SOME INTEGRAL EQUATIONS INVOLVING FINITE PARTS OF DIVERGENT INTEGRALS

by A. ERDÉLYI

(Received 29 June, 1966)

1. Introduction. In recent years, a number of special integral equations of the first kind was discussed by several authors (see [1]–[4], [6], [7], [9]–[18]). The kernels of these integral equations are special functions of the hypergeometric family, and it was necessary to restrict the parameters appearing in these functions to secure convergence of the integrals. If these restrictions are removed, the integral fails to converge but it may possess a finite part (in Hadamard's sense), and the question arises whether the methods used in the restricted case will also apply in the new situation. Indeed, one could pose the more general problem of Volterra integral equations involving finite parts of divergent integrals [19].

Volterra integral equations of convolution type involving finite parts of divergent integrals have been solved by Butzer [5] by means of Mikusinski's operational calculus. The special integral equations we have in mind here are not of convolution type, but they are sufficiently closely related to convolutions to allow the application of related techniques. Presumably, very general Volterra integral equations involving finite parts of divergent integrals would require research into repeated finite part integrals; in the case of the special integral equations we have in mind here, one can use the known connection of hypergeometric functions with integration of fractional order, and the distribution theory of fractional integrals, to solve the integral equations.

In this note, one of the numerous special integral equations has been selected for detailed discussion. It is believed that the methods applied here could be used for the solution of the others.

The integral equation

$$F(g) \equiv \int_a^x (x^2-t^2)^{\lambda/2} P_\nu^{-\lambda}\left(\frac{x}{t}\right) g(t)\,dt = f(x) \quad (0 < a \leqq x < b), \tag{1}$$

where $P_\nu^{-\lambda}$ is the Legendre function, was investigated in [7]. Since $P_\nu^{-\lambda} = P_{-\nu-1}^{-\lambda}$, $\operatorname{Re}\nu \geqq \frac{1}{2}$ may be assumed without loss of generality; and the additional condition $\operatorname{Re}\lambda > -1$ was imposed in [7] in order to secure convergence of the integral in (1). If this last condition is violated, the integral will be divergent but it will possess a *finite part* (in the sense of Hadamard) if $g(t)$ is differentiable to a sufficiently high order.

It was shown in [7] that, under the conditions assumed there, (1) could be transformed into

$$I_x^{\lambda-\nu} I_{x^2}^{\nu+1}\{(2x)^{-\nu-1} g(x)\} = f(x), \tag{2}$$

where $I_{x^n}^{\alpha}$ denotes the operator of integration of (fractional) order α with respect to x^n. If $\operatorname{Re}\alpha > 0$,

$$I_{x^n}^{\alpha} f(x) = \frac{n}{\Gamma(\alpha)} \int_a^x (x^n - u^n)^{\alpha-1} f(u) u^{n-1}\, du, \tag{3}$$

while for $\operatorname{Re}\alpha < 0$, I^{α} is the inverse of $I^{-\alpha}$.

If $\operatorname{Re}\alpha \leqq 0$, and f is sufficiently smooth, the divergent integral in (3) will possess a finite part, and define the operator I^{α}. This interpretation of I^{α} for $\operatorname{Re}\alpha \leqq 0$ is consistent with $I^{\alpha} = (I^{-\alpha})^{-1}$, and we wish to show, without placing any restriction on λ that, with this interpretation, (1) and (2) are equivalent. Actually, it will be more convenient to use the theory of generalized functions (distributions) and work with pseudo-functions and the *regularization* of integrals [**8**, Chapter I, § 3].

2. Pseudo-functions and fractional integration. We shall use largely the notations and terminology of [**8**] and for the sake of convenience abbreviate " [**8**, Chapter I, § 2, section 3] " as I.2.3.

For $\operatorname{Re}\alpha > 0$,

$$p_{\alpha}(x) = \frac{x_+^{\alpha-1}}{\Gamma(\alpha)} = \begin{cases} \dfrac{x^{\alpha-1}}{\Gamma(\alpha)} & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x \leqq 0, \end{cases} \tag{4}$$

defines a locally integrable function on R which generates a (regular) distribution. This distribution is an analytic function of α and can be continued analytically to an entire function of α (I.3.2. and I.3.5.). The distribution thus defined for all α is a pseudo-function of x: if $\alpha \neq 0, -1, -2, \ldots$, the support of $p_{\alpha}(x)$ is $[0, \infty[$, and $p_{\alpha}(x) = x^{\alpha-1}/\Gamma(\alpha)$ on $]0, \infty[$; if $\alpha = -n$ and n is a non-negative integer, $p_{-n}(x) = \delta^{(n)}(x)$.

