restriction will forbid the control function to switch freely at any time that it wants, and Yong felt that this assumption is unnatural (and we agree) even though the assumption will support uniqueness. On the other hand, our approach yields uniqueness by assuming an Isaacs-type condition, which we felt is more natural. So it appears to us that our results of the zero-switching-cost case supplement the results in [14–17], and all together they form a quite complete investigation for differential games with switchings. We shall comment more on the results of [14–17] in the final remark after the presentation of our results.

As for this paper, dynamic programming identities are proved for our differential game. The associated upper and lower Elliott-Kalton value functions are shown to be the viscosity solution of some Hamilton-Jacobi-Isaacs equations. Comparison principles and convergence theorems for the Hamilton-Jacobi-Isaacs equations may be proved by modifying the steps in [4, 5]. The existence of the value of the differential game is proved by assuming certain conditions analogous to the classical Isaacs condition.

The time interval that we shall be working on is either $[0, \infty)$ or $[t, T]$ with $0 \le t \le T$. We shall refer to the former as the infinite horizon case and the later as the finite horizon case. There are a lot of similarities between treatments of the infinite and finite horizon cases. We shall use the same terms of definitions and the same notation for both of them. All definitions and results are stated and proved mostly for the infinite horizon case but only briefly mentioned for the finite horizon case since those of the latter can easily be deduced from the former. We will treat the infinite and finite horizon cases concurrently until it is necessary to separate them.

In a differential game problem, two players (usually called player 1 and player 2) are involved in a conflicting situation. One (player 1) wants to minimize the cost functional while the other (player 2) wants to maximize it through their own control functions. More than that, each player can respond to their opponent's control function by altering their own control function after knowing their opponent's move. This type of response will be called a strategy whose rigorous definition together with others are defined next. Since the role of player 2 is symmetric with that of player 1, corresponding definitions for player 2 will be very brief. We begin with the definition of the control function.

Let H and V be two real Hilbert spaces and suppose that I and J are two arbitrary index sets. Let $\{Q_i : i \in I\}$ and $\{Z_j : j \in J\}$ be two collections of subsets of V. We always assume that

$$\text{for } i \in I,\ j \in J, A(i, j) \text{ is a linear and densely defined maximal monotone operator in } H. \tag{A}$$

DEFINITION 1 (controls). An I-indexed control $\alpha := (q, N, \Delta_N)$ for the infinite horizon problem is defined to consist of:

(i) a strongly measurable function $q : [0, +\infty) \to V$,

(ii) a natural number N,

(iii) an I-indexed partition of $[0, +\infty)$, $\Delta_N := \{(r_n, i_n)\}_{n=1}^N \subset [0, +\infty) \times I$ such that $0 = r_1 < r_2 < \cdots < r_N < +\infty$ and $q([r_n, r_{n+1})) \subset Q_{i_n}$.

The set of all these I-indexed controls α is denoted by $\mathscr{C}$.

For the finite horizon case, $0 \le t \le T$, an I-indexed control $\alpha := (q, N, \Delta_N)$ is defined to consist of:

(1) a strongly measurable function $q : [t, T] \to V$,

(2) a natural number N,

(3) an I-indexed partition of $[t, T]$, $\Delta_N := \{(r_n, i_n)\}_{n=1}^N \subset [t, T] \times I$ such that $t = r_1 < r_2 < \cdots < r_N < T$ and $q([r_n, r_{n+1})) \subset Q_{i_n}$.

The set of these I-indexed controls is denoted by $\mathscr{C}[t, T]$. In both the infinite and finite horizon cases, we will write the control as either $\alpha = (q, N, \Delta_N)$ or $\alpha = (q, N, \{(r_n, i_n)\}_{n=1}^{N})$.

For convenience, whenever we say r_{N+1} hereafter while we are dealing with Δ_N, we will always mean the convention $r_{N+1} := +\infty$ (or respectively, $r_{N+1} := T$ for the finite horizon case).

If the given index set is J, we may define the J-indexed partition $\Omega_M := \{(s_m, j_m)\}_{m=1}^{M}$ ($M \in \mathbb{N}$) and the J-indexed control $\beta := (z, M, \Omega_M)$ analogously. The set of all J-indexed controls is denoted by $\mathscr{D}$ and $\mathscr{D}[t, T]$ respectively for the infinite and finite horizon cases.

In the sequel, given an I-indexed partition $\Delta_N := \{(r_n, i_n)\}_{n=1}^{N}$ and a J-indexed partition $\Omega_M := \{(s_m, j_m)\}_{m=1}^{M}$, we shall make heavy use of the following two functions to simplify our notation:

$$i(\tau) := \sum_{n=1}^{N} i_n \chi_{[r_n, r_{n+1})}(\tau), \qquad j(\tau) := \sum_{m=1}^{M} j_m \chi_{[s_m, s_{m+1})}(\tau)$$

and we caution the readers not to be confused with a single index $i \in I$ or $j \in J$.

From now on, we shall concentrate on player 1 only. Exact parallel arguments for player 2 can be deduced by obvious modifications. We next define the nonanticipating strategy, which is the response of player 1 to his opponent's control function. Interested readers may consult [6] for the motivation for the classical nonanticipating strategy.

DEFINITION 2 (Non-anticipating strategies). A player 1 strategy for the infinite (and finite) horizon problem is any mapping $\xi : \mathscr{C} \to \mathscr{D}$ (and respectively any mapping $\xi : \mathscr{C}[t, T] \to \mathscr{D}[t, T]$). The value of the strategy ξ will be denoted by $\xi(\alpha) = (z(\alpha), M(\alpha), \Omega_M(\alpha))$ (or simply by (z, M, Ω_M) for brevity).

Let $\alpha := (q, N, \{(r_n, i_n)\}_{n=1}^{N})$ and $\hat{\alpha} := (\hat{q}, \hat{N}, \{(\hat{r}_n, \hat{i}_n)\}_{n=1}^{\hat{N}})$ be two controls and let $\xi(\alpha)$ and $\xi(\hat{\alpha})$ be the values of the strategy ξ respectively. The player 1 strategy ξ is called nonanticipating if for any $s > 0$ (or any $s \in [t, T]$), properties

(i) $q = \hat{q}$ a.e. on $[r_1, s]$,
(ii) for all except finitely many of those $h \in [r_1, s]$, $i(h) = \hat{i}(h)$

imply that

(1) $z = \hat{z}$ a.e. on $[r_1, s]$,
(2) for all except finitely many of those $h \in [r_1, s]$, $j(h) = \hat{j}(h)$.

A player 2 nonanticipating strategy $\zeta : \mathscr{D} \to \mathscr{C}$ ($\zeta : \mathscr{D}[t, T] \to \mathscr{C}[t, T]$) is defined analogously. We will denote the collection of all nonanticipating strategies by $\mathscr{B}$ in the infinite horizon case and by $\mathscr{B}[t, T]$ in the finite horizon case. Corresponding nonanticipating player 2 strategies are denoted by $\mathscr{A}$ and $\mathscr{A}[t, T]$.

The definition of trajectories is a little complicated since there are two players involved and hence, two partitions of $[0, +\infty)$ (or of $[t, T]$) occur. We first define the union of two partitions.

The cost functionals of the differential game that the two players seek to extremize over all control $\alpha = (q, N, \{(r_n, i_n)\}_{n=1}^{N})$ in $\mathscr{C}$ (and respectively $\mathscr{C}[t, T]$) and all strategy $\xi = (z, M, \Omega_M)$ in $\mathscr{B}$ (and respectively $\mathscr{B}[t, T]$) are respectively

$$\int_0^\infty L(X(s; x), q(s), z(s); i(s), j(s))e^{-s}\, ds$$

and

$$\int_t^T L(s, X(s; x), q(s), z(s); i(s), j(s))\, ds + \psi(X(T; x)).$$

Here $L(\cdot; i, j)$ is the running cost defined on $H \times Q_i \times Z_j$ (or $[0, T] \times H \times Q_i \times Z_j$) for each $i \in I$ and each $j \in J$, X is a trajectory on $[0, +\infty)$ (or respectively on $[t, T]$) from the initial point $x \in H$ and its definition is as follows.

DEFINITION 3 (Trajectories). Given any control $\alpha = (q, N, \Delta_N) \in \mathscr{C}$ and any strategy $\xi \in \mathscr{B}$, we set $\xi(\alpha) = (z, M, \Omega_M)$. For any $x \in H$, we associate a trajectory $X(t; x)$ to α and $\xi(\alpha)$ as the mild solution of

$$\begin{aligned} &\frac{d}{ds}X(s; x) + A(i(s), j(s))X(s; x) \\ &\qquad = f(X(s; x), q(s), z(s); i(s), j(s)) \quad \text{on } (r_1, r_{N+1}) \end{aligned}$$

with $X(r_1; x) = x$.

Trajectories of the finite horizon case is defined similarly and we shall refer to the trajectory defined as "the trajectory corresponding to control α, strategy ξ and the starting point x". If necessary, we will denote $X(t; x)$ simply as $X(t)$ dropping the starting point.

Notice that, under condition (A) and the following assumptions:

$$\left.\begin{array}{l}\text{there exists } \kappa > 0 \text{ such that } \psi(\cdot) \text{ are Lipschitz continuous} \\ \text{with modulus } \kappa \text{ and also bounded by } \kappa \text{ on } H\end{array}\right\} \tag{1.1}$$

and

$$\left.\begin{array}{l}\text{there exists } \kappa > 0 \text{ such that for each } i \in I \text{ and each } j \in J, \\ \text{both } L(\cdot; i, j), \text{ and } f(\cdot; i, j) \text{are Lipschitz continuous with} \\ \text{modulus } \kappa \text{ and also bounded by } \kappa \text{ on } H \times Q_i \times Z_j \text{ (or} \\ \text{on } [0, T] \times H \times Q_i \times Z_j \text{ for the finite horizon problems),}\end{array}\right\} \tag{1.2}$$

the cost functional is finite and we have the following estimates of X:

$$\|X(s;x)\| \leq \|x\| + Cs, \tag{1.3}$$

for some constant $C \geq 0$ and for all $s \geq 0$ (or $s \in [t, T]$) and $x \in H$.

