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# Uncertainty Principles on Weighted Spheres, Balls, and Simplexes

Han Feng

*Abstract.* This paper studies the uncertainty principle for spherical $h$-harmonic expansions on the unit sphere of $\mathbb{R}^d$ associated with a weight function invariant under a general finite reflection group, which is in full analogy with the classical Heisenberg inequality. Our proof is motivated by a new decomposition of the Dunkl–Laplace–Beltrami operator on the weighted sphere.

## 1 Introduction

The uncertainty principle is a fundamental result in quantum mechanics, and it can be formulated in the Euclidean space $\mathbb{R}^d$, in the form of the classical Heisenberg inequality, as

$$\inf_{a\in\mathbb{R}^d}\int_{\mathbb{R}^d}\|x-a\|^2|f(x)|^2dx\int_{\mathbb{R}^d}|\nabla f(x)|^2dx\geq\frac{d^2}{4}\Big(\int_{\mathbb{R}^d}|f(x)|^2\Big)^2,\tag{1.1}$$

where $\nabla$ is the gradient operator. There are many papers devoted to the study of this inequality and its various generalizations; see, for instance, [3, 9, 10].

In particular, on the unit sphere, F. Dai and Y. Xu [3] established the analogue result, which states that if $f\colon\mathbb{S}^{d-1}\to\mathbb{R}$ satisfies

$$\int_{\mathbb{S}^{d-1}}f(x)\,d\sigma(x)=0\quad\text{and}\int_{\mathbb{S}^{d-1}}|f(x)|^2d\sigma(x)=1,$$

then

$$\Big(\min_{y\in\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\big(1-\langle x,y\rangle\big)|f(x)|^2\,d\sigma(x)\Big)\Big(\int_{\mathbb{S}^{d-1}}|\nabla_0 f|^2d\sigma(x)\Big)\geq C_d>0.\tag{1.2}$$

In a recent paper [11], with a weight function $h_\kappa^2(x)$ invariant under a group $G$, Xu studied the uncertainty principle on the unit sphere $\mathbb{S}^{d-1}$. By introducing a weighted analogue $\nabla_{\kappa,0}$ of the tangential gradient $\nabla_0$, he proved in [11, Theorem 4.1] that if $f\colon\mathbb{S}^{d-1}\to\mathbb{R}$ is invariant under the group $G$ and satisfies

$$\int_{\mathbb{S}^{d-1}}f(x)h_\kappa^2(x)\,d\sigma(x)=0\quad\text{and}\quad\int_{\mathbb{S}^{d-1}}|f(x)|^2h_\kappa^2(x)d\sigma(x)=1,$$

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then

$$(1.3)\quad \Big(\min_{1\le i\le d}\int_{\mathbb{S}^{d-1}}\big(1-\langle x,e_i\rangle\big)|f(x)|^2h_\kappa^2(x)\,d\sigma(x)\Big)\Big(\int_{\mathbb{S}^{d-1}}|\nabla_{\kappa,0}f|^2h_\kappa^2(x)d\sigma(x)\Big)\ge C_{\kappa,d}>0,$$

where $e_i$, $i=1,\dots,d$, is the standard vector; namely, only the $i$-th coordinate is nonzero 1, and $C_{\kappa,d}$ is a constant that depends only on parameters $\kappa$ and $d$, and $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^d$.

The purpose of this paper is to show that inequality (1.3), with minimum being taken over all $y\in\mathbb{S}^{d-1}$ rather than the finite subset $\{e_1,\dots,e_d\}$, remains true without the extra assumption that $f$ is $G$-invariant.

Recall that the geodesic distance on the sphere is defined by $d(x,y)=\arccos\langle x,y\rangle$, so that

$$1-\langle x,y\rangle=2\sin^2\frac{d(x,y)}{2}\sim d(x,y)^2$$

with $A\sim B$ meaning $\frac{1}{c}A\le B\le cA$ for some $c>0$. It implies that (1.2) and (1.3) can be regarded as close analogies of (1.1).

Let $G\subset O(d)$ be a finite reflection group on $\mathbb{R}^d$. For $v\in\mathbb{R}^d\setminus\{0\}$, we denote by $\sigma_v$ the reflection with respect to the hyperplane perpendicular to $v$; that is,

$$\sigma_v x=x-\frac{2\langle x,v\rangle}{\|v\|^2}v,\quad x\in\mathbb{R}^d,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbb{R}^d$ and $\|x\|:=\sqrt{\langle x,x\rangle}$. Let $\mathcal{R}$ be the root system of $G$, normalized so that $\langle v,v\rangle=2$ for all $v\in\mathcal{R}$, and fix a positive subsystem $\mathcal{R}_+$ of $\mathcal{R}$, such that $\mathcal{R}=\mathcal{R}_+\cup(-\mathcal{R}_+)$. From the general theory of reflection groups (see, *e.g.*, [8]), the set of reflections in $G$ associates with $\{\sigma_v : v\in\mathcal{R}_+\}$, which also generates the group $G$. Let $\kappa\colon\mathcal{R}\to[0,\infty)$, $v\mapsto\kappa_v=\kappa(v)$ be a nonnegative multiplicative function on $\mathcal{R}$; that is, $\kappa$ is a nonnegative $G$-invariant function on $\mathcal{R}$. Let $h_\kappa$ denote the weight function on $\mathbb{R}^d$ defined by

$$(1.4)\qquad h_\kappa(x):=\prod_{v\in\mathcal{R}_+}|\langle x,v\rangle|^{\kappa_v},\quad x\in\mathbb{R}^d.$$

It is $G$-invariant and homogeneous of degree $|\kappa|:=\sum_{v\in\mathcal{R}_+}\kappa_v$.