If $\operatorname{Re}\alpha > -n$, $f^{(n-1)}(x)$ exists (in the ordinary sense) and is absolutely continuous, the (possibly) divergent integral

$$\int_0^c p_{\alpha}(x) f(x)\, dx \tag{5}$$

is interpreted as

$$\int_0^c \frac{x^{\alpha-1}}{\Gamma(\alpha)} \left\{ f(x) - \sum_{m=0}^{n-1} f^{(m)}(0) \frac{x^m}{m!} \right\} dx + \sum_{m=0}^{n-1} \frac{f^{(m)}(0) c^{\alpha+m}}{m!(\alpha+m)\Gamma(\alpha)}, \tag{6}$$

and this is the finite part of (5). Similarly,

$$\int_a^b p_{\alpha}(x-t) f(x)\, dt \tag{7}$$

possesses a finite part for almost every x in $[a, b]$, if $f^{(n-1)}(t)$ exists and is absolutely continuous in $[a, b]$.

More generally, the convolution $p_{\alpha} * f$ (in the distribution sense) is well defined whenever f is a distribution whose support is bounded on the left; and

$$I_x^{\alpha} f = p_{\alpha} * f \tag{8}$$

defines, for all values of α, the integral of order α of the left-bounded distribution f (I.5.5.). The rules

$$I_x^0 f = f, \qquad I_x^\alpha(I_x^\beta f) = I_x^{\alpha+\beta} f \tag{9}$$

follow immediately. If f is an integrable function and $\operatorname{Re}\alpha > 0$, the definition (8) coincides with that of the Riemann-Liouville integral of order α.

For $x, t > 0$ and $\operatorname{Re}\lambda > \operatorname{Re}\nu > -1$, we have [7, equation (3.8)]

$$(2t)^{-\nu} I_x^{\lambda-\nu} p_{\nu+1}(x^2-t^2) = \left[(x^2-t^2)^{\lambda/2} P_\nu^{-\lambda}\left(\frac{x}{t}\right)\right]_{x>t}, \tag{10}$$

where the notation on the right hand side is intended to indicate a function that has the given value if $x > t$ and vanishes otherwise.

With the definition (8), the left-hand side of (10) has a meaning for all λ and is an entire function of λ. Moreover, the equation

$$P_\nu^{-\lambda}\left(\frac{x}{t}\right) = \frac{1}{\Gamma(1+\lambda)}\left(\frac{x-t}{x+t}\right)^{\lambda/2} {}_2F_1\left(-\nu, \nu+1; 1+\lambda; \frac{t-x}{2t}\right)$$

shows that the right-hand side of (10) may be written in the form

$$\sum_{k=0}^{n-1} c_k t^{-k} p_{\lambda+k+1}(x-t) + t^{-n} p_{\lambda+n+1}(x-t) q_n\left(\frac{x-t}{t}, \lambda\right), \tag{11}$$

where n is any positive integer,

$$c_k = \frac{\Gamma(\nu+k+1)}{2^k k!\,\Gamma(\nu-k+1)},$$

and $q_n(z, \lambda)$ is analytic for $|\arg z| < \pi$ and $\operatorname{Re}\lambda > -n$. Equation (11) defines a distribution which is an analytic function of λ when $\operatorname{Re}\lambda > -n$, and by the theory of analytic continuation, (10) holds in this half-plane. Since n is arbitrary, (10) holds for all λ.

From (9), $I_x^\alpha I_x^{-\alpha} f = f$, so that from (10) we also have

$$I_x^{\nu-\lambda}\left[(x^2-t^2)^{\lambda/2} P_\nu^{-\lambda}\left(\frac{x}{t}\right)\right]_{x>t} = (2t)^{-\nu} p_{\nu+1}(x^2-t^2) \quad (x > 0) \tag{12}$$

for all λ.

3. The integral equation. Let g be a locally integrable function on $[a, \infty[\; \subset\;]0, \infty[$, and extend g to be equal to zero on $]-\infty, a[$. Then g defines a regular distribution with left-bounded support, and we shall denote this distribution also by g. We shall now show that the left-hand side of (1) can be interpreted for such g.

Let n be a positive integer such that $\operatorname{Re}\lambda > -n$. Then

$$\int_a^x p_{\lambda+k+1}(x-t) t^{-k} g(t)\, dt \quad (k = 0, 1, \ldots, n-1)$$

https://doi.org/10.1017/S0017089500000070 Published online by Cambridge University Press

exists as the convolution $p_{\lambda+k+1}(x) * [x^{-k}g(x)]$, since x^{-k} is infinitely differentiable on the support of g, and $x^{-k}g(x)$ is a distribution. Also,

$$\int_a^x p_{\lambda+n+1}(x-t)q_n\left(\frac{x-t}{t},\lambda\right)t^{-n}g(t)\,dt$$

exists for almost every x, since $p_{\lambda+n+1}(x-t)$ and $g(t)$ are locally integrable, and the other factors are bounded and continuous. Thus, (11) leads to a definition of the left-hand side of (1), and this definition is independent of n, since whenever Re $\lambda+k>0$,

$$p_{\lambda+k+1}(x) * [x^{-k}g(x)] = \int_0^x p_{\lambda+k+1}(x-t)t^{-k}g(t)\,dt \tag{13}$$

for almost every x.

Furthermore, with this interpretation, the left-hand sides of both (1) and (2) are entire functions of λ. Since they are equal when Re $\lambda>-1$, they must be equal for all values of λ.

