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ON THE STRUCTURE OF A CLASS OF EQUIVARIANT MAPS

M.J. Field

Let G be a compact Lie group and M be a compact G-manifold. We investigate the class of equivariant diffeomorphisms of M covering the identity map on the orbit space M/G .

Introduction

Let H be a closed subgroup of the compact Lie group G . Denote the normaliser and centraliser of H in G by $N(H)$ and $C(H)$ respectively. If we let h, $n(h)$ and $c(h)$ denote the Lie algebras of H, $N(H)$ and $C(H)$ it is an elementary and well known result that $n(h) = h + c(h)$. An immediate consequence is that every element in the identity component $N(H)^0$ of $N(H)$ can be written, not necessarily uniquely, as a product ch , where $c \in C(H)^0$, $h \in H^0$. The main aim of Section 1 is to provide an alternative, differential geometric proof, of this result. In Section 2 we show how our techniques may be used to give information about G-equivariant diffeomorphisms covering the identity map on the orbit space.

We should point out that the techniques and results used in this paper are elementary. In particular, in Section 2, we make no serious use of homotopy theory and stop short of using any of the deeper results of smooth invariant theory as developed by Schwarz in [9].

1. Centralisers and normalisers

Let G be a group. Recall that the semi-direct product structure on

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$G \times G$ is defined by

$$(a, b).(c, d) = \left(ac, bada^{-1}\right), \quad (a, b), (c, d) \in G \times G .$$

We have an action of $G \times G$ on G defined by

$$(a, b)g = baga^{-1}, \quad (a, b) \in G \times G, \quad g \in G .$$

From now on we suppose G is a compact Lie group. Using Haar measure on $G \times G$, we may average any riemannian metric for G over $G \times G$ to obtain a $G \times G$-equivariant riemannian metric on G. Let $d(\, , \,)$ denote the corresponding distance function on G. Note that if $x, y \in G$ we have

(A) $$d(gx, gy) = d(x, y), \quad g \in G ,$$

(B) $$d\left(gxg^{-1}, gyg^{-1}\right) = d(x, y), \quad g \in G .$$

Indeed, these relations follow immediately from the $G \times G$-invariance of the metric d.

LEMMA 1. *If U is an open d-disc, centre e, in G, then*

$$gUg^{-1} = U, \quad g \in G .$$

Proof. Immediate from Property (B) of d. Compare Bredon [2, Chapter 0, Proposition 1-10]. □

LEMMA 2. *Let H be a closed subgroup of G. Then there exists an open d-disc U, centre e, in G and smooth map $\chi : U \to U$ such that:*

(1) $\chi(g) \in gH$, $g \in U$;

(2) $d(e, gH) = d\left(e, \chi(g)\right)$ *and* $\chi(g)$ *is the unique point in* gH *satisfying this relation;*

(3) $\chi\left(hgh^{-1}\right) = h\chi(g)h^{-1}$, $h \in H$, $g \in U$;

(4) $\chi(g) \in C(H)$, *if* $g \in N(H) \cap U$.

Proof. For all $g \in G$, gH is a submanifold of G. Choose a sufficiently small open d-disc neighbourhood U of the identity in G such that for every $g \in U$ there exists a unique minimising geodesic between e and gH (see Bishop and Crittenden [1] and note that our statement amounts to choosing a tubular neighbourhood of H in G, relative to the metric d). Property (A) of d implies that

$d(e,\ gH) = d(H,\ gH)$, $g \in U$. Define $\chi(g)$ to be the unique point in gH minimising distance between e and gH . Note that $\chi(g) \in U$ and that $\chi(g)$ depends smoothly on g , $g \in U$. By Property (B) of d we have

$$d(e,\ gH) = d\left(e,\ hgHh^{-1}\right) = d\left(e,\ h\chi(g)h^{-1}\right)\ ,\quad h \in G\ .$$

But $d\left(e,\ hgHh^{-1}\right) = d\left(e,\ \left(hgh^{-1}\right)\left(hHh^{-1}\right)\right)$ and so if $h \in H$ we see that $d\left(e,\ hgHh^{-1}\right) = d\left(e,\ hgh^{-1}H\right)$ and so $d\left(e,\ h\chi(g)h^{-1}\right) = d\left(e,\ \chi\left(hgh^{-1}\right)\right)$. By uniqueness of χ , it follows that

$$\chi\left(hgh^{-1}\right) = h\chi(g)h^{-1}\ ,\quad h \in H\ ,\quad g \in U\ .$$

Now if $g \in N(H) \cap U$, $hgh^{-1}H = gH$ and so

$$h\chi(g)h^{-1} = \chi(g)\ ,\quad h \in H\ ,\quad g \in N(H) \cap U\ ,$$

proving (4). □

COROLLARY TO LEMMA 1. *The identity component of* $N(H)$ *is generated by the identity components of* H *and* $C(H)$. *Moreover,* $C(H)$ *meets* H *transversally at* e *, in* $N(H)$.