Let $\Delta_{\kappa,0}$ be the weighted analogy of the Laplace–Beltrami operator $\Delta_0$ on $\mathbb{S}^{d-1}$, whose precise definition will be given in next section. Then our main result can be stated as follows:

***Theorem 1.1*** *Let $f\in C^1(\mathbb{S}^{d-1})$ be such that*

$$\int_{\mathbb{S}^{d-1}}f(x)h_\kappa^2(x)\,d\sigma(x)=0\quad\text{and}\quad\int_{\mathbb{S}^{d-1}}|f(x)|^2h_\kappa^2(x)\,d\sigma(x)=1.$$

*Then*

$$(1.5)\quad\Big[\min_{y\in\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}(1-\langle x,y\rangle)|f(x)|^2h_\kappa^2(x)\,d\sigma(x)\Big]\times\Big[\int_{\mathbb{S}^{d-1}}|\sqrt{-\Delta_{\kappa,0}}f(x)|^2h_\kappa^2(x)\,d\sigma(x)\Big]\ge C_{d,\kappa}>0,$$

*where $C_{d,\kappa}$ is a constant depending only on $d$ and $\kappa$.*

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As a direct corollary, we obtain the following improvement of [11, Theorems 4.1 and 4.2].

***Corollary 1.2*** *If $f \in C^1(\mathbb{S}^{d-1})$ satisfies that*

$$\int_{\mathbb{S}^{d-1}} f(x) h_\kappa^2(x)\, d\sigma(x) = 0 \quad \textit{and} \quad \int_{\mathbb{S}^{d-1}} |f(x)|^2 h_\kappa^2(x)\, d\sigma(x) = 1,$$

*then*

$$\Big( \min_{y \in \mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \big(1 - \langle x, y\rangle\big) |f(x)|^2 h_\kappa^2(x)\, d\sigma(x)\Big)\Big( \int_{\mathbb{S}^{d-1}} |\nabla_{\kappa,0} f|^2 h_\kappa^2 d\sigma(x)\Big) \geq C_{\kappa,d} > 0.$$

Note that the improvement by taking the minimum over all $y \in \mathbb{S}^{d-1}$ instead of $\{e_1, \ldots, e_d\}$ is nontrivial since the weight $h_\kappa^2$ is not invariant under all rotations. And obviously, the requirement of the $G$-invariance of $f$ turns out not to be necessary.

Finally, we shall also establish similar results for the weighted orthogonal polynomial expansions (WOPEs) with respect to the weight function

$$W_\kappa^B(x) := \Big(\prod_{v \in \mathcal{R}_+} |\langle x, v\rangle|^{2\kappa_v}\Big)\big(1 - \|x\|^2\big)^{\mu - 1/2}, \quad \mu \geq 0 \tag{1.6}$$

on the unit ball $\mathbb{B}^d := \{x \in \mathbb{R}^d : \|x\| \leq 1\}$, where $\mathcal{R}_+$, $\kappa$ are adopted as before, as well as for the WOPEs with respect to the weight function

$$W_\kappa^T(x; \mathbb{Z}_2^d) := \Big(\prod_{i=1}^d x_i^{\kappa_i - 1/2}\Big)(1 - |x|)^{\kappa_{d+1} - 1/2}, \quad \min_{1 \leq i \leq d+1} \kappa_i \geq 0, \tag{1.7}$$

or

$$W_{\kappa,\mu}^T(x; H_d) = \prod_{i=1}^d x_i^{\kappa' - 1/2} \prod_{1 \leqslant i < j \leqslant d} |x_i - x_j|^\kappa (1 - |x|)^{\mu - 1/2}, \quad \min\{\kappa', \kappa, \mu\} \geq 0, \tag{1.8}$$

on the simplex $\mathbb{T}^d := \{x \in \mathbb{R}^d : x_j \geq 0, \ldots, x_d \geq 0, 1 - |x| \geq 0\}$, here, and in what follows, $|x| := \sum_{j=1}^d |x_j|$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

## 2 Preliminaries

### 2.1 The Dunkl Theory

This theory of spherical $h$-harmonics was initially developed by C. F. Dunkl in [5–7]. For details, one can refer to [2, 8]. Let $\mathcal{R}$ be a fixed root system in $\mathbb{R}^d$ normalized so that $\langle v, v\rangle = 2$ for all $v \in \mathcal{R}$, and $G$ the associated reflection group. Let $\kappa\colon \mathcal{R} \to [0, \infty)$ be a multiplicity function on $\mathcal{R}$.