The structural assumptions (1.1) and (1.2) may be relaxed without affecting the results of the first two sections if we incorporate the convergence result developed in later sections. Please consult the discussion at the end of this paper for details of this relaxation.

To enhance smooth presentation of our main results, we gather together some technical definitions that we will need in later sections. We shall define switching between two controls, strategies and trajectories. The geometric pictures of the definitions are very clear. We just switch from an original to a new one at a given time. We also define left time-shifting of controls, strategies and trajectories.

For brevity, we use the same symbol $\oplus_h$ for the switchings of different objects, but it should be clear by the use which definition the notation implies.

DEFINITION 4 (Switching of controls). Given $h > 0$, let $\alpha = (q, N, \Delta_N)$, $\hat{\alpha} = (\hat{q}, \hat{N}, \Delta_{\hat{N}}) \in \mathscr{C}$ with $\Delta_N = \{(r_n, i_n) : n = 1, \cdots, N\}$ and $\Delta_{\hat{N}} = \{(\hat{r}_n, \hat{i}_n) : n = 1, \cdots, \hat{N}\}$. We define a new control $\alpha \oplus_h \hat{\alpha}$ by $\alpha \oplus_h \hat{\alpha} := (q \oplus_h \hat{q}, N \oplus_h \hat{N}, \Delta_N \oplus_h \hat{\Delta}_N)$ where

(i)

$$(q \oplus_h \hat{q})(s) := \begin{cases} q(s), & 0 \leq s < h \\ \hat{q}(s-h), & s \geq h \end{cases}$$

(ii) the natural number $N \oplus_h \hat{N}$ and the I-indexed partition $\Delta_N \oplus_h \hat{\Delta}_N$ respectively are the total number of partition points (that are different from r_{N+1}) and the I-indexed partition associated with the step function

$$i_h(s) := \begin{cases} i(s), & 0 \leq s < h \\ \hat{i}(s-h), & s \geq h. \end{cases}$$

Switching on $\mathscr{C}[t, T]$ at $t + h$ is defined similarly and so are those on $\mathscr{D}$ and $\mathscr{D}[t, T]$.

It is easy to see that the newly defined $\alpha \oplus_h \hat{\alpha}$ belongs to $\mathscr{C}$. Indeed, it is the same as α when $0 \leq t \leq h$ but switches to $\hat{\alpha}$ when $t > h$ along with an adjustment in the t-scale by an amount of h. The same is true for the finite horizon case. We define another operation on the control functions, namely the left shifting of a control function.

DEFINITION 5 (Left shifting of controls). Given any control $\alpha = (q, N, \{(r_n, i_n)\}_{n=1}^N) \in \mathscr{C}$ and any $h > 0$, a new control $\alpha^h = (\tilde{q}, \tilde{N}, \Delta_{\tilde{N}}) \in \mathscr{C}$ with $\Delta_{\tilde{N}} = \{(\tilde{r}_l, \tilde{i}_l) : l = 1, \cdots, \tilde{N}\}$ is defined by

(i) $\tilde{q}(t) = q(t+h)$;

(ii) $\tilde{N}$ and $\Delta_{\tilde{N}}$ are respectively the total number of partition points (that are different from r_{N+1}) as well as the I-indexed partition associated with $\tilde{i}(t) := i(t+h)$.

The left shifting of control is defined similarly for the finite horizon case.

DEFINITION 6 (Switching of strategies). Let ξ, $\hat{\xi} \in \mathscr{B}$ be two strategies with $\xi(\alpha) := (z(\alpha), M(\alpha), \Omega_M(\alpha))$ and $\hat{\xi}(\alpha) := (\hat{z}(\alpha), \hat{M}(\alpha), \Omega_{\hat{M}}(\alpha))$ on a control $\alpha = (q, N, \Delta_N) \in \mathscr{C}$. We define their switching $\xi \oplus_h \hat{\xi}$, at a given time $h > 0$, by

$$(\xi \oplus_h \hat{\xi})(\alpha) := (z(\alpha) \oplus_h \hat{z}(\alpha^h), N(\alpha) \oplus_h \hat{N}(\alpha^h), \Omega_M(\alpha) \oplus_h \Omega_{\hat{M}}(\alpha^h)),$$

where α^h is the left shifted α by an amount of h (see Definition 5). The switching between two strategies is defined similarly for the finite horizon problem.

Roughly speaking, the resultant strategy in the above definition has the same response as strategy ξ when $t < h$ but switches to the response of $\hat{\xi}$ when $t \leq h$ along with an adjustment in the time scale. It is easy to check that $\xi \oplus_h \hat{\xi}$ belongs to $\mathscr{B}$, i.e. it is also a nonanticipating strategy. The same remarks are also true for the finite horizon case.

DEFINITION 7 (Left shifting of strategies). Given any $h > 0$ and $\xi \in \mathscr{B}$, we define a strategy $\xi^h \in \mathscr{B}$ as follows: for any control $\alpha \in \mathscr{C}$, let $\xi(\alpha) = (z(t), M, \{(s_m, j_m)\}_{m=1}^M)$, then $\xi^h(\alpha) := (\tilde{z}(t), \tilde{M}, \{(\tilde{s}_l, \tilde{j}_l)\}_{l=1}^{\tilde{M}})$ is given by

(i) $\tilde{z}(t) = z(t+h)$;

(ii) $\tilde{M}$ and $\{(\tilde{s}_l, \tilde{j}_l)\}_{l=1}^{\tilde{M}})$ are respectively the total number of partition points (that are different from s_{M+1}) as well as the J-indexed partition associated with $\tilde{j}(t) := j(t+h)$.

The shifting is defined similarly for the finite horizon case.

DEFINITION 8 (Switching of trajectories). Let $X(t)$ be the trajectory corresponding to control α, strategy ξ and starting point x. Let $\hat{X}(t)$ be the trajectory corresponding to control $\hat{\alpha}$, strategy $\hat{\xi}$ and starting point $X(h)$ for an given $h > 0$. Then a new trajectory Y on $[0, +\infty)$ denoted by $X \oplus_h \hat{X}$ is defined by

$$Y(t) = \begin{cases} X(t), & 0 \leq t \leq h \\ \hat{X}(t-h), & t \geq h. \end{cases}$$

For the finite horizon case, the switching is defined similarly.

2. Value function and dynamic programming identity

We again give description for player 1 only. Those for player 2 are summarized in Remark 2.

DEFINITION 9 (Value Function). We treat both horizon cases concurrently. Let I and J be two arbitrary index sets. Suppose that (1.1) and (1.2) hold and let $X(s;x)$ be the trajectory corresponding to control $\alpha = (q, N, \Delta_N)$, strategy ξ (with $\xi(\alpha) = (z, M, \Omega_M)$) and starting point x. The function $u : H \to \mathbb{R}$ given by

$$u(x) = \sup_{\xi\in\mathscr{B}} \inf_{\alpha\in\mathscr{C}} \left\{ \int_0^\infty L(X(s;x), q(s), z(s); i(s), j(s)) e^{-s}\, ds \right\} \tag{2.1}$$

is called the upper (Elliott-Kalton) value function of the infinite horizon differential game problem.

For the finite horizon case, the function

$$u(t,x) = \sup_{\xi\in\mathscr{B}[t,T]} \inf_{\alpha\in\mathscr{C}[t,T]} \left\{ \int_t^T L(s, X(s;x), q(s), z(s); i(s), j(s))\, ds + \psi(X(T;x)) \right\}, \tag{2.2}$$

mapping $[0, T] \times H$ to $\mathbb{R}$, is called the upper (Elliott-Kalton) value function of the finite horizon differential game problem.

Notice that u would be the value function of a conventional optimal control problem if both L and f are independent of z, i and j.

NOTATION. To avoid clumsy integral representation and in view of the classical non-switching differential game cases, we will use the following short-hand integral notation:

$$u(x) = \sup_{\xi\in\mathscr{B}} \inf_{\alpha\in\mathscr{C}} \int_0^\infty L(X(s;x), \alpha(s), \xi(\alpha)(s)) e^{-s}\, ds,$$

$$u(t,x) = \sup_{\xi\in\mathscr{B}[t,T]} \inf_{\alpha\in\mathscr{C}[t,T]} \int_t^T L(s, X(s;x), \alpha(s), \xi(\alpha)(s))\, ds + \psi(X(T;x))$$

in the integral representations of (2.1), (2.2) and other integrals if necessary.

We are now ready to prove dynamic programming identities for the value function u.

THEOREM 1 (Dynamic Programming Identities). *Suppose that* (A), (1.1) *and* (1.2) *hold. Then, for all* $h \geq 0$,

$$u(x) = \sup_{\xi \in \mathscr{B}} \inf_{\alpha \in \mathscr{C}} \{ \int_0^h L(X(s;x), \alpha(s), \xi(\alpha)(s)) e^{-s} \, ds + u(X(h;x)) e^{-h} \}, \quad (2.3)$$

and for $0 \leq h \leq T - t$,

$$u(t,x) = \sup_{\xi \in \mathscr{B}[t,T]} \inf_{\alpha \in \mathscr{C}[t,T]} \left\{ \int_t^{t+h} L(s, X(s;x), \alpha(s), \xi(\alpha)(s)) \, ds \right. \\ \left. + u(t+h, X(t+h;x)) \right\}, \quad (2.4)$$

where $X(s;x)$ *is the trajectory corresponding to control* α, *strategy* ξ *and starting point* x.