Proof. Immediate from (4) of Lemma 2. □

LEMMA 3. *If* $H_1, \ldots, H_p$ *are closed subgroups of* G *then there exists an open* d*-disc* U *, centre* e *, in* G *such that for all* $g \in U \cap N\left(H_j\right)$ *,* $j = 1, \ldots, p$ *, there exists a unique* $\chi(g) \in gH_j \cap C\left(H_j\right)$ *minimising distance between* e *and* gH_j . *Moreover, if* $k \in G$ *,* $g \in U \cap N\left(H_j\right)$ *,* $k\chi(g)k^{-1} \in \left(kgk^{-1}\right)\left[kH_jk^{-1}\right] \cap C\left[kH_jk^{-1}\right] \cap U$ *and is the unique point minimising distance between* e *and* kgH_jk^{-1} .

Proof. The first part follows by choosing a sufficiently small disc at e which works for $H_1, \ldots, H_p$. The second part is immediate from Property (B) of the metric together with the invariance of U under the adjoint action (Lemma 1). □

REMARK. Let Σ denote the cut locus of the exponential map of $h^{\perp} \subset n(h)$ in G (see Bishop and Crittenden [1, pp. 237-241]). It is well known that Σ is always a sphere (topologically) though Σ may have singularities. Two questions naturally arise. Are the singularities of Σ always rational? To what extent can we regard the singularities of Σ as obstructions to expressing $N(H)^0$ as a product of H^0 and $C(H)^0$ and how might these singularities relate to cohomological invariants of this extension problem?

2. Structure of equivariant diffeomorphisms which are trivial mod G

Let M be a compact G-manifold. Throughout this section we assume M is connected and G acts smoothly on M ("smooth" will always mean C^∞). We follow the notation of Bredon [2] (see also Field [3, Section 1]). Thus if $x \in M$, $G(x)$ will denote the G-orbit through x and G_x the isotropy subgroup of G at x. We partition M into points of the same G-orbit type and write

$$M = \bigcup_{i=1}^{N} M_i ,$$

where all points in M_i have the same G-orbit type and if there exists $x \in M_i$, $y \in M_j$ such that $G_x \supsetneqq G_y$ then $i < j$. We refer to M_N as the principal G-orbit type and remark that M_N is open and dense in M and M_N/G is connected.

Let $F^r(M)$ denote the group of C^r equivariant diffeomorphisms of M satisfying $f(x) \in G(x)$ for all $x \in M$. Give $F^r(M)$ the C^r topology. We say that a C^r map $\chi : M \to G$ is C^r skew G-equivariant if

$$\chi(gx) = g\chi(x)g^{-1} , \quad g \in G , \quad x \in M .$$

Denote the set of C^r skew equivariant maps of M into G by $S^r(M)$. Observe that $S^r(M)$ has the structure of a group with composition defined by

$$(\chi_1 \cdot \chi_2)(x) = \chi_1(x)\chi_2(x) ,$$

$\chi_1, \chi_2 \in S^r(M)$, $x \in M$. We have a natural group homomorphism

$\gamma : S^r(M) \to F^r(M)$ defined by

$$\gamma(\chi)(x) = \chi(x)x \ , \quad x \in M \ .$$

In future we write $\gamma(\chi) = \hat{\chi}$, $\chi \in S^r(M)$ and set $\Gamma^r(M) = \gamma\big(S^r(M)\big)$.

LEMMA 1. *$\Gamma^r(M)$ is a normal subgroup of $F^r(M)$.*

Proof. Let $\chi \in S^r(M)$, $f \in F^r(M)$. Then

$$\big(f^{-1}\hat{\chi}f\big)(x) = \chi\big(f(x)\big)x \ , \quad x \in M \ .$$

Defining $\beta \in S^r(M)$ by $\beta(x) = \chi\big(f(x)\big)$, we see that

$$f^{-1}\hat{\chi}f = \hat{\beta} \in \Gamma^r(M)$$

and so $\Gamma^r(M)$ is normal in $F^r(M)$. □

We let $F_0^r(M)$ denote the subgroup of $