The Dunkl operators associated with $G$ and $\kappa$ are defined by

$$\mathcal{D}_i f(x) = \partial_i f(x) + \sum_{v \in \mathcal{R}_+} \kappa_v \langle v, e_i\rangle \frac{f(x) - f(\sigma_v x)}{\langle x, v\rangle}, \quad i = 1, \ldots, d, \quad f \in C^1(\mathbb{R}^d),$$

where $\partial_i = \frac{\partial}{\partial x_i}$, $\mathcal{R}_+$ is a fixed positive subsystem of $\mathcal{R}$. Here we use the notation $g \circ f(x) := f(gx)$ for $g \in G$, $f \in C(\mathbb{S}^{d-1})$ and $x \in \mathbb{S}^{d-1}$.

The $\kappa$-Laplacian on $\mathbb{R}^d$ is defined by $\Delta_\kappa := \sum_{j=1}^d \mathcal{D}_j^2$. The operator $\Delta_\kappa$ is $G$-invariant; that is, $g \circ \Delta_\kappa = \Delta_\kappa \circ g$ for all $g \in G$. Similarly, the $\kappa$-gradient is defined by $\nabla_\kappa = (\mathcal{D}_1, \dots, \mathcal{D}_d)$. Furthermore, by restricting to the unit sphere, the weighted analogue $\Delta_{\kappa,0}$ of the Laplace–Beltrami operator $\Delta_0$ and analogue $\nabla_{\kappa,0}$ of the tangential gradient $\nabla_0$ are defined as follows:

$$\Delta_{\kappa,0} f(x) := \Delta_\kappa F(z)|_{z=x}, \qquad x \in \mathbb{S}^{d-1}$$

and

$$\nabla_{\kappa,0} f(x) := \nabla_\kappa F(z)|_{z=x}, \qquad x \in \mathbb{S}^{d-1},$$

where $F(z) = f(\frac{z}{\|z\|})$.

## 2.2 $h$-harmonic Expansions

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ denote the unit sphere of $\mathbb{R}^d$ equipped with the usual Haar measure $d\sigma(x)$, and the weight function $h_\kappa$ given in (1.4). For $1 < p < \infty$, recall that

$$\|f\|_{\kappa,p} := \Big( \int_{\mathbb{S}^{d-1}} |f(y)|^p h_\kappa^2(y) d\sigma(y) \Big)^{1/p}.$$

We denote by $\Pi_n^d$ the space of all spherical polynomials of degree at most $n$ on $\mathbb{S}^{d-1}$, and by $\mathcal{H}_n^d(h_\kappa^2)$ the space of all spherical $h$-harmonics of degree $n$ on $\mathbb{S}^{d-1}$. Thus, $\mathcal{H}_n^d(h_\kappa^2)$ is the orthogonal complement of $\Pi_{n-1}^d$ in the space $\Pi_n^d$ with respect to the inner product

$$\langle f, g \rangle_\kappa := \int_{\mathbb{S}^{d-1}} f(x)\overline{g(x)} h_\kappa^2(x)\, d\sigma(x),$$

and each function $f \in L^2(h_\kappa^2; \mathbb{S}^{d-1})$ has a spherical $h$-harmonic expansion

$$f = \sum_{n=0}^\infty \operatorname{proj}_n(h_\kappa^2; f)$$

converging in the norm of $L^2(h_\kappa^2; \mathbb{S}^{d-1})$.

Here $\operatorname{proj}_n(h_\kappa^2) \colon L^2(h_\kappa^2; \mathbb{S}^{d-1}) \to \mathcal{H}_n^d(h_\kappa^2)$ is the orthogonal projection. Also, the projection $\operatorname{proj}_n(h_\kappa^2; f)$ can be extended to all $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$ in the sense that

$$\operatorname{proj}_n(h_\kappa^2; f)(x) = \int_{\mathbb{S}^{d-1}} f(y) P_n(x, y) h_\kappa^2(y) d\sigma(y),\ f \in L^1(h_\kappa^2; \mathbb{S}^{d-1}),$$

with $P_n(h_\kappa^2; x, y)$ being the reproducing kernel of $\mathcal{H}_n^d(h_\kappa^2)$.