PROOF. We only prove (2.3). Let the right hand side of the assertion be w, that is,

$$w(x) := \sup_{\xi \in \mathscr{B}} \inf_{\alpha \in \mathscr{C}} \left\{ \int_0^h L(X(s;x), \alpha(s), \xi(\alpha)(s)) e^{-s} \, ds + u(X(h;x)) e^{-h} \right\}. \quad (2.5)$$

We want to show that $w(x) \leq u(x)$. Given $\epsilon > 0$, from (2.5) there exists strategy $\xi_1 \in \mathscr{B}$ so that for all control $\alpha \in \mathscr{C}$,

$$w(x) \leq \int_0^h L(X(s;x), \alpha(s), \xi_1(\alpha)(s)) e^{-s} \, ds + u(X(h;x)) e^{-h} + \epsilon, \quad (2.6)$$

where $X(t;x)$ is the trajectory corresponding to control α, strategy ξ_1 and starting point x. If we let $\sigma = X(h;x)$, then from (2.1) there exists $\xi_\sigma \in \mathscr{B}$ so that for all $\alpha \in \mathscr{C}$,

$$u(\sigma) = u(X(h;x)) \leq \int_0^\infty L(Y(s;\sigma), \alpha(s), \xi_\sigma(\alpha)(s)) e^{-s} \, ds + \epsilon, \quad (2.7)$$

where $Y(t;\sigma)$ is the trajectory corresponding to control α, strategy ξ_σ and starting point σ. Let $\hat{\xi} = \xi_1 \oplus_h \xi_\sigma$ (see Definition 6). For this $\hat{\xi}$, (2.6) and (2.7) imply that for all control $\alpha \in \mathscr{C}$, we have

$$w(x) \leq \int_0^\infty L(\hat{X}(s;x), \alpha(s), \hat{\xi}(\alpha)(s)) e^{-s} \, ds + 2\epsilon,$$

where $\hat{X}(s;x) := X(s;x) \oplus_h Y(s;\sigma)$ is the associated trajectory. Thus, letting $\epsilon \downarrow 0$,

$$w(x) \leq \sup_{\xi \in \mathscr{B}} \inf_{\alpha \in \mathscr{C}} \int_0^\infty L(X(s;x), \alpha(s), \xi(\alpha)(s)) e^{-s} \, ds = u(x).$$

We next verify the opposite side $w \geq u$. From (2.1), there exists $\tilde{\xi} \in \mathscr{B}$ such that for all $\alpha \in \mathscr{C}$,

$$u(x) \leq \int_0^\infty L(X(s;x), \alpha(s), \tilde{\xi}(\alpha)(s))e^{-s}\, ds + \epsilon, \tag{2.8}$$

where $X(s;x)$ is the trajectory corresponding to control α, strategy $\tilde{\xi}$ and starting point x. Since $w(x)$ is an infimum over $\mathscr{C}$, so for strategy $\tilde{\xi}$ there exist $\alpha_1 \in \mathscr{C}$ so that

$$w(x) \geq \int_0^h L(X(s;x), \alpha_1(s), \tilde{\xi}(\alpha_1)(s))e^{-s}\, ds + u(X(h;x))e^{-h} - \epsilon, \tag{2.9}$$

where $X(t;x)$ is the trajectory corresponding to control α_1, strategy $\tilde{\xi}$ and starting point x.

On the other hand, if we define a new strategy ξ_2 by $\xi_2(\alpha) := \tilde{\xi}^h(\alpha_1 \oplus_h \alpha)$ (see Definition 7 and Definition 4), then (2.1) implies that there exists $\alpha_2 \in \mathscr{C}$ for this strategy ξ_2 such that

$$u(X(h;x)) \geq \int_0^\infty L(Y(s), \alpha_2(s), \xi_2(\alpha)(s))e^{-s}\, ds - \epsilon, \tag{2.10}$$

where $Y(s)$ above is the trajectory corresponding to control α_2, strategy ξ_2 $\tilde{\alpha} := \alpha_1 \oplus_h \alpha_2$ and $\tilde{X} = X \oplus_h Y$, then (2.9), (2.10) and (2.8) imply that

$$w(x) \geq \int_0^\infty L(\tilde{X}(s;x), \tilde{\alpha}(s), \tilde{\xi}(\alpha)(s))e^{-s}\, ds - 2\epsilon \geq u(x) - 3\epsilon$$

for any $\epsilon \geq 0$. Hence, $w \geq u$ and the proof is completed.

3. Viscosity solution of Hamilton-Jacobi-Isaacs equation

The viscosity solution was introduced in [2] and [3]. One of the virtues of this solution notion is that comparison principle and (hence) uniqueness can be achieved quite easily. By modifying slightly the framework of [5], we can achieve a nice viscosity solution theory for both the infinite and finite horizon problems. Corresponding convergence theorems under this modified framework may be obtained by adapting the proofs in [5] and we shall refer to [5] from time to time for preliminary results and motivations. Similar to [5], we impose further restriction on $A(i, j)$ that is not too restrictive to exclude important applications. In addition to (A), we always assume that condition (B0) holds:

(B0) there exists an operator B on H that is bounded, linear, self-adjoint and positive-definite such that for each $i \in I$ and each $j \in J$, $A^*(i, j)B$ is a bounded linear operator on H and $\sup\{\|A^*(i,j)B\| : i \in I, j \in J\} < +\infty$. Furthermore, we also assume that one of the following conditions hold:

(B1) for the operator B in condition (B0), there exists a constant $C_0 \in \mathbb{R}$ such that for all $i \in I$, $j \in J$ and all $x \in H$, $\langle (A^*(i,j)B + C_0 B)x, x\rangle \geq \|x\|^2$;

$(\mathrm{B}1)_w$ for the operator B in condition (B0), there exists a constant $C_0 \in \mathbb{R}$ such that for all $i \in I$, $j \in J$, and all $x \in H$, $\langle (A^*(i,j)B + C_0 B)x, x\rangle \geq 0$.

Following [5], we shall refer to the strong B and the weak B case when respectively condition (B1) and condition $(\mathrm{B}1)_w$ hold. Let us remark that when $A(i,j) \equiv 0$ ($i \in I$, $j \in J$), conditions (B0), (B1) and $(\mathrm{B}1)_w$ hold with B chosen to be the identity operator of H. For the nontrivial cases, like $\{A(i,j) : i \in I,\ j \in J\}$ which corresponds to a family of elliptic differential operators or respectively wave operators, we can also choose B appropriately so that (B0), (B1) and $(\mathrm{B}1)_w$ hold. We will use some simple examples to illustrate this.

Let $\mathscr{O}$ be a C^1 bounded open domain of $\mathbb{R}^n$ (an element of $\mathbb{R}^n$ is denoted as $(\rho_1, \cdots, \rho_n)$) and let $H := L^2(\mathscr{O})$. Suppose that $A(i,j)$ is the operator on $L^2(\mathscr{O})$ associated with the elliptic operator

$$-\sum_{l,k=1}^{n} a_{lk}^{ij} \frac{\partial^2}{\partial\rho_l \partial\rho_k} + c^{ij} \qquad \forall\, i,\ j \in \mathbb{N} \tag{3.1}$$

subject to Dirichlet boundary value and the coefficients of these operators satisfy

$$\left.\begin{aligned}
&\forall\, i \in \mathbb{N}, \quad \forall\, k,\ l = 1, \cdots, n, \quad a_{lk}^{ij} = a_{kl}^{ij} \in \mathbb{R},\ c^{ij} \geq 0;\\
&\exists \theta > 0,\ \forall \eta = (\eta_1, \cdots, \eta_n) \in \mathbb{R}^n,\ \sum_{l,k=1}^{n} a_{lk}^{ij}\eta_l\eta_k \geq \theta|\eta|^2,\ \forall i, j \in \mathbb{N};\\
&\exists M > 0 \text{ such that } |a_{lk}^{ij}| \leq M \text{ and } c^{ij} \leq M,\ \forall\, i,\ j \in \mathbb{N},\ \forall k, l = 1, \cdots, n.
\end{aligned}\right\} \tag{3.2}$$

We let $D(A(i,j)) := W_0^{1,2}(\mathscr{O}) \cap W^{2,2}(\mathscr{O})$ and clearly condition (A) holds. By taking $B := (-\Delta)^{-1}$, we can easily see that condition (B0) is true while condition (B1) follows as a consequence of the Sobolevskii inequality [12] (see also [10]).

Notice that this particular $B = (-\Delta)^{-1}$ is compact on $H = L^2(\mathscr{O})$. In contrast, (B1) can not hold when $\{A(i,j)\}$ is a family of skew-symmetric operators for any compact operator B. This includes, in particular, the case that $\{A(i,j)\}$ is a family of densely defined maximal monotone operators corresponding to some wave operators. To see what could go wrong, suppose that the index sets I and J are singleton sets and $A := A(i,j)$ is skew symmetric with B being compact. Let $v \in H$ be an eigenvector of this compact operator B with eigenvalue ν. If (B1) holds then

$$\|v\|^2 \leq -\nu\langle Av, v\rangle + \nu C_0\|v\|^2 \leq \nu C_0\|v\|^2;$$

this prohibits the eigenvalues of B from tending to zero, and hence B could not be compact.

