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INRADIUS AND CIRCUMRADIUS FOR PLANAR CONVEX BODIES CONTAINING NO LATTICE POINTS

P.R. SCOTT AND P.W. AWYONG

Let K be a planar convex body containing no points of the integer lattice. We give a new inequality relating the inradius and circumradius of K.

1. INTRODUCTION

Let K be a convex body in the Euclidean plane E^2, and let Γ denote the integer lattice. Denote by $\mathcal{K}_0$ the set of all such convex bodies K which contain no point of Γ as an interior point. Associated with K are a number of well-known functionals including the diametr $d(K) = d$, the width $w(K) = w$, the inradius $r(K) = r$ and the circumradius $R(K) = R$. (For definitions see, for example, [**3**].) A number of inequalities betwee these various functionals have been extablished. Examples are:

$$w \leqslant \frac{1}{2}\left(2+\sqrt{3}\right) \approx 1.866, \tag{1}$$

$$(w-1)(d-1) \leqslant 1, \tag{2}$$

$$2R - d \leqslant \frac{1}{3}, \tag{3}$$

$$(2r-1)(d-1) < 1, \tag{4}$$

and

$$(w-1)R \leqslant \frac{1}{\sqrt{3}}w. \tag{5}$$

These inequalities are all best possible. We define the following sets in $\mathcal{K}_0$:
$\mathcal{P}_0$: an infinite strip of width 1;
$\mathcal{T}_0$: a triangle with a longest side on the x-axis, and unit intercept by the line $y = 1$;
$\mathcal{E}_0$: the equilateral triangle in the set $\{\mathcal{T}_0\}$.
Then $\mathcal{P}_0$ is the extremal set for inequality (4) [**2**]; $\mathcal{E}_0$ is the extremal set for inequalities (1) [**4**], (3) [**1**], and (5) [**6**]; and $\mathcal{T}_0$ is the extremal set for inequality (2) [**5**].

In this paper we establish a pretty new inequality relating the quantities r and R.

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Theorem 1. *If $K \in \mathcal{K}_2$ then*

$$(2r-1)(2R-1) < 1. \tag{6}$$

This bound cannot be improved as we see by taking $K = \mathcal{T}_0$ with its longest side (the base) becoming large.

2. Setting up the problem

By translating K through a suitable lattice vector, we may take the centre of the incircle of K to lie within the square with vertices $A(0,0)$, $B(1,0)$, $C(1,1)$, $D(0,1)$. It is clear that (6) is trivially satisfied if $2r \leqslant 1$. We therefore assume that $2r > 1$. Since K is convex, K is bounded by lines through the points A, B, C and D. If these lines form a convex quadrilateral Q, then Q contains no lattice points in its interior, and we may assume that K is Q. On the other hand these lines may determine a triangular region T, as for example, a degenerate quadrilateral, or when a line through D separates K from C. Such a region T may contain interior lattice points; nevertheless it will be sufficient for us to establish the theorem for T. Arguing as in [5], we may assume that T has an edge along the x-axis. A further possibility is that $Q(T)$ may degenerate into an infinite strip of width 1.

Let us first assume then that K is the quadrilateral Q. Let quadrilateral Q have vertices L, M, N, P, and edges LM, MN, NP, PL passing through C, B, A, D respectively. By reflecting Q in the line $x = 1/2$ if necessary, we may assume that L lies in the strip $1/2 \leqslant x \leqslant 1$.

The circumcircle of Q may be determined by two boundary points of Q which are endpoints of a diameter of the circle. In this case we have $d = 2R$. If Q is non-degenerate, then since $w \geqslant 2r$, and noting that (2) holds with equality only for a triangle $\mathcal{T}_0$, our result follows immediately from:

$$(2r-1)(2R-1) \leqslant (w-1)(d-1) < 1. \tag{7}$$

On the other hand, if Q degenerates to a triangle, then

$$(2r-1)(2R-1) < (w-1)(d-1) \leqslant 1. \tag{8}$$

The other possibility is that the circumcircle of Q is determined by three points on the boundary of Q forming the vertices of an acute-angled triangle. Take this triangle to be $T = \triangle LMP$. We observe that $\angle MNP$ will be obtuse. The incircle of Q will touch edges LM, LP and at least one of the edges MN, PN. In fact, we may assume Q is such that the incircle touches all four edges. For if necessary, we can rotate PN

in an anti-clockwise direction about A, or MN in a clockwise direction abut B until these edges of Q are tangents to the incircle, making contact on the short arc AB. This operation leaves the value of r unchanged, and increases the value of R. A consequence of this construction is that we may assume that the incircle intercepts each side of square $ABCD$.

The following results will be useful.