A crucial point in the theory of $h$-harmonics is that the space $\mathcal{H}_n^d(h_\kappa^2)$ can also be seen as an eigenspace of a second order differential-difference operator $\Delta_{\kappa,0}$ corresponding to the eigenvalue $-n(n + 2\lambda_\kappa)$. Here and throughout the paper,

$$\lambda_\kappa := \frac{d-2}{2} + |\kappa|.$$

Given $\alpha \in \mathbb{R}$, we define the fractional power $(-\Delta_{\kappa,0})^\alpha$ of $(-\Delta_{\kappa,0})$, in a distributional sense, by

$$\operatorname{proj}_n\big( h_\kappa^2; (-\Delta_{\kappa,0})^\alpha f \big) = \big( n(n + 2\lambda_\kappa) \big)^\alpha \operatorname{proj}_n(h_\kappa^2; f), \quad n = 0, 1, \dots.$$

Next we introduce a first order differential operator on suitable functions defined on $\mathbb{R}^d$:

$$D_{i,j}f(x) = x_j\partial_i f(x) - x_i\partial_j f(x), \quad 1 \le i,j \le d$$

and

$$E_\nu f(x) = \frac{f(x) - f(\sigma_\nu x)}{\langle x, \nu\rangle}, \ \ \nu \in \mathbb{R}^d \setminus \{0\}.$$

The proof of our main result relies on a decomposition of $(-\Delta_{\kappa,0})$ and a practical estimate of $\|(-\Delta_{\kappa,0})^{1/2} f\|_{\kappa,p}$ in [1], which is stated as the following theorem.

***Theorem 2.1*** ([1]) *For $f \in C^1(\mathbb{S}^{d-1})$, with the notation given above,*

$$\|(-\Delta_{\kappa,0})^{1/2} f\|_{\kappa,2}^2 = \sum_{1\le i<j\le d} \|D_{i,j} f\|_{\kappa,2}^2 + \sum_{\nu\in\mathcal{R}_+} \kappa_\nu \|E_\nu f\|_{\kappa,2}^2. \tag{2.1}$$

Particularly, in the unweighted setting, namely when $\kappa = 0$, this theorem will go back to the classical result (see for instance [4, Section 1.8]) that for $f \in C^1(\mathbb{S}^{d-1})$,

$$\|(-\Delta_0)^{1/2} f\|_2^2 = \|\nabla_0 f\|_2^2 = \sum_{1\le i<j\le d} \|D_{i,j} f\|_2^2, \tag{2.2}$$

where $\|g\|_2^2 = \int_{\mathbb{S}^{d-1}} |g(x)|^2 d\sigma(x)$, $g \in L^2(\mathbb{S}^{d-1})$.

## 3 The Proof of Corollary 1.2

For the moment, we take Theorem 1.1 for granted and proceed with the proof of Corollary 1.2.

**Proof** By (1.5), it suffices to show that

$$\|\sqrt{-\Delta_{\kappa,0}} f\|_{\kappa,2} \le \|\nabla_{\kappa,0} f\|_{\kappa,2}. \tag{3.1}$$

Indeed, noticing [11, (3.15), (3.13)], we have that

$$\|\sqrt{-\Delta_{\kappa,0}} f\|_{\kappa,2}^2 = \|\nabla_{h,0} f\|_{\kappa,2}^2 - \frac{2\lambda_\kappa}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} (\xi \cdot \nabla_{h,0} f(\xi)) f(\xi) h_\kappa^2(\xi)\, d\sigma(\xi), \tag{3.2}$$

where $\omega_d^\kappa = \int_{\mathbb{S}^{d-1}} h_\kappa^2(x)\, d\sigma(x)$. Here it should be pointed out that the last two terms in [11, (3.15)] in fact can be cancelled out by realising that

$$(I - \sigma_\nu)^2 = 2(I - \sigma_\nu), \quad \nu \in \mathcal{R}_+.$$

Furthermore, by [11, (3.3)], we obtain

$$\int_{\mathbb{S}^{d-1}} \big(\xi \cdot \nabla_{h,0} f(\xi)\big) f(\xi) h_\kappa^2(\xi)\, d\sigma(\xi) = \sum_{\nu\in\mathcal{R}_+} \kappa_\nu \int_{\mathbb{S}^{d-1}} \big(f(\xi) - f(\sigma_\nu \xi)\big) f(\xi) h_\kappa^2(\xi)\, d\sigma(\xi).$$

However, by the Cauchy–Schwartz inequality,

$$\int_{\mathbb{S}^{d-1}} f(x) f(\sigma_\nu x) h_\kappa^2(x)\, d\sigma(x) \le \|f\|_{\kappa,2}^2, \quad \nu \in \mathcal{R}_+.$$

Thus,

$$\int_{\mathbb{S}^{d-1}} \Big( (\xi \cdot \nabla_{h,0}) f(\xi) \Big) f(\xi) h_\kappa^2(\xi)\, d\sigma(\xi) \geq 0.$$

The desired inequality (3.1) then follows by (3.2). ∎

## 4 The Proof of Theorem 1.1

Now we turn to the proof of Theorem 1.1. Recall that $\lambda_\kappa = \frac{d-2}{2} + |\kappa|$ and $|\kappa| = \sum_{\alpha \in \mathcal{R}_+} \kappa_\alpha$.
Our proof relies on the following lemma.

