geometry of the algebraic theory are discussed by *Bôcher* and *Salmon*, in many text-books on Projective Geometry like *Veblen and Young*, and in a *Cambridge Tract* by *Bromwich*.

The chief difference between the examples of the present note and the theory of the books mentioned, lies in the use of Cartesian co-ordinates in place of the more usual homogeneous co-ordinates. While the latter system is preferable in projective and advanced general work, Cartesian co-ordinates should surely not be entirely neglected, and indeed they have certain advantages for the present purpose. Their use is more elementary, and likely to be more familiar to the students for whose instruction the above examples are suggested. Also ideas of length and perpendicularity, and the geometrical interpretation of the orthogonal transformation are simpler in rectangular Cartesian co-ordinates than in any other. It will be found that all the examples discussed above are based on these ideas.

JAMES HYSLOP.

Conditions obtained by Multiplication of Determinants.

The condition that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ should represent two straight lines can be obtained very easily by considering the product

$$\begin{vmatrix} l, & l', & 0 \\ m, & m', & 0 \\ n, & n', & 0 \end{vmatrix} \begin{vmatrix} l', & l, & 0 \\ m', & m, & 0 \\ n', & n, & 0 \end{vmatrix},$$

which is equal to

$$\begin{vmatrix} 2ll' & , \; lm' + l'm, & nl' + n'l \\ lm' + l'm, & 2mm' \;\;\; , & mn' + m'n \\ nl' + n'l, & mn' + m'n, & 2nn' \end{vmatrix},$$

and is identically zero.

For if $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is equivalent to

$$(lx + my + n)(l'x + m'y + n') = 0,$$

then $$\frac{ll'}{a} = \frac{mm'}{b} = \frac{nn'}{c} = \frac{mn' + m'n}{2f} = \frac{nl' + n'l}{2g} = \frac{lm' + l'm}{2h},$$

and hence

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0.$$

A former student, Mr H. M. Taylor, Clare College, Cambridge, sends me the following neat application of the same idea to the case of finding the condition that $ax^4 + 4\,bx^3y + 6\,cx^2y^2 + 4\,dxy^3 + ey^4 = 0$ should represent the rays of a harmonic pencil. If the pairs of conjugate rays are

$$lx^2 + 2mxy + ny^2 = 0,$$
$$l'x^2 + 2m'xy + n'y^2 = 0,$$

we have, for a *H.P.*, $nl' + n'l = 2\,mm'$. ....(1)

$$\therefore \quad \frac{ll'}{a} = \frac{nn'}{e} = \frac{mn' + m'n}{2d} = \frac{lm' + l'm}{2b} = \frac{nl' + n'l + 4\,mm'}{6c}$$

$$= \frac{nl' + n'l}{2c} = \frac{mm'}{c}, \quad \because \text{ by (1)}.$$

$$\therefore \qquad \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix} = 0.$$

The same idea is employed in the course of the following method of proving that if $ax^2 + by^2 + cz^2 + 2fyz + 2\,gzx + 2\,hxy$ transforms into $\lambda_1\xi^2 + \lambda_2\eta^2 + \lambda_3\zeta^2$ by a rotation of rectangular axes, then λ_1, λ_2, λ_3 are the roots of the discriminating cubic. If $\lambda_1\xi^2 + \lambda_2\eta^2 + \lambda_3\zeta^2$ is transformed, it becomes, with the usual notation,

$$\lambda_1\ (l_1x + m_1y + n_1z)^2 + \lambda_2\ (l_2x + m_2y + n_2z)^2 + \lambda_3\ (l_3x + m_3y + n_3z)^2;$$

$$\begin{aligned} \therefore \quad a &= \lambda_1 l_1^2 + \lambda_2 l_2^2 + \lambda_3 l_3^2, & f &= \lambda_1 m_1 n_1 + \lambda_2 m_2 n_2 + \lambda_3 m_3 n_3, \\ b &= \lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2, & g &= \lambda_1 n_1 l_1 + \lambda_2 n_2 l_2 + \lambda_3 n_3 l_3, \\ c &= \lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2, & h &= \lambda_1 l_1 m_1 + \lambda_2 l_2 m_2 + \lambda_3 l_3 m_3, \end{aligned}$$

whence $a + b + c = \lambda_1 + \lambda_2 + \lambda_3$.