Lemma 1. *Let l, m be two non-orthogonal lines meeting in P, and let C be a point interior to one of the acute angles formed by l and m. Let $\mathcal{T}$ denote the set of all non-obtuse-angled triangles $T = \triangle LMP$ havign L on l, M on m, and LM through C. Then $R(T)$ is maximal when $T \in \mathcal{T}$ is a right-angled triangle.*

Proof: Let $T = \triangle LMP$ be an acute-angled triangle. (See Figure 1.) We may assume that $CL \leqslant CM$, and that line m is the x-axis.

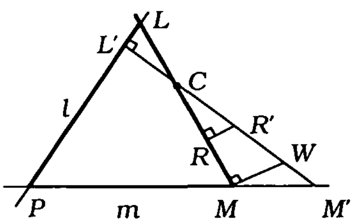

Figure 1. The triangle with largest circumcircle

Take $P' = P$, and L' on LP, M' on the x-axis so that L', C and M' are collinear, and $\angle P'L'M'$ is a right-angle. Denote by T' the right-angled triangle $\triangle P'L'M'$. We claim that $R(T') > R(T)$. To show this will be sufficient to show that $L'M' > LM$. For recalling that $P' = P$, the sine rule then gives

$$2R(T') = \frac{L'M'}{\sin \angle P'} > \frac{LM}{\sin \angle P} = 2R(T).$$

Noting that $CL' < CL \leqslant CM$, we take points R, R' on CM, CM' respectively such that $\triangle CL'L \cong \triangle CRR'$. Choose point W on CM' such that $MW//RR'$. We now have

$$LM = LC + CR + RM \leqslant CR' + L'C + R'W < CR' + L'C + R'W + WM' = L'M'.$$

Hence for $T \in \mathcal{T}$, $R(T)$ is maximal when T is a right-angled triangle. This completes the proof of the lemma. ∎

Lemma 2. *Let A, B, C, D be points defined as previously, and let XC be the line with equation $4x + 3y - 7 = 0$, making an angle of $53.13°$ with the x-axis. Then*

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any circle which intercepts segment AB and does not contain C, D in its interior, does not intercept line XC in the halfplane $y > 1$.

PROOF: It is easily checked that the circle F which just touches the x-axis and passes through points C, D has equation

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{5}{8}\right)^2 = \frac{25}{64}.$$

The angle which the radius of this circle to C makes with the x-axis is now arctan $3/4 = 36.87°$; hence the angle which the tangent to the circle at C makes with the x-axis is $53.13°$. Thus XC is the tangent to F at C.

Let F_S denote the segment of circle F which lies above CD. Let F' be any other circle satisfying the conditions of the lemma. If the radius of F' exceeds the radius of F, then the centre of F' lies further from CD than the centre of F, and the portion of F' lying above CD is contained in segment F_S. If the radius of F' is smaller than the radius of F, then the cente F' lies closer to the x-axis than the centre of F, and again the portion of F' lying above CD is contained in segment F_S. Hence in all cases the circle fails to intercept the line XC in the half-plane $y > 1$, and the lemma is proved. ∎

COMMENT. It folows that if the edge LCM of Q makes an angle of more than $53.13°$ with the x-axis, then it will meet the incircle on the short arc CB. The contrapositive is that if LCM meets the incircle on the short are CD, then LCM makes an angle of not more than $53.13°$ with the x-axis.

3. PROOF OF THE THEOREM

Suppose that Q is either a non-degenerate quadrilateral or an acute-angled triangle $\triangle LMP$ with edge MP along the x-axis for which inequality (6) is *not* satisfied. From our setting up, vertex L lies in the half-strip $1/2 \leqslant x \leqslant 1$, $y \geqslant 1$. Since $\angle LMP$ is acute, L is exterior to the semicircle on CD as diameter defined by $(x - 1/2)^2 + (y - 1)^2 = 1/4$, $y \geqslant 1$.

Let now X be the intersection of the given line XC of Lemma 2 with the semicircle on CD as diameter, and let DX meet the line $x = 1$ in E. Denote by U the 'triangular' region bounded by arc XC and line segments XE, EC (see Figure 2).

L cannot lie in U. For in this case, by Lemma 2, edge LM touches the incircle of Q on the short arc BC. Let $\triangle X'E'C$ be the (point) reflection of $\triangle XEC$ in C, and let line t through B be the reflection of line XC in the line $y = 1/2$. Since XC and t meet on the mirror line $y = 1/2$, $\triangle X'E'C$ lies in the half-plane bounded by t which contains C. We know that edge MN of Q meets the incircle on the short arc AB.