***Lemma 4.1*** *If $f \in C^1(\mathbb{S}^{d-1})$ and $y \in \mathbb{S}^{d-1}$, then*

$$\begin{aligned}(4.1)\quad \Big(\frac{d-1}{2} + |\kappa|\Big) &\int_{\mathbb{S}^{d-1}} \langle x, y\rangle |f(x)|^2 h_\kappa^2(x)\, d\sigma(x) \\ &= \sum_{\alpha \in \mathcal{R}_+} \kappa_\alpha \langle y, \alpha\rangle \int_{\mathbb{S}^{d-1}} \frac{|f(x)|^2 h_\kappa^2(x)}{\langle x, \alpha\rangle}\, d\sigma(x) \\ &\qquad - \int_{\mathbb{S}^{d-1}} \Big[ \sum_{i=1}^{d} \sum_{j=1}^{d} x_j y_i D_{i,j} f(x) \Big] f(x) h_\kappa^2(x)\, d\sigma(x),\end{aligned}$$

*where $x_j = \langle x, e_j\rangle$ and $y_j = \langle y, e_j\rangle$.*

**Proof** By noticing that for $f, g \in C^1(\mathbb{S}^{d-1})$ and $i \neq j$,

$$\int_{\mathbb{S}^{d-1}} f(x) D_{i,j} g(x) d\sigma(x) = -\int_{\mathbb{S}^{d-1}} D_{i,j} f(x) g(x) d\sigma(x),$$

we obtain that for $2 \leq j \leq d$,

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} \Big[ x_j D_{1,j} f(x) \Big] f(x) h_\kappa^2(x)\, d\sigma(x) &= -\int_{\mathbb{S}^{d-1}} f(x) \Big[ D_{1,j} f(x) \Big] x_j h_\kappa^2(x)\, d\sigma(x) \\ &\quad - \int_{\mathbb{S}^{d-1}} |f(x)|^2 \Big[ D_{1,j}\big( x_j h_\kappa^2(x) \big) \Big]\, d\sigma(x).\end{aligned}$$

A straightforward calculation shows that

$$D_{1,j}\big( x_j h_\kappa^2(x) \big) = \Bigg( x_1 + x_1 \sum_{\alpha \in \mathcal{R}_+} \frac{2\kappa_\alpha x_j \alpha_j}{\langle x, \alpha\rangle} - x_j^2 \sum_{\alpha \in \mathcal{R}_+} \frac{2\kappa_\alpha \alpha_1}{\langle x, \alpha\rangle} \Bigg) h_\kappa^2(x),$$

where $\alpha_j = \langle \alpha, e_j\rangle$. Thus,

$$\begin{aligned}2 &\int_{\mathbb{S}^{d-1}} \big[ x_j D_{1,j} f(x) \big] f(x) h_\kappa^2(x)\, d\sigma(x) \\ &= \int_{\mathbb{S}^{d-1}} |f(x)|^2 x_j^2 \Bigg( \sum_{\alpha \in \mathcal{R}_+} \frac{2\kappa_\alpha \alpha_1}{\langle x, \alpha\rangle} \Bigg) h_\kappa^2(x)\, d\sigma(x) \\ &\quad - \int_{\mathbb{S}^{d-1}} |f(x)|^2 \Big[ x_1 + x_1 \sum_{\alpha \in \mathcal{R}_+} \frac{2\kappa_\alpha x_j \alpha_j}{\langle x, \alpha\rangle} \Big] h_\kappa^2(x)\, d\sigma(x)\end{aligned}$$

Summing this last equation over $j = 2, \dots, d$ yields

$$\int_{\mathbb{S}^{d-1}} \Big[ \sum_{j=2}^{d} x_j D_{1,j} f(x) \Big] f(x) h_\kappa^2(x)\, d\sigma(x) = \int_{\mathbb{S}^{d-1}} |f(x)|^2 \sum_{\alpha \in \mathcal{R}_+} \frac{\kappa_\alpha \alpha_1}{\langle x, \alpha \rangle} h_\kappa^2(x)\, d\sigma(x) - \Big( |\kappa| + \frac{d-1}{2} \Big) \int_{\mathbb{S}^{d-1}} x_1 |f(x)|^2 h_\kappa^2(x)\, d\sigma(x).$$

In general, for $1 \le i \le d$, recalling $D_{i,i} = 0$, and using symmetry, we obtain

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \Big[ \sum_{j=1}^{d} x_j D_{i,j} f(x) \Big] f(x) h_\kappa^2(x)\, d\sigma(x) &= \int_{\mathbb{S}^{d-1}} |f(x)|^2 \sum_{\alpha \in \mathcal{R}_+} \frac{\kappa_\alpha \alpha_i}{\langle x, \alpha \rangle} h_\kappa^2(x)\, d\sigma(x) \\ &\quad - \Big( |\kappa| + \frac{d-1}{2} \Big) \int_{\mathbb{S}^{d-1}} x_i |f(x)|^2 h_\kappa^2(x)\, d\sigma(x) d\sigma(x). \end{aligned} \tag{4.2}$$

Multiplying both sides of (4.2) by $y_i$ and summing the resulting equation over $i = 1, \dots, d$ yield the desired identity (4.1). ∎

We are now in a position to prove Theorem 1.1 .