Rather than giving up our freedom of choosing compact B, we change to the weaker assumption $(\mathrm{B}1)_w$. It holds, for instance, when $H = W_0^{1,2}(\mathcal{O}) \times L^2(\mathcal{O})$ and

$$A(i, j) = \begin{pmatrix} 0 & -Id \\ E(i, j) & 0 \end{pmatrix},$$

where Id is the identity operator on $L^2(\mathcal{O})$, $E(i, j)$ is given by 3.1 and 3.2 on $\mathcal{O}$ and 0 is the corresponding zero operator on $W_0^{1,2}(\mathcal{O})$ and $L^2(\mathcal{O})$. In this case, B can be chosen as

$$\begin{pmatrix} \lambda(-\Delta)^{-1} & 0 \\ 0 & \mu(-\Delta)^{-1} \end{pmatrix}$$

and $(\mathrm{B}1)_w$ follows from the Sobolevskii inequality with suitably large positive constants λ and μ. The two simple examples above are chosen only for illustration purposes. Treatments of maximal monotone operators that are associated with more sophisticated differential operators can be found in the discussions of [4].

In terms of the operator B, a special kind of continuity, called B-continuity, will be used in our study. Its definition and various related terminologies are adopted from [5].

Let B be a positive, self-adjoint, bounded linear operator on H. We define, for each $s > 0$, a new inner product and a corresponding new norm on H, namely $\langle x, y\rangle_{-s} = \langle B^s x, y\rangle$ and $\|x\|_{-s} = \langle B^s x, x\rangle^{1/2}$ and denote the completion of H under $\|x\|_{-s}$ by H_{-s}. Using this operator B, we define a notion of continuity on H which, in general, is weaker than weak sequential continuity but stronger than the usual continuity on H.

DEFINITION 10 (B-continuity). We give a definition for the finite horizon case. The corresponding definition for the infinite horizon case can be obtained by regarding u to be independent of t. Let E be an arbitrary subset of $[0, T] \times H$ and let B be an operator on H. A function $u : E \to \mathbb{R}$ is B-upper-semicontinuous (respectively, B-lower-semicontinuous) on E if $\limsup_{n\to+\infty} u(t_n, x_n) \le u(t, x)$ (respectively $\liminf_{n\to+\infty} u(t_n, x_n) \ge u(t, x)$) for every sequence $\{t_n, x_n\} \subset E$ and any $(t, x) \in E$ that satisfies $x_n \rightharpoonup x$ in H and $(Bx_n, t_n) \to (Bx, t)$ in $[0, T] \times H$.

We call u B-continuous on E if it is both B-upper-semicontinuous and B-lower-semicontinuous on E. These two semicontinuities will be abbreviated as B-usc and B-lsc.

We refer readers to [5] for the following facts about B-continuity.

LEMMA 1. *Let E be an arbitrary subset of $[0, T] \times H$ and let B be a positive, self-adjoint, bounded linear operator on H. Then we have the following.*

(i) *If u is B-continuous on E, then u is continuous on E.*

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(ii) *If u is weakly sequentially continuous on E, then u is B-continuous on E.*

(iii) *If B is compact, then u is B-continuous on E if and only if u is weakly sequentially continuous on E.*

(iv) *For each $t \in [0, T]$, $u(t, \cdot)$ is B-continuous on E if and only if $u(t, \cdot)$ is continuous on each ball B_R ($R > 0$) with respect to the H_{-2} norm.*

(v) *If u is B^s-continuous on E for some $s > 0$, then u is $B^{s'}$-continuous on E for all $s' > 0$.*

(vi) *The statements in (i) – (v) are true if we replace all involved continuities by upper semicontinuities (or lower semicontinuities).*

(vii) *Let $(H_{-s})'$ denote the dual space of H_{-s}. Then, for all $\xi \in (H_{-s})'$, $s > 0$, there exists $\eta \in H$ such that for all $x \in H$, $\xi(x) = \langle B^{s/2}\eta, x\rangle$ and $\|\xi\|_{(H_{-s})'} = \|\eta\|$.*

One of our goals is to show that the value functions are solutions respectively of some Hamilton-Jacobi-Isaacs equations on H and $[0, T) \times H$ in a viscosity solution sense that is modified from [5] to suit our problems. The modified definitions are as follows:

DEFINITION 11 (Viscosity Solution). We give the definition for the finite horizon case and the corresponding definition for the infinite horizon case can be obtained by regarding all the functions involved to be independent of t.

Given $0 < T < +\infty$ and let Ω be an open subset of H. Let $u \in C([0, T] \times \Omega)$ and let $F(\cdot; i, j)$ be a real-valued function on $[0, T] \times \Omega \times \mathbb{R} \times H$ for each $i \in I$ and $j \in J$. Then u is a viscosity subsolution (respectively, supersolution) of

$$-u_t(t,x) + \sup_{i\in I}\inf_{j\in J}\{\langle A(i,j)x, Du(t,x)\rangle + F(t,x,u(t,x),Du(t,x);i,j)\} = 0$$

on $[0, T) \times \Omega$ if for every local maximum (respectively, minimum) point $(s, y) \in [0, T) \times \Omega$ of $u - \varphi - g$ (respectively, $u + \varphi + g$) where $\varphi : [0, T] \times \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ satisfy, respectively, condition (φ) and condition (g) below:

$$\left.\begin{array}{l}\varphi \text{ is B-lower-semicontinuous on } (0,T]\times\Omega;\ D\varphi \text{ exists and is continuous}\\ \text{on } [0,T]\times\Omega \text{ with Range } (D\varphi) \subset \bigcap_{i\in I, j\in J} \text{Domain } (A^*(i,j));\\ \text{and also the mapping } (t,x)\mapsto\langle x, A^*(i,j)D\varphi(t,x)\rangle \text{ from } [0,T]\times\Omega\\ \text{to}\mathbb{R} \text{ is equi-continuous in } i\in I \text{ and } j\in J\end{array}\right\} \quad (\varphi)$$

$$\left.\begin{array}{l}\text{there exists } h:[0,+\infty)\to\mathbb{R} \text{ such that}\\ h \text{ is nondecreasing, } C^1,\ h'(0+)=0 \text{ and}\\ g(x)=h(\|x\|) \qquad \forall x\in H,\end{array}\right\} \quad (g)$$

we have

$$-\varphi_t(s,y) + \sup_{i\in I}\inf_{j\in J}\{\langle y, A^*(i,j)D\varphi(s,y)\rangle + F(s,y,u(s,y),D\varphi(s,y) + Dg(y);i,j)\} \le 0$$

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(respectively,

$$\varphi_t(s, y) + \sup_{i\in I}\inf_{j\in J}\{\langle y, -A^*(i, j)D\varphi(s, y)\rangle + F(s, y, u(s, y), -D\varphi(s, y) - Dg(y); i, j)\} \geq 0).$$

We call u a viscosity solution on $[0, T) \times \Omega$ if it is both a viscosity subsolution and a supersolution on $[0, T) \times \Omega$.

We prove a technical lemma for later use.

LEMMA 2. *Suppose that* (A), (1.1) *and* (1.2) *hold. Given control* $\alpha = (q, N, \{(r_n, i_n)\}_{n=1}^N) \in \mathscr{C}$, *strategy* $\xi \in \mathscr{B}$ *(with* $\xi(\alpha) = (z(t), M, \Omega_M)$*) and a point* $x \in H$, *let* $X(t) := X(t; x)$ *be the corresponding trajectory. Let* $\varphi : H \to \mathbb{R}$ *satisfy condition* (φ) *and* $g : H \to \mathbb{R}$ *satisfy condition condition* (g).

(1) *For each* $h \in [0, \infty)$, *we have*

$$\int_0^h e^{-t}\{\varphi(X(t)) + \langle X(t), A^*(i(t), j(t))D\varphi(X(t))\rangle - \langle D\varphi(X(t)), f(X(t), q(t), z(t); i(t), j(t))\rangle\}\,dt = \varphi(X(0)) - \varphi(X(h))e^{-h}.$$

(2) *For* $h \in [0, \infty)$, *we have*

$$\int_0^h \langle Dg(X(t)), f(X(t), q(t), z(t); i(t), j(t))\rangle\,dt \leq g(X(h)) - g(X(0)).$$

(3) *Suppose in addition that* (B0) *and* (B1) *hold. Let* $Y(t) := Y(t; y)$ *be the trajectory corresponding to the same control* α *and same strategy* ξ *of* X *but different starting point* y, *then there exists a positive constant* C *that depends only on* C_0, κ *(the Lipschitzian modulus of* f*) and* $\|B\|$ *such that*

$$\int_0^t \|X(s) - Y(s)\|^2\,ds \leq e^{Ct}\langle B(x - y), x - y\rangle, \tag{3.3}$$

$$\langle B(X(t) - Y(t)), X(t) - Y(t)\rangle \leq e^{Ct}\langle B(x - y), x - y\rangle, \tag{3.4}$$

and

$$t\|X(t) - Y(t)\|^2 \leq \langle B(x - y), x - y\rangle e^{2\kappa t}\left\{1 + \int_0^t Ce^{(C-2\kappa)s}\,ds\right\} \tag{3.5}$$

for each $t \in [0, \infty)$.