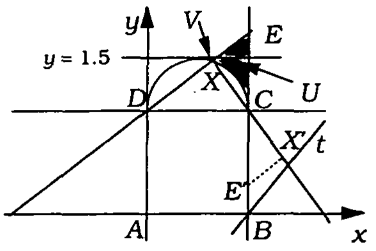

Figure 2. Restricting the position of L

Hence applying the Coment after Lemma 2 to edge NBM, this edge makes an angle of at most $53.13°$ with the x-axis. It follows that $LC \leqslant CM$, and we can apply Lemma 1 to edge LM (running L down LP and M along MN) to obtain a new right-angled quadrilateral Q^* with $r(Q^*) \geqslant r(Q)$ and $R(Q^*) > R(Q)$. The right-angle of Q^* tells us that $d(Q^*) = 2R(Q^*)$, and inequality (6) follows from (7). Hence we may assume that L lies outside the semicircle and above the line DE.

L cannot lie on or above the line $y = 1.5$. Since trivially $2r \leqslant \sqrt{2}$, it is easy to check that (6) is satisfied when $2R \leqslant 2+\sqrt{2}$. Hence we may assume that $2R > 2+\sqrt{2}$. From inequality (3), it follows that we may assume that $d(Q) > d(T) > 3$. Let LM, LP meet the x-axis in M^*, P^* respectively, and let $T^* = \Delta LM^*P^*$. Since $T \subseteq T^*$, we have $d(T) \leqslant d(T^*)$. Suppose that L lies on or above the line $y = 3/2$. Then we claim that $d(T^*) = M^*P^* \leqslant 3$. By a simple similarity argument, this is certainly true if L has x-coordinate $x = 1/2$. As L moves to the right along $y = 3/2$, the length of P^*M^* remains the same, and LP^* increases, first assuming the value d when $\angle LP^*M^* = 30°$. But then L lies in the triangular region U considered above (since $\angle XDC = 36.87°$). Hence $d(T) \leqslant 3$ for L on or above the line $y = 1.5$. This contradiction allows us to assume that L lies in the small 'triangular' region V bounded by the semi-circular arc, the line DX and the line $y = 1.5$.

L cannot lie in V. The coordinates of X are easily found to be $(16/25, 111/75) = (0.64, 1.48)$. It would be nice to adapt the argument of the above paragraph to the line $y = 1.48$, but unfortunately the bound obtained is not tight enough to give a contradiction. But we observe that in [1] inequality (3) is deduced from the more general inequality $2R - d \leqslant (2/3)(2-\sqrt{3})w$ which holds for general convex sets with no lattice point constraints. Since we now have $w \leqslant 1.5$, we can replace inequality (3) by the tighter bound $2R - d \leqslant 2 - \sqrt{3}$, whence we may assume that $d(Q) \geqslant d(T) \geqslant 2R - (2-\sqrt{3}) > (2+\sqrt{2}) - (2-\sqrt{3}) = 3.146$. By repeating the similarity argument of the previous paragraph with L lying on or above the line $y = 1.48$, we obtain $d(T^*) = P^*M^* \leqslant 3.084$. This contradiction establishes that L cannot lie in V.

In summary, we have shown that there are three possible classes of extremal set: the non-degenerate quadrilateral Q, the acute-angled triangle ΔLMP with edge MP

along the x-axis, and the infinite strip $0 \leqslant y \leqslant 1$. The above argument shows that there is no set in the first two classes for which inequality (6) does not hold. Regarding the infinite strip as the limit of $T = \triangle LMP$ as $R \to \infty$, we have $2r < w$, $2r \to w$, $2R = d$, and

$$(2r-1)(2R-1) < (w-1)(d-1) \leqslant 1.$$

Hence in every case, inequality (6) is satisfied, and the bound of 1 cannot be improved.

4. Final comments

We observe that there are nice similarities between the inequalities (2), (4) and (6). The final likely combination of two of d, $2r$, $2R$ and w,

$$(w-1)(2R-1) < 1$$

is false, as can be checked using the equilateral triangle $\mathcal{E}_0$. In fact using inequalities (5) and (1) we have

$$(w-1)(2R-1) \leqslant \frac{2w}{\sqrt{3}} - w + 1 = \left(\frac{2-\sqrt{3}}{\sqrt{3}}\right) w + 1 \leqslant \frac{\sqrt{3}}{6} + 1 \approx 1.289,$$

with equality for the triangle $\mathcal{E}_0$.

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Department of Pure Mathematics
University of Adelaide
South Australia 5005
Australia
e-mail: pscott@maths.adelaide.edu.au

Division of Mathematics
National Institute of Education
469 Bukit Timah Rd
Singapore 259756
e-mail: awyongpw@nievax.nie.ac.sg