**Proof of Theorem 1.1** Let $\varepsilon \in (0,1)$ be a small absolute constant to be specified later. If

$$\int_{\mathbb{S}^{d-1}} \langle x, y \rangle |f(x)|^2 h_\kappa^2(x)\, d\sigma(x) \le 1 - \varepsilon,$$

then

$$\int_{\mathbb{S}^{d-1}} |f(x)|^2 (1 - \langle x, y \rangle) h_\kappa^2(x)\, d\sigma(x) \ge \varepsilon,$$

and (1.5) holds trivially, as $\|\sqrt{-\Delta_{\kappa,0}} f\|_{\kappa,2} \ge \|f\|_{\kappa,2} = 1$. Thus, without loss of generality, we may assume that

$$\int_{\mathbb{S}^{d-1}} \langle x, y \rangle |f(x)|^2 h_\kappa^2(x)\, d\sigma(x) > 1 - \varepsilon. \tag{4.3}$$

We will use the identity (4.1). Indeed, it will be shown that

$$\begin{aligned} J_1 &:= \Big| \int_{\mathbb{S}^{d-1}} \Big[ \sum_{i=1}^{d} \sum_{j=1}^{d} y_i x_j D_{i,j} f(x) \Big] f(x) h_\kappa^2(x)\, d\sigma(x) \Big| \\ &\le C \|\nabla_0 f\|_{\kappa,2} \Big( \int_{\mathbb{S}^{d-1}} |f(x)|^2 (1 - \langle x, y \rangle) h_\kappa^2(x)\, dx \Big)^{\frac{1}{2}} \end{aligned} \tag{4.4}$$

and that for each $\alpha \in \mathcal{R}_+$ with $\kappa_\alpha > 0$,

$$\begin{aligned} J_2(\alpha) &:= \Big| \langle y, \alpha \rangle \int_{\mathbb{S}^{d-1}} \frac{|f(x)|^2 h_\kappa^2(x)}{\langle x, \alpha \rangle}\, d\sigma(x) \Big| \\ &\le \frac{1}{1-\varepsilon} + \frac{C}{\varepsilon} \|E_\alpha f\|_{\kappa,2} \Big( \int_{\mathbb{S}^{d-1}} |f(x)|^2 (1 - \langle x, y \rangle) h_\kappa^2(x)\, d\sigma(x) \Big)^{\frac{1}{2}}. \end{aligned} \tag{4.5}$$

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Once (4.4) and (4.5) are proved, then using (4.1), (4.3), and (2.1), we obtain

$$(1-\varepsilon)\Big(|\kappa|+\frac{d-1}{2}\Big)\le\frac{C|\kappa|}{\varepsilon}\|\sqrt{-\Delta_{\kappa,0}}f\|_{\kappa,2}\Big(\int_{\mathbb{S}^{d-1}}|f(x)|^2(1-\langle x,y\rangle)h_\kappa^2(x)\,d\sigma(x)\Big)^{\frac12}+\frac{|\kappa|}{1-\varepsilon}.$$

Thus, choosing $\varepsilon\in(0,1)$ small enough so that

$$(1-\varepsilon)\Big(|\kappa|+\frac{d-1}{2}\Big)-\frac{1}{1-\varepsilon}|\kappa|\ge C_{d,\kappa}>0,$$

we deduce the desired inequality (1.5).

It remains to show (4.4) and (4.5). For the proof of (4.4), we first note that for $x\in\mathbb{S}^{d-1}$,

$$\sum_{i=1}^d\sum_{j=1}^d x_ix_jD_{i,j}=\sum_{i=1}^d\sum_{j=1}^d(x_i^2x_j\partial_j-x_ix_j^2\partial_i)=0.$$

Thus,

$$\begin{aligned}J_1&=\Big|\int_{\mathbb{S}^{d-1}}\Big[\sum_{i=1}^d\sum_{j=1}^d(y_i-x_i)x_jD_{i,j}f(x)\Big]f(x)h_\kappa^2(x)\,d\sigma(x)\Big|\\&\le\Big(\int_{\mathbb{S}^{d-1}}\frac{|\sum_{i,j=1}^d(y_i-x_i)x_jD_{i,j}f(x)|^2}{1-\langle x,y\rangle}h_\kappa^2(x)\,d\sigma(x)\Big)^{\frac12}\\&\quad\times\Big(\int_{\mathbb{S}^{d-1}}|f(x)|^2(1-\langle x,y\rangle)h_\kappa^2(x)\,d\sigma(x)\Big)^{\frac12}.\end{aligned}$$

But, by the Cauchy–Schwartz inequality,

$$\begin{aligned}\Big|\sum_{i=1}^d\sum_{j=1}^d(y_i-x_i)x_jD_{i,j}f(x)\Big|^2&\le\Big[\sum_{i,j=1}^d|x_j|^2(y_i-x_i)^2\Big]\Big[\sum_{i,j=1}^d|D_{i,j}f(x)|^2\Big]\\&=4(1-\langle x,y\rangle)\Big[\sum_{1\le i<j\le d}|D_{i,j}f(x)|^2\Big].\end{aligned}$$

It follows that

$$J_1\le2\Big(\sum_{1\le i<j\le d}\int_{\mathbb{S}^{d-1}}|D_{i,j}f(x)|^2h_\kappa^2(x)\,d\sigma(x)\Big)^{\frac12}\Big(\int_{\mathbb{S}^{d-1}}|f(x)|^2(1-\langle x,y\rangle)h_\kappa^2(x)\,d\sigma(x)\Big)^{\frac12},$$

which implies (4.4) by (2.2).