PROOF. (1) We denote the semigroup of linear operators generated by $A(i, j)$ by $e^{-A(i,j)t}$ for $t > 0$. We want to show that

$$\int_0^h e^{-t}\{\varphi(X(t)) + \langle X(t), A^*(i(t), j(t))D\varphi(X(t))\rangle \\ - \langle D\varphi(X(t)), f(X(t), q(t), z(t); i(t), j(t))\rangle\}\, dt = \varphi(X(0)) - \varphi(X(h))e^{-h},$$

where

$$X(t) = X(0) + \int_0^t e^{-A(i(s),j(s))(t-s)} f(X(s), q(s), z(s); i(s), j(s))\, ds,$$

for $t \in [0, \infty)$. The assertion is true if each $A(i, j)$ is a linear bounded operator defined on all of H because for almost all $t \in [0, \infty)$,

$$\begin{aligned} -\frac{d}{dt}[e^{-t}\varphi(X(t))] &= e^{-t}[\varphi(X(t)) - \langle \frac{d}{dt}X(t), D\varphi(X(t))\rangle] \\ &= e^{-t}[\varphi(X(t)) + \langle A(i(t), j(t))X(t), D\varphi(X(t))\rangle \\ &\quad - \langle f(X(t), q(t), z(t); i(t), j(t)), D\varphi(X(t))\rangle] \end{aligned}$$

and an integration over $[0, h]$ proves the result.

To prove the case when $A(i, j)$ is unbounded, we consider the Yosida approximation of $A(i, j)$, $A_\lambda(i, j) := \lambda^{-1}(I + \lambda A(i, j))^{-1}$, which is a bounded operator for all $\lambda > 0$. If we let

$$X_\lambda(t) := X_\lambda(0) + \int_0^t e^{-A_\lambda(i(s),j(s))(t-s)} f(X_\lambda(s), q(s), z(s); i(s), j(s))ds,$$

then the previous argument holds and

$$\int_0^h e^{-t}\{\varphi(X_\lambda(t)) + \langle X_\lambda(t), (A_\lambda^*(i(t), j(t)))D\varphi(X_\lambda(t))\rangle \\ - \langle D\varphi(X_\lambda(t)), f(X_\lambda(t), q(t), z(t); i(t), j(t))\rangle\}\, dt = \varphi(X_\lambda(0)) - \varphi(X_\lambda(h))e^{-h}.$$

Let $\lambda \downarrow 0$ and then using routine estimations, we can conclude (1).

(2) We may use the Yosida approximation similarly. Just as in (1), we can assume that each $A(i, j)$ is bounded and defined on all H since we can always replace $A(i, j)$ by $A_\lambda(i, j)$ and let $\lambda \downarrow 0$. Letting

$$X(t) = X(0) + \int_0^t e^{-A(i(s),j(s))(t-s)} f(X(s), q(s), z(s); i(s), j(s))\, ds,$$

we then have

$$\begin{aligned} -\frac{d}{dt}[e^{-t}g(X(t))] =& e^{-t}[g(X(t)) + \langle A(i(t), j(t))X(t), Dg(X(t))\rangle \\ & - \langle f(X(t), q(t), z(t); i(t), j(t)), Dg(X(t))\rangle] \end{aligned}$$

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almost everywhere on $[0, \infty)$. Therefore, the claim follows from an integration over $[0, h]$ and the fact that $0 \le \langle A(i(t), j(t))X(t), Dg(X(t))\rangle$.

(3) We begin with the proof of (3.3) and (3.4) and keep in mind that we may treat all the monotone operators involved as bounded operators by using Yosida's approximation as before. Proceed as in [5], we differentiate $\langle B(X(s) - Y(s)), X(s) - Y(s)\rangle$. From (B1) and the Lipschitz continuity of g, we obtain

$$\begin{aligned}\frac{d}{dt}&\langle B(X(t) - Y(t)), X(t) - Y(t)\rangle\\ &\le 2C_0\langle B(X(t) - Y(t)), X(t) - Y(t)\rangle - 2\|X(t) - Y(t)\|^2\\ &\quad + 2\kappa\|B(X(t) - Y(t))\|\|X(t) - Y(t)\|\\ &\le C\langle B(X(t) - Y(t)), X(t) - Y(t)\rangle - \|X(t) - Y(t)\|^2\end{aligned}$$

for some positive constant C. In the above, we have also used the inequality $\|Bz\|^2 \le \|B\|\langle Bz, z\rangle$ for $z \in H$. Multiplying by an integrating factor, we have

$$\frac{d}{dt}\left\{e^{-Ct}\langle B(X(t) - Y(t)), X(t) - Y(t)\rangle\right\} \le -e^{-Ct}\|X(t) - Y(t)\|^2$$

and

$$\begin{aligned}e^{-Ct}\int_0^t \|X(s) - Y(s)\|^2\,ds + e^{-Ct}\langle B(X(t) - Y(t)), X(t) - Y(t)\rangle\\ \le \langle B(x - y), x - y\rangle\end{aligned}$$

Splitting it into two inequalities gives (3.3) and (3.4).

To show (3.5), we differentiate and then integrate $\langle X(\sigma) - Y(\sigma), X(\sigma) - Y(\sigma)\rangle$ on $(s, t) \subset [0, \infty)$ and conclude that

$$\|X(t) - Y(t)\|^2 \le \|X(s) - Y(s)\|^2 + 2\kappa\int_s^t \|X(\sigma) - Y(\sigma)\|^2\,d\sigma$$

via the monotonicity of $A(i, j)$'s and the Lipschitz continuity of f. Integrating both sides with respect to s on $(0, t)$ gives

$$t\|X(t) - Y(t)\|^2 \le \int_0^t \|X(s) - Y(s)\|^2\,ds + 2\kappa\int_0^t\int_s^t \|X(\sigma) - Y(\sigma)\|^2\,d\sigma ds.$$

Making use of (3.3) and interchanging the order of integration, we have

$$t\|X(t) - Y(t)\|^2 \le \langle B(x - y), x - y\rangle e^{Ct} + 2\kappa\int_0^t \sigma\|X(\sigma) - Y(\sigma)\|^2\,d\sigma.$$

By treating this condition as a generalized Gronwall-type inequality (see [8, p.4]), we conclude that

$$t\|X(t) - Y(t)\|^2 \le \langle B(x - y), x - y\rangle e^{2\kappa t}\left\{1 + \int_0^t Ce^{(C-2\kappa)s}ds\right\}.$$

We now prove the main results of this section by starting with the infinite horizon case first.

THEOREM 2. *Suppose that* (A), (B0) *and* (1.2) *hold. Then, the value function* u *given by* (2.1) *is a bounded Lipschitz continuous viscosity solution of the stationary Hamilton-Jacobi-Isaacs equation*

$$\begin{aligned} u(x) + \sup_{i\in I}\inf_{j\in J}\Bigg\{&\langle A(i,j)x, Du(x)\rangle \\ &+ \sup_{q\in Q_i}\inf_{z\in Z_j}\{-\langle Du(x), f(x,q,z;i,j)\rangle - L(x,q,z;i,j)\}\Bigg\} = 0. \end{aligned} \tag{3.6}$$

Moreover, if (B0) *and* (B1) *hold, then* u *is* B*-continuous on* H *and can be extended so that* u *is bounded and uniformly continuous on* H_{-1}.

If (B0) *and* $(\mathrm{B1})_w$ *hold instead of* (B0) *and* (B1), *then* u *is continuous in the norm of* H_{-1} *on every bounded subset of* H *and hence,* B*-continuous on* H. *Furthermore,* u *can be extended to be bounded and uniformly continuous on* H_{-1} *if we assume in addition that there exists* $\kappa \geq 0$ *such that for each* $i \in I$, $j \in J$, $q \in Q_i$ *and* $z \in Z_j$,

$$|L(x,q,z;i,j) - L(y,q,z;i,j)| \leq \kappa\|x-y\|_{-1} \tag{3.7}$$

for all $x,\ y \in H$.

PROOF. It is easy to see that u is bounded and Lipschitz continuous. To show that u is a viscosity solution, we first show that $u(x)$ is a supersolution by contradiction. Assume that $u(a) + \varphi(a) + g(a) \leq u(x) + \varphi(x) + g(x)$ in a neighborhood of a, or even further we can assume that $\varphi(a) = -u(a)$, $g(a) = 0$ and $\varphi(x) + g(x) \geq -u(x)$ near a. Suppose that there exists $\theta > 0$ such that

$$\begin{aligned} u(a) + \sup_{i\in I}\inf_{j\in J}\Big\{&\langle A(i,j)a, -D\varphi(a)\rangle \\ &+ \sup_{q\in Q_i}\inf_{z\in Z_j}\{\langle D\varphi(a) + Dg(a), f(a,q,z;i,j)\rangle - L(a,q,z;i,j))\}\Big\} < -4\theta. \end{aligned}$$

Thus, for each $i \in I$ there exists $j = j_i \in J$ such that

$$\begin{aligned} &-\varphi(a) - g(a) - \langle A(i,j_i)a, D\varphi(a)\rangle \\ &\quad + \inf_{z\in Z_j}\{\langle D\varphi(a) + Dg(a), f(a,q,z;i,j_i)\rangle - L(a,q,z;i,j_i))\} < -3\theta \end{aligned}$$

for all $q \in Q_i$. By continuity of L, for each $i \in I$, there exists $z \in Z_j$ (z depends on i) and there exists an open set containing q, such that for all p in that set of q,

$$\begin{aligned} &\varphi(a) + g(a) + \langle A(i,j_i)a, D\varphi(a)\rangle \\ &\quad - \langle D\varphi(a) + Dg(a), f(a,p,z;i,j_i)\rangle + L(a,p,z;i,j_i) > 2\theta. \end{aligned}$$

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Engaging a partition of unity argument, we can find, for each $i \in I$, a continuous function $\eta^i : Q_i \to Q_i$ so that for all $q \in Q_i$,

$$\varphi(a) + g(a) + \langle A(i, j_i)a, D\varphi(a)\rangle - \langle D\varphi(a) + Dg(a), f(a, q, \eta^i(q); i, j_i)\rangle + L(a, q, \eta^i(q); i, j_i) > 2\theta. \tag{3.8}$$

Given any control $\alpha = (q, N, \{(r_n, i_n)\}_{n=1}^N) \in \mathscr{C}$, define a new strategy $\hat{\xi} \in \mathscr{B}$ with $\hat{\xi}(\alpha) = (\hat{z}(t), \hat{M}, \Omega_{\hat{M}})$ as follows:

(i) for $n = 1, \cdots, N$ and $t \in [r_n, r_{n+1})$, let $\hat{z}(t) := \eta^{i_n}(q(t))$;
(ii) let $\hat{M} := N$ and for $m = 1, \cdots, \hat{M}$, let $(\hat{s}_m, \hat{j}_m) := (r_m, j_{i_m})$;
(iii) let $\Omega_{\hat{M}} := \{(\hat{s}_m, \hat{j}_m) : m = 1, \cdots, \hat{M}\}$.