For a locally integrable function g with left-bounded support in $]0,\infty[$, (1) or (2) define a distribution f whose support is also bounded on the left, but f will not in general be a regular distribution, or even a pseudo-function.

One could ask: for which distributions f does (1) possess a solution g that is a regular distribution on $[a,\infty[$? Now, it was proved in [7, section 6], that corresponding to each g, there is a unique regular distribution h such that

$$I_{x^2}^{\nu+1}(2x)^{-\nu-1}g(x) = I_x^{\nu+1}h(x), \tag{14}$$

and conversely. We then have $f(x) = I_x^{\lambda+1}h(x)$, and hence the necessary and sufficient condition for a solution of (1) to exist is that $I_x^{-\lambda-1}f$ be a regular distribution. Under these circumstances the values (in the sense of distributions) of g on $[a,b[$ depend only on the values of f on $[a,b[$.

If $g^{(n-1)}(x)$ exists and is absolutely continuous on $[a,b]$, and Re $\lambda>-n$, then the integral in (1) possesses a finite part, and this finite part coincides, on $[a,b[$, with the interpretation based on the theory of distributions. It is seen from [7, equation (6.5)] that this situation will arise if and only if h, determined by (14), is $(n-1)$ times differentiable, with $h^{(n-1)}(x)$ absolutely continuous on $[a,b]$. In this case,

$$h(x) = p_n(x) * \{H(x-a)h^{(n)}(x)\} + \sum_{k=0}^{n-1} h^{(k)}(a)p_{k+1}(x-a)$$

and hence

$$f(x) = I_x^{\lambda+1}h(x) = p_{\lambda+n+1}(x) * \{H(x-a)h^{(n)}(x)\} + \sum_{k=0}^{n-1} h^{(k)}(a)p_{\lambda+k+2}(x-a).$$

From here one sees that (1) will have a solution g such that $g^{(n-1)}$ exists and is absolutely continuous on $[a,b]$ if and only if f is of the form

$$f(x) = \sum_{m=1}^{n} c_m p_{\lambda+m+1}(x-a) + f_1(x), \tag{15}$$

where the c_m are constants, Re $\lambda+n>-1$, and $f_1 \in B_{\lambda+n+1}$ (this class is defined in [7, section 6]); and in this case the finite part of the integral must be taken. Solution formulae corresponding to those given in [7] can be based on the representation (15).

REFERENCES

1. R. G. Buschman, An inversion integral for a Legendre transformation, *Amer. Math. Monthly* **69** (1962), 288–289.

2. R. G. Buschman, An inversion integral, *Proc. Amer. Math. Soc.* **13** (1962), 675–677.

3. R. G. Buschman, An inversion integral for a general Legendre transformation, *SIAM Rev.* **5** (1963), 232–233.

4. R. G. Buschman, Convolution equations with generalized Laguerre polynomial kernels, *SIAM Rev.* **6** (1964), 166–167.

5. P. L. Butzer, Singular integral equations of Volterra type and the finite part of divergent integrals, *Arch. Rational Mech. Anal.* **3** (1959), 194–205.

6. A. Erdélyi, An integral equation involving Legendre's polynomial, *Amer. Math. Monthly* **70** (1963), 651–652.

7. A. Erdélyi, An integral equation involving Legendre functions, *J. Soc. Indust. Appl. Math.* **12** (1964), 15–30.

8. I. M. Gelfand and G. E. Shilov, *Generalized functions*, vol. I (Academic Press, New York and London, 1964).

9. T. P. Higgins, An inversion integral for a Gegenbauer transformation, *J. Soc. Indust. Appl. Math.* **11** (1963), 886–893.

10. T. P. Higgins, A hypergeometric function transform, *J. Soc. Indust. Appl. Math.* **12** (1964), 601–612.

11. Ta Li, A new class of integral transforms. *Proc. Amer. Math. Soc.* **11** (1960), 290–298.

12. E. R. Love, Some integral equations involving hypergeometric functions; to appear in *Proc. Edinburgh Math. Soc.*

13. K. N. Srivastava, On some integral transforms, *Math. Japon.* **6** (1961–62), 65–72.

14. K. N. Srivastava, A class of integral equations involving ultraspherical polynomials as kernel, *Proc. Amer. Math. Soc.* **14** (1963), 932–940.

15. K. N. Srivastava, Inversion integrals involving Jacobi's polynomials, *Proc. Amer. Math. Soc.* **15** (1964), 635–638.

16. K. N. Srivastava, On some integral transforms involving Jacobi functions, *Ann. Polon. Math.* **16** (1965), 195–199.

17. K. N. Srivastava, On integral equations involving Whittaker's function, *Proc. Glasgow Math. Assoc.* **7** (1966), 125–127.

18. Jet Wimp, Two integral transform pairs involving hypergeometric functions, *Proc. Glasgow Math. Assoc.* **7** (1965), 42–44.

19. Klaus Wiener, Lineare Integralgleichungen mit Hadamard-Integralen, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe* **11** (1962), 567–580.

MATHEMATICAL INSTITUTE OF THE UNIVERSITY,
20, CHAMBERS STREET,
EDINBURGH, 1.