Finally, we prove (4.5). Splitting the integral $\int_{\mathbb{S}^{d-1}}\cdots$ into two parts, we get

$$J_2(\alpha)\le J_{2,1}(\alpha)+J_{2,2}(\alpha),\tag{4.6}$$

where

$$J_{2,1}(\alpha) := \Big| \langle y, \alpha \rangle \int_{|\langle x,\alpha\rangle| > (1-\varepsilon)|\langle y,\alpha\rangle|} \frac{|f(x)|^2 h_\kappa^2(x)}{\langle x, \alpha \rangle} \, d\sigma(x) \Big|,$$

$$J_{2,2}(\alpha) := \Big| \langle y, \alpha \rangle \int_{|\langle x,\alpha\rangle| \leqslant (1-\varepsilon)|\langle y,\alpha\rangle|} \frac{|f(x)|^2 h_\kappa^2(x)}{\langle x, \alpha \rangle} \, d\sigma(x) \Big|.$$

A straightforward calculation shows that

$$J_{2,1}(\alpha) \le \frac{1}{1-\varepsilon} \int_{\mathbb{S}^{d-1}} |f(x)|^2 h_\kappa^2(x) \, d\sigma(x) = \frac{1}{1-\varepsilon}. \tag{4.7}$$

To estimate the term $J_{2,2}(\alpha)$, we first note that for any $t \in (0,1)$ and $\alpha \in \mathcal{R}_+$,

$$\int_{|\langle x,\alpha\rangle| \leqslant t} \frac{|f(x)|^2}{\langle x, \alpha \rangle} h_\kappa^2(x) \, d\sigma(x) = \int_{|\langle x,\alpha\rangle| \leqslant t} \Big( E_\alpha f(x) \Big) f(x) h_\kappa^2(x) \, d\sigma(x).$$

Thus,

$$\begin{aligned} J_{2,2}(\alpha) &= \Big| |\langle y, \alpha \rangle| \int_{|\langle x,\alpha\rangle| \leqslant (1-\varepsilon)|\langle y,\alpha\rangle|} \Big( E_\alpha f(x) \Big) f(x) h_\kappa^2(x) \, d\sigma(x) \Big| \\ &\le \frac{1}{\varepsilon} \Big| \int_{\mathbb{S}^{d-1}} \|x - y\| \Big( E_\alpha f(x) \Big) f(x) h_\kappa^2(x) \, d\sigma(x) \Big| \\ &\le \frac{\sqrt{2}}{\varepsilon} \|E_\alpha f\|_{\kappa,2} \Big( \int_{\mathbb{S}^{d-1}} |f(x)|^2 (1 - \langle x, y \rangle) h_\kappa^2(x) \, d\sigma(x) \Big)^{\frac{1}{2}}, \end{aligned} \tag{4.8}$$

where the second step uses the fact that if $|\langle x, \alpha \rangle| \leqslant (1-\varepsilon)|\langle y, \alpha \rangle|$, then

$$\varepsilon |\langle y, \alpha \rangle| \leqslant |\langle y, \alpha \rangle| - |\langle x, \alpha \rangle| \leqslant \|x - y\|.$$

Now a combination of (4.6), (4.7) and (4.8) yields estimate (4.5).

This completes the proof of Theorem 1.1. ■

## 5 Uncertainty Principle on the Unit Ball and the Simplex

In this section, we will derive uncertainty principles for weighted orthogonal polynomial expansions on the unit ball and the simplex from results established in the last section.

Our argument is based on a close relationship among analysis on the unit sphere, the unit ball, and the simplex (see, *e.g.,* [8], [1, Sections 9,10]). More precisely, given two changes of variables $y = \phi(x)$, $z = \psi(x)$ with

$$\phi : \mathbb{B}^d \to \mathbb{S}^d, \quad x \in \mathbb{B}^d \mapsto (x, \sqrt{1 - \|x\|^2}) \in \mathbb{S}^d,$$

$$\psi : \mathbb{B}^d \to \mathbb{T}^d, \quad x \in \mathbb{B}^d \mapsto (x_1^2, x_2^2, \dots, x_d^d) \in \mathbb{T}^d,$$

we have that

$$\int_{\mathbb{S}^d} f(y) d\sigma(y) = \int_{\mathbb{B}^d} \Big[ f(x, \sqrt{1 - \|x\|^2}\,) + f(x, -\sqrt{1 - \|x\|^2}\,) \Big] \frac{dx}{\sqrt{1 - \|x\|^2}} \tag{5.1}$$

and

$$\int_{\mathbb{B}^d} g\big( \psi(x) \big) \, dx = \int_{\mathbb{T}^d} g(z) \, \frac{dz}{|z_1 \cdots z_d|}. \tag{5.2}$$