Let $X(t)$ be the trajectory corresponding to control $\alpha = (q, N, \Delta_N)$, strategy $\hat{\xi}$ and starting point a. Then, since f is uniformly continuous and estimates (1.3) of $X(t)$ is independent of α, there exists $h > 0$, so that for all $t \in [0, h]$,

$$\begin{aligned} e^{-t}\{&\varphi(X(t)) + g(X(t)) + \langle X(t), A^*(i(t), j_{i(t)})D\varphi(X(t))\rangle \\ &- \langle D\varphi(X(t)) + Dg(X(t)), f(X(t), q(t), \hat{z}(t); i(t), j_{i(t)})\rangle \\ &- L(X(t), q(t), \hat{z}(t); i(t), j_{i(t)})\} > \theta. \end{aligned} \tag{3.9}$$

From (1) of Lemma 2, we conclude that for all $h > 0$

$$\int_0^h e^{-t}\{\varphi(X(t)) + \langle X(t), A^*(i(t), \hat{j}_{i(t)})D\varphi(X(t))\rangle - \langle D\varphi(X(t)), f(X(t), q(t), \hat{z}(t); i(t), j_{i(t)})\rangle\}\, dt = \varphi(X(0)) - \varphi(X(h))e^{-h}.$$

This and an integration of (3.9) over $[0, h]$ yield

$$-e^{-h}\{\varphi(X(h)) + g(X(h))\} + \varphi(a) + \int_0^h e^{-t}L(X(t), \alpha(t), \hat{\xi}(\alpha)(t))dt + G(h) > h\theta$$

with

$$G(h) := g(X(h))e^{-h} + \int_0^h e^{-t}\{g(X(t)) - \langle Dg(X(t)), f(X(t), \alpha(t), \hat{\xi}(\alpha)(t))\rangle\}\, dt.$$

From (2) of Lemma 2, we know that $G(h) \le g(a) = 0$. Hence, for all control $\alpha = (q, N, \Delta_N)$,

$$\begin{aligned} -u(a) &= \varphi(a) \\ &> e^{-h}\{\varphi(X(h)) + g(X(h))\} - \int_0^h e^{-t}L(X(t), \alpha(t), \hat{\xi}(\alpha)(t))dt + h\theta \\ &\ge -e^{-h}u(X(h)) - \int_0^h e^{-t}L(X(t), \alpha(t), \hat{\xi}(\alpha)(t))dt + h\theta \end{aligned}$$

$$u(a) < \sup_{\xi \in \mathscr{B}} \inf_{\alpha \in \mathscr{C}}\{\int_0^h L(X(t), \alpha(t), \xi(\alpha)(t))e^{-t}dt + u(X(h))e^{-h}\} = u(a)$$

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due to the dynamic programming identity and we have a contradiction!

We next show that $u(x)$ is a subsolution by a similar contradiction. Assume that $u(a) - \varphi(a) - g(a) \geq u(x) - \varphi(x) - g(x)$ in a neighbourhood of a and $\varphi(a) = u(a)$, $g(a) = 0$, so $\varphi(x) + g(x) \geq u(x)$ near a. Suppose that there exists $\theta > 0$ such that

$$u(a) + \sup_{i\in I}\inf_{j\in J}\Big\{\langle A(i,j)a, D\varphi(a)\rangle + \sup_{q\in Q_i}\inf_{z\in Z_j}\{-\langle D\varphi(a) + Dg(a), f(a,q,z;i,j)\rangle$$
$$- L(a,q,z;i,j)\}\Big\} > 3\theta.$$

Hence there exists $\hat{i} \in I$ and $p \in Q_i$ so that for any $j \in J$ and any $z \in Z_j$, we have

$$\varphi(a) + g(a) + \langle A(\hat{i},j)a, D\varphi(a)\rangle$$
$$- \langle D\varphi(a) + Dg(a), f(a,p,z;\hat{i},j)\rangle - L(a,p,z;\hat{i},j) > 2\theta.$$

Let $\hat{\alpha} = (\hat{q}, 1, \{(0,\hat{i})\})$ with $\hat{q}(t) \equiv p$ and let $X(t)$ be the trajectory corresponding to control $\hat{\alpha}$, strategy ξ and starting point a. Let $\xi(\hat{\alpha}) := (z(t), M, \Omega_M)$ with $\Omega_M := \{(s_m, j_m) : m = 1, \cdots, M\}$. Then, there exists $h > 0$ so that $[0,h] \subset [0, s_2)$ and

$$\begin{aligned} e^{-t}\{&\varphi(X(t)) + g(X(t)) + \langle X(t), A^*(\hat{i}, j_2)D\varphi(X(t))\rangle \\ &- \langle D\varphi(X(t)) + Dg(X(t)), f(X(t), \hat{q}(t), z(t); \hat{i}, j_2)\rangle \\ &- L(X(t), \hat{q}(t), z(t); \hat{i}, j_2)\} > \theta \end{aligned} \tag{3.10}$$

for all $t \in [0,h]$. Integrating (3.10) over $[0,h]$ and using Lemma 2 again as in the supersolution case, we conclude that for all strategy $\xi \in \mathscr{B}$,

$$\begin{aligned} u(a) &= \varphi(a) \\ &> e^{-h}\{\varphi(X(h)) + g(X(h))\} + \int_0^h e^{-t}L(X(t), \hat{\alpha}(t), \xi(\hat{\alpha})(t))dt + h\theta \\ &\geq e^{-h}u(X(h)) + \int_0^h e^{-t}L(X(t), \hat{\alpha}(t), \xi(\hat{\alpha})(t))dt + h\theta \\ u(a) &> \sup_{\xi\in\mathscr{B}}\inf_{\alpha\in\mathscr{C}}\{\int_0^h L(X(t), \alpha(t), \xi(\alpha)(t))e^{-t}dt + u(X(h))e^{-h}\} = u(a), \end{aligned}$$

hence, u is a subsolution.

Next, we want to show that u is B-continuous on H assuming, in addition, that (B1) holds. Let x, $y \in H$ be given and let $X(s;x)$ and $Y(s;y)$ respectively be the trajectory corresponding to a control $\alpha \in \mathscr{C}$, a strategy $\xi \in \mathscr{B}$ but different starting

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point x and y respectively. From the definition of u and (1.2), there exists constant κ, $M > 0$, such that

$$|u(x) - u(y)| \leq \kappa \sup_{\xi \in \mathscr{B}} \sup_{\alpha \in \mathscr{C}} \left\{ \int_0^t \|X(s; x) - Y(s; y)\| e^{-s}\, ds + \kappa M e^{-t} \right\}$$
$$\leq \frac{\kappa}{2} \sup_{\xi \in \mathscr{B}} \sup_{\alpha \in \mathscr{C}} \left[\int_0^t \|X(s; x) - Y(s; y)\|^2 ds]^{1/2} + \kappa M e^{-t} \right]. \tag{3.11}$$

Notice that the second term on the right hand side would be made small by choosing t large and for this chosen t, the first term would also be small if $\langle B(x - y), x - y\rangle$ is small enough due to (3.3) of Lemma 2. Thus, u is B-continuous on H and can be extended to a bounded uniformly continuous function in H_{-1}.

In the case that $(\mathrm{B1})_w$ holds rather than (B1), we can argue similarly. By replacing each $A(i, j)$ by its Yosida approximation $A_\lambda(i, j)$, the $\|X - Y\|$ terms inside the supremum of (3.11) can be approximated arbitrarily by $[\int_0^t \|X_\lambda(s; x) - Y_\lambda(s; y)\|^2 ds]^{\frac{1}{2}}$. We also notice that each $A_\lambda(i, j)$ also satisfies $(\mathrm{B1})_w$ with a different constant :

$$\langle (A_\lambda^*(i, j)B + \frac{C_0}{1 - \lambda} B)x, x\rangle \geq 0.$$

So we may assume that all the maximal monotone operators $A(i, j)$ that generate the trajectories in (3.11) are bound operators. On the other hand, for each $n > 0$ and each $i \in I$, we have

$$\left\langle \left(A^*(i, j) \left(B + \frac{Id}{n} \right) + C_0 \left(B + \frac{Id}{n} \right) \right) x, x \right\rangle \geq \frac{C_0}{n} \|x\|^2$$

for all $x \in H$. So, putting all these together, we can go through the estimations as in (3.5) to conclude that there exists $C \geq 0$ such that for all $\lambda > 0$,

$$\int_0^t \|X_\lambda(s; x) - Y_\lambda(s; y)\|^2\, ds \leq e^{Ct} \left[\langle B(x - y), x - y\rangle + \tfrac{1}{n} \|x - y\|^2 \right].$$

Thus, u is continuous in the norm of H_{-1} on every bounded subset of H and hence, B-continuous on H. The last assertion follows easily from (3.7).

The corresponding result for the finite horizon case is as follows.