Again $bc - f^2 = \Sigma\,\lambda_2\lambda_3\,(m_2n_3 - m_3n_2)^2$
$= \Sigma\,\lambda_2\lambda_3\,l_1^2$,

and similarly $ca - g^2 = \Sigma\,\lambda_2\lambda_3 m_1^2$, $ab - h^2 = \Sigma\,\lambda_2\lambda_3 n_1^2$.

$$\therefore \quad bc + ca + ab - f^2 - g^2 - h^2 = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2.$$

Further

$$\begin{vmatrix} l_1, & l_2, & l_3 \\ m_1, & m_2, & m_3 \\ n_1, & n_2, & n_3 \end{vmatrix} \begin{vmatrix} \lambda_1 l_1, & \lambda_2 l_2, & \lambda_3 l_3 \\ \lambda_1 m_1, & \lambda_2 m_2, & \lambda_3 m_3 \\ \lambda_1 n_1, & \lambda_2 n_2, & \lambda_3 n_3 \end{vmatrix} = \begin{vmatrix} \Sigma\lambda_1 l_1^2, & \Sigma\lambda_1 l_1 m_1, & \Sigma\lambda_1 n_1 l_1 \\ \Sigma\lambda_1 l_1 m_1, & \Sigma\lambda_1 m_1^2, & \Sigma\lambda_1 m_1 n_1 \\ \Sigma\lambda_1 n_1 l_1, & \Sigma\lambda_1 m_1 n_1, & \Sigma\lambda_1 n_1^2 \end{vmatrix}.$$

$$\therefore \quad \lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

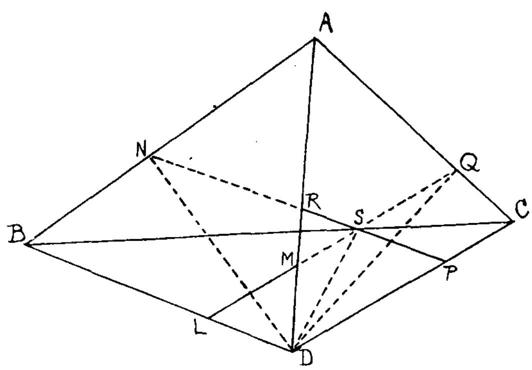
Using the values of a, b, ... above, it is easy to show that

$$\frac{al_1 + hm_1 + gn_1}{l_1} = \frac{hl_1 + bm_1 + fn_1}{m_1} = \frac{gl_1 + fm_1 + cn_1}{n_1} = \lambda_1$$

and so to obtain the result, but the direct method is of some interest.

R. J. T. Bell.

A Geometrical Proof for Hero's Formula.



The following proof is designed to link up Hero's formula geometrically with the formulae for the trigonometrical functions of $\frac{1}{2}A$ in a triangle.

From the bisector of angle A let AD be cut off equal to the mean proportional between AB and AC, and let N be the projection of D on AB. The formulae $AN = \sqrt{s(s-a)}$, $ND = \sqrt{(s-b)(s-c)}$ are easily established geometrically, and are assumed here. Thus triangle AND gives directly the formulae for $\sin \frac{1}{2}A$, $\cos \frac{1}{2}A$, $\tan \frac{1}{2}A$. Hero's formula is thus represented by $AN \,.\, ND$. It is required to prove therefore that twice the area of triangle AND is equal to the area of triangle ABC.

Join BD, DC and draw the pedal triangles LMN, PQR of the triangles ABD, ADC, which are similar since $AB : AD = AD : AC$. These triangles are then divided by their pedal triangles into similar component pairs, and the three triangles round a pedal triangle

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