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Recall that $G$ is a finite reflection group on $\mathbb{R}^d$ with a root system $\mathcal{R} \subset \mathbb{R}^d$; $\kappa: \mathcal{R} \to [0, \infty)$ is a nonnegative multiplicity function on $\mathcal{R}$; the weight functions $W^B_{\kappa,\mu}$ on $\mathbb{B}^d$ and $W^T_{\kappa,\mu}$ on $\mathbb{T}^d$ are given in (1.6) and (1.7), (1.8), respectively.

Let $\Delta^B_{\kappa,\mu}$ and $\Delta^T_{\kappa,\mu}$ be the analogues of the Dunkl-Laplace-Beltrami operator $\Delta_{\kappa,0}$ on $\mathbb{B}^d$ and $\mathbb{T}^d$, respectively. They are second order differential-difference operators, and their precise definitions can be found in [8, Sections 8.1, 8.2]. Here, we simply emphasize the relations among the three operators. First, for a function $f$ on $\mathbb{B}^d$, the identity

$$(-\Delta^B_{\kappa,\mu})^\alpha f(x) = (-\Delta_{\tilde{\kappa},0})^\alpha \widetilde{f}(\phi(x)), \quad x \in \mathbb{B}^d, \quad \alpha \in \mathbb{R}, \tag{5.3}$$

holds in a distributional sense, where the weight associated with $\Delta_{\tilde{\kappa},0}$ is

$$h_{\tilde{\kappa}}(x) = |x_{d+1}|^\mu \prod_{v \in \mathbb{R}_+} |\langle x, v\rangle|^{\kappa_v}, \quad x \in \mathbb{S}^d$$

and $\tilde{f}(x, x_{d+1}) = f(x)$. Second, for a function $f$ on $C^2(\mathbb{T}^d)$,

$$\big((-\Delta^T_{\kappa,\mu})^\alpha f\big) \circ \psi(x) = 4^{-\alpha}(-\Delta^B_{\kappa,\mu})^\alpha (f \circ \psi)(x), \quad x \in \mathbb{B}^d, \quad \alpha \in \mathbb{R}. \tag{5.4}$$

Then the following results on the unit ball and the simplex, which are similar to that of Theorem 1.1 on the sphere, are immediate consequences of (5.1) ,(5.3) and (5.2),(5.4).

***Theorem 5.1*** *Let $f \in C^1(\mathbb{B}^d)$ be such that*

$$\int_{\mathbb{B}^d} f(x) W^B_{\kappa,\mu}(x)\, dx = 0 \quad \textit{and} \quad \int_{\mathbb{B}^d} |f(x)|^2 W^B_{\kappa,\mu}(x) dx = 1.$$

*Then*

$$\Big[\min_{y \in \mathbb{B}^d} \int_{\mathbb{B}^d} (1 - \langle x, y\rangle) |f(x)|^2 W^B_{\kappa,\mu}(x)\, d(x)\Big] \times \Big[\int_{\mathbb{B}^d} \big|\sqrt{-\Delta^B_{\kappa,\mu}} f(x)\big|^2 W^B_{\kappa,\mu}(x)\, d(x)\Big] \geq C_{d,\kappa,\mu} > 0.$$

***Theorem 5.2*** *Let $f \in C^1(\mathbb{T}^d)$ be such that*

$$\int_{\mathbb{T}^d} f(x) W^T_{\kappa,\mu}(x)\, dx = 0 \quad \textit{and} \quad \int_{\mathbb{T}^d} |f(x)|^2 W^T_{\kappa,\mu}(x) dx = 1.$$

*Then*

$$\Big[\min_{y \in \mathbb{T}^d} \int_{\mathbb{T}^d} (1 - \langle \psi^{-1}(x), \psi^{-1}(y)\rangle) |f(x)|^2 W^T_{\kappa,\mu}(x)\, d(x)\Big] \times \Big[\int_{\mathbb{T}^d} \big|\sqrt{-\Delta^T_{\kappa,\mu}} f(x)\big|^2 W^T_{\kappa,\mu}(x)\, d(x)\Big] \geq C_{d,\kappa,\mu} > 0,$$

*where we recall that $\psi^{-1}(x) = (\sqrt{x_1}, \sqrt{x_2}, \dots, \sqrt{x_d})$.*

We remark that Theorems 5.1 and 5.2 improve the corresponding results obtained recently in [11]. In fact, [11, Theorems 5.1, 5.2, 6.1, 6.2, and Corollary 5.3] follow directly from the above two theorems. We also note that equivalently, we can take the minimums in the above two theorems over the sphere $\mathbb{S}^d$ rather than the ball $\mathbb{B}^d$.

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*Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1*
*e-mail:* hfeng3@ualberta.ca