THEOREM 3. *Let* $T > 0$. *Suppose that* (A), (1.1) *and* (1.2) *hold. Then the value function* u *given by* (2.2) *is bounded, Lipschitz continuous in* x *uniformly in* t, *uniformly continuous in* t *uniformly for bounded* x. *Moreover,* u *is a viscosity solution on*

$[0, T) \times H$ *of*

$$-u_t(t,x) + \sup_{i\in I}\inf_{j\in J}\Big\{\langle A(i,j)x, Du(t,x)\rangle$$
$$+ \sup_{q\in Q_i}\inf_{z\in Z_j}\{-\langle Du(t,x), f(t,x,q,z;i,j)\rangle - L(t,x,q,z;i,j)\}\Big\} = 0 \quad (3.12)$$

and for each bounded subset $S \subset H$,

$$\lim_{t\uparrow T}\Big(\sup_{x\in S}\{|u(t,x) - \psi(x)|\}\Big) = 0. \quad (3.13)$$

In addition, if (B0) *and* (B1) *holds, then* u *is B-continuous in* $[0, T) \times H$ *and for each* $t \in [0, T)$, $u(t, \cdot)$ *is uniformly continuous in* H *under the* H_{-1} *norm with the same modulus of continuity uniformly in* t.

If $(\mathrm{B1})_w$ *holds rather than* (B1), *then for each* $t \in [0, T)$, $u(t, \cdot)$ *is uniformly continuous in the norm of* H_{-1} *on every bounded subset of* H *with modulus uniformly in* t *and hence, B-continuous on* $[0, T) \times H$. *Furthermore, if we assume in addition that there exists* $\kappa \geq 0$ *such that for each* $i \in I$, $j \in J$, $q \in Q_i$, $z \in Z_j$ *and each* $t \in [0, T]$,

$$|L(t,x,q,z;i,j) - L(t,y,q,z;i,j)| \leq \kappa\|x - y\|_{-1} \quad (3.14)$$

for all x, $y \in H$, *then* u *can be extended to a bounded and uniformly continuous on* $[0, T) \times H_{-1}$.

PROOF. First, from (1.1), (1.3) and (1.2), it follows that u is bounded, Lipschitz continuous in x uniformly in t, uniformly continuous in t uniformly for bounded x and satisfies the terminal condition (3.13). The verification of the assertion that u is a viscosity solution of (3.12) is similar to that of Theorem 2 and is so omitted.

To show that u is B-continuous, let x, $y \in H$ and let $X(s;x)$ and $Y(s;y)$ respectively be the trajectory corresponding to a control $\alpha \in \mathscr{C}[t, T]$ and strategy $\xi \in \mathscr{B}[t, T]$ but different starting point x and y respectively. Since $u(\cdot, x)$ is Lipschitz continuous, it suffices to show that $u(t, \cdot)$ is B-continuous on H for each $t \in [0, T)$. From the definition of u and (1.2), there exists constant $\kappa > 0$, such that

$$|u(t,x) - u(t,y)| \leq \quad (3.15)$$
$$\kappa \sup_{\xi\in\mathscr{B}[t,T]}\sup_{\alpha\in\mathscr{C}[t,T]}\left\{\int_t^T \|X(s;x) - Y(s;y)\|ds + \|X(T;x) - Y(T;y)\|\right\}.$$

In view of (3.5), we conclude that u is B-continuous in $[0, T) \times H$. Indeed, for each $t \in [0, T)$, $u(t, \cdot)$ is uniformly continuous in H under the H_{-1} norm with the same modulus of continuity uniformly in t.

In the case that $(\mathrm{B1})_w$ holds rather than (B1), we can argue similarly as in Theorem 2.

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REMARK 1. Notice that all our assumptions and definitions ((A), (B0), (B1), $(B1)_w$, (1.2), ...) are direct modifications (uniform in i and j) of those in [5]. So the comparison principles in [5] are directly applicable to our Hamilton-Jacobi-Isaacs equations. Thus, the upper value function u given by (2.1) (and respectively (2.2)) is the unique viscosity solution of (3.6) (and respectively (3.12) that satisfies the terminal condition (3.13)).

In fact, even more is true. The proof of the comparison principles in [5] can be modified (as pointed out in [5]) to yield convergence theorems – namely that if the Hamiltonian of the Hamilton-Jacobi-Isaacs equation is approximated locally uniformly on bounded sets by regularized Hamiltonians, each of which satisfies some structural hypotheses ((F0), (F1), (F2), (F3), (F4) in [5]), then the viscosity solutions corresponding to these regularized Hamiltonians will converge to a viscosity solution of the original Hamilton-Jacobi-Isaacs equation. This convergence result can be used to extend our results obtained so far to more general differential games problems. We shall further discuss this in Section 4.

REMARK 2. Although our emphases are on player 1, results for player 2 can be deduced from those of player 1 if we replace the cost functions L and ψ by $-L$ and $-\psi$. We summarize them as follows (using the short-handed notation in Section 2).

(1) The lower Elliott-Kalton value of the infinite (and respectively finite) horizon differential game problem is defined as

$$v(x) = \inf_{\zeta\in\mathscr{A}} \sup_{\beta\in\mathscr{D}} \int_0^\infty L(X(s;x), \zeta(\beta)(s), \beta(s))e^{-s}\,ds$$

(and respectively

$$v(t,x) = \inf_{\zeta\in\mathscr{A}[t,T]} \sup_{\beta\in\mathscr{D}[t,T]} \int_t^T L(s, X(s;x), \zeta(\beta)(s), \beta(s))\,ds + \psi(X(T;x))).$$

(2) They satisfy the following dynamic programming identities respectively:

$$v(x) = \inf_{\zeta\in\mathscr{A}} \sup_{\beta\in\mathscr{D}} \left\{ \int_0^h L(X(s;x), \zeta(\beta)(s), \beta(s))e^{-s}\,ds + u(X(h;x))e^{-h} \right\}, \quad h \geq 0,$$

$$v(t,x) = \inf_{\zeta\in\mathscr{A}[t,T]} \sup_{\beta\in\mathscr{D}[t,T]} \left\{ \int_t^{t+h} L(s, X(s;x), \zeta(\beta)(s), \beta(s))\,ds + u(t+h, X(t+h;x)) \right\}, \quad 0 \leq h \leq T-t.$$

(3) They are viscosity solutions of respectively

$$v(x) + \inf_{j\in J}\sup_{i\in I}\Big\{\langle A(i,j)x, Dv(x)\rangle + \inf_{z\in Z_j}\sup_{q\in Q_i}\{-\langle Dv(x), f(x,q,z;i,j)\rangle - L(x,q,z;i,j)\}\Big\} = 0,\ x\in H,$$

$$-v_t(t,x) + \inf_{j\in J}\sup_{i\in I}\Big\{\langle A(i,j)x, Dv(t,x)\rangle + \inf_{z\in Z_j}\sup_{q\in Q_i}\{-\langle Dv(t,x), f(t,x,q,z;i,j)\rangle - L(t,x,q,z;i,j)\}\Big\} = 0, \quad (t,x)\in[0,T)\times H.$$

In view of the uniqueness results of viscosity solution in [5], we may deduce the existence of the Elliott-Kalton value of the differential game by assuming that

$$A^*(i,j) \text{ has a common domain which is denoted by } \mathscr{S}, \tag{3.16}$$

and

$$\sup_{i\in I}\inf_{j\in J}\Big\{\langle x, A^*(i,j)p\rangle + \sup_{q\in Q_i}\inf_{z\in Z_j}\{-\langle p, f(t,x,q,z;i,j)\rangle - L(t,x,q,z;i,j)\}\Big\} = \inf_{j\in J}\sup_{i\in I}\Big\{\langle x, A^*(i,j)p\rangle + \inf_{z\in Z_j}\sup_{q\in Q_i}\{-\langle p, f(t,x,q,z;i,j)\rangle - L(t,x,q,z;i,j)\}\Big\} \tag{3.17}$$

for all $x\in H$, $p\in\mathscr{S}$ and $t\in[0,+\infty)$ (or $t\in[0,T]$).

Condition (3.17) is analogous to the classical one. Condition (3.16) is necessary for the postulation of (3.17) but not too restrictive in view of the two examples in Section 3.

THEOREM 4. *Suppose that* (A), (3.16), (B0), (1.1), (1.2) *and* (3.17) *hold and let* $0 < t < T$. *If* (B1) *or* $(\mathrm{B1})_w$ *and* (3.14) *hold, then* $u(x) = v(x)$ *for all* $x\in H$ *and* $u(t,x) = v(t,x)$ *for all* $(t,x)\in[0,T]\times H$.

4. Relaxation of the structural hypotheses

In order to present our result in a more direct way, the structural assumptions (1.1) and (1.2) of the problem are postulated in a simple way and may be too restrictive to include some practical applications. These structural assumptions may be relaxed further without affecting the validity of our results. Roughly speaking, any structural assumptions that the differential game problem makes sense on, our results will still hold. To be more precise, whenever the value function is well-defined and the dynamic programming identity hold under a set of structural assumptions, the rest of our results would hold on them. As an example, we take out two sets of assumptions from [5] and use them to illustrate how our results could be extended. They are as follows.

(1) There exist positive constants $\kappa_0, \kappa_1, \kappa_2$ and $m \in [0, \kappa_0^{-1})$ such that

$$\begin{cases} |f(t,x,q,z;i,j) - f(t,y,q,z;i,j)| \leq \kappa_0 \|x-y\| , \\ |f(t,0,q,z;i,j)| \leq \kappa_1 , \\ |L(t,x,q,z;i,j)| \leq \kappa_2 (1+\|x\|)^m , \end{cases}$$

$\forall i \in I, \forall j \in J, \forall q \in Q_i, \forall z \in Z_j, \forall x,\ y \in H$ and $\forall t$.

(2) There exist positive constants $\kappa_0, \kappa_1, \kappa_2$ and $m \in [0, \kappa_2^{-1})$ such that

$$\begin{cases} |f(t,x,q,z;i,j) - f(t,y,q,z;i,j)| \leq \kappa_0 \|x-y\| , \\ |f(t,x,q,z;i,j)| \leq \kappa_1 , \\ |L(t,x,q,z;i,j)| \leq \kappa_2 e^{m\|x\|} , \end{cases}$$

$\forall i \in I, \forall j \in J, \forall q \in Q_i, \forall z \in Z_j, \forall x,\ y \in H$ and $\forall t$.

Both the finite and infinite horizon problems make sense on these assumptions and the dynamic programming identities hold (see details in [5]).

A way of extending our results to these structural assumptions is to approximate f, L and ψ by more regular ones until they fit the structural hypotheses (1.1) and (1.2) in our setting, and we can deduce the desired results on these regularized functions. The results would then be extended to that of the original setting via some convergence theorems together with a stability result saying that limit of viscosity solutions is a viscosity solution of the limiting problem. These convergence theorems can be proved by modifying the proof of the comparison principle in [5] and indeed, had been remarked in [5]. Readers who are interested in the detailed modifications may follow the comments in [5] or consult [18, 19] (where the whole procedure is exhibited but will double the size of this paper if included here).

Some regularizations on f that we could use in extending our results to other structural assumptions are (with $n \in \mathbb{N}$):

(1) $f_n := \max\{\min\{f, n\}, -n\}$. It transforms an unbounded f to a bounded f_n.

(2) $f_n(t,x,u,p) := \chi_n(\|x\|)\chi_n(|\xi|)\chi_n(\|p\|) f(t,x,u,p)$ where χ_n is a cutoff function satisfying $\chi_n(r) = \chi(\frac{r}{n})$ with $\chi \in C_c^\infty(\mathbb{R}),\ 0 \leq \chi \leq 1,\ \chi(r) \equiv 1$ for $|r| \leq 1$. It transforms a bounded locally uniformly continuous f to a bounded uniformly continuous f_n.

(3) $f_n(t,x,u,p) := \inf\{f(s,y,v,q) + n|s-t| + n\|x-y\| + n|u-v| + n\|p-q\|$

$$: (s,y,v,q) \in [t_0, T] \times H \times \mathbb{R} \times H\} .$$

It transforms a bounded uniformly continuous f to a Lipschitz continuous f_n with Lipschitzian modulus n.

These regularizations can also be applied on L and ψ. Suppose f, L and ψ are regularized in one of the above ways by a sequence of f_n, L_n and ψ_n. Let F be the sup-inf term over Q_i and Z_j in (3.12) and let F_n be that of f_n and L_n. Then, each F_n will satisfy hypotheses (F0) – (F4) (see [5] or [18, 19] for their statements) with the same parameters as those of F, and $F_n \to F$ as well as $\psi_n \to \psi$ uniformly on bounded sets as $n \to +\infty$. Thus, the convergence theorems in [5] (or [18, 19]) can be applied and we have a limit of viscosity solutions. This limit is the viscosity solution of the unregularized problem due to the following stability result quoted from [5] with slight modifications, and proofs there can easily be modified to the present cases. We state only that the finite horizon case and the infinite horizon case can be obtained by regarding every involved function to be independent of t.

PROPOSITION 1. *Suppose that* (A) *and* (B0) *hold and let* $\{u_n\}_{n=1}^{+\infty}$ *be a sequence of* B*-continuous viscosity subsolutions* (*respectively supersolutions*) *on* $[0, T) \times H$ *of*

$$-u_t(t, x) + \sup_{i\in I} \inf_{j\in J} \Big\{ \langle A(i, j)x, Du(t, x)\rangle + F_n\big(t, x, u(t, x), Du(t, x); i, j\big)\Big\} = 0$$

where $F_n(\cdot; i, j) \in C([0, T] \times H \times \mathbb{R} \times H)$ *for each* $i \in I$ *and* $j \in J$. *Let* $F(\cdot; i, j) \in C([0, T] \times H \times \mathbb{R} \times H)$ *and let* $u \in C([0, T] \times H)$ *be a* B*-continuous function on* $[0, T) \times H$ *such that*

$$F_n(\cdot; i, j) \to F(\cdot; i, j) \text{ locally uniformly in } [0, T] \times H \times \mathbb{R} \times H \\ \text{and uniformly in } i \in I$$

and

$$\forall x \in H,\ \exists \rho > 0,\ \text{such that } u_n \to u \text{ uniformly on } [0, T] \times B(x, \rho).$$

Then u *is a viscosity subsolution* (*respectively, supersolution*) *of*

$$-u_t(t, x) + \sup_{i\in I} \inf_{j\in J} \Big\{ \langle A(i, j)x, Du(t, x)\rangle + F\big(t, x, u(t, x), Du(t, x); i, j\big)\Big\} = 0$$

on $[0, T) \times H$.

REMARK 3. Finally, let us compare the results of J. Yong in [14–17] with ours. The problems considered there have positive switching costs but L and f have no dependence in z and q. Infinite and finite horizon problems in finite dimensional Euclidean space were studied in [14, 15] and differential games with impulse controls were

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studied in [17]. We will narrow our comparison down to [16] since the results there (almost) cover those in [14, 15] and is closer to our framework than [17].

The underlining space in [16] is a Banach space and only infinite horizon problems were studied. The main result is the existence of the value to the differential game, which was shown by first regularizing the differential game, showing existence of the value for the regularized game and then establishing the convergence of the regularized game value to the value of the original game. In doing so, a "finite-loop" condition (condition (G5) in [16]) was initially assumed and then eliminated. So in the end, the uniqueness of the value was established without assuming any condition. However, as Yong pointed out in [16], the "finite-loop" condition cannot be eliminated for the zero switching cost case (i.e. our case) unless the restriction on the choices of partition on $[0, \infty)$ was put on the control function, and this restriction makes the game unnatural. In other words, the existence of value holds without assuming any Isaacs-type condition for the differential games with positive switching, but does not hold for the zero switching cost limiting case unless restrictions are put on the switching time of the control functions.

On the other hand, our working space is a Hilbert space and we need to assume an Isaacs-type condition in order to establish the existence of the value for the game of zero switching cost. So in this sense, the uniqueness result that we obtain here may be viewed as a supplement for those in [14–17] for the zero-switching-cost cases.

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References

[1] N. U. Ahmed and K. L. Teo, *Optimal control of Distributed Parameter Systems* (North Holland, Amsterdam, 1981).

[2] M. G. Crandall and P. L. Lions, "Condition d'unicité pour les solutions generalisées des équations de Hamilton-Jacobi du premier ordre", *C.R. Acad. Sci. Paris* **292** (1981) 183–186.

[3] M. G. Crandall and P. L. Lions, "Viscosity solutions of Hamilton-Jacobi equations", *Trans. Amer. Math. Soc.* **277** (1983) 1–42.

[4] M. G. Crandall and P. L. Lions, "Hamilton-Jacobi equations in infinite dimensions: Part IV. Unbounded linear terms", *J. Func. Anal.* **90** (1990) 237–283.

[5] M. G. Crandall and P. L. Lions, "Hamilton-Jacobi equations in infinite dimensions: Part V. *B*-continuous solutions", *J. Func. Anal.*, **7** (1991) 417-465.

[6] R. J. Elliott and N. J. Kalton, "The existence of value in differential games", *Mem. Amer. Math. Soc.* **126** (1972) 1–67.

[7] A. Friedman, *Differential games* (John Wiley, New York, 1971).

[8] B. Z. Guo and Z. H. Luo, "Initial-boundary value problem and exponential decay for a flexible beam vibration with gain adaptive direct strain feedback control", to appear in *Nonlinear Analysis*.

[9] N. N. Krasovskii and A. I. Subbotin, *Game-theoretical control problems* (Springer-Verlag, New York, 1988).

[10] P. L. Lions, "Une inegalite pour les operateurs elliptiques du second ordre", *Ann. Mat. Pura Appl.* **127** (1981) 1–11.

[11] A. J. van der Schaft, "Nonlinear state space H^∞ control theory" in *Essays on control: perspectives in the theory and its applications*, Ed. H. L. Trentelman and J. C. Willems (Birkh'auser, Boston, Basel, Berlin, 1993).

[12] P. E. Sobolevsky, "Equations with operators constituting an acute angle", *Dokl. Akad. Nauk USSR* **116** (1957) 754–757 [In Russian].

[13] J. Song and J. Y. Yu, *Population Control theory* (Springer-Verlag, New York, 1987).

[14] J. Yong, "A zero-sum differential game in a finite duration with switching strategies", *SIAM J. Control and Optim.* **28** (1990) 1234–1250.

[15] J. Yong, "Differential games with switching strategies", *J. of Math. Anal. Appl.* **145** (1990) 455–467.

[16] J. Yong, "Existence of the value for a differential game with switching strategies in a Banach space", *System Sci. & Math.* **4** (1991) 321–340.

[17] J. Yong, "Zero-sum differential games involving impulse controls", *App. Math. Optim.* **29** (1994) 243–261.

[18] S. P. Yung, "Results on infinite dimensional Hamilton-Jacobi equations", Ph.D. thesis, University of Wisconsin-Madison (1991).

[19] S. P. Yung, "Uniqueness of a Hamilton-Jacobi equation on Hilbert space", Research report HKUM-92-3, University of Hong Kong.

[20] M. I. Zelikin and V. F. Borisov, *Theory of chattering control : with applications to astronautics, robotics, economics, and engineering* (Birkhauser, Boston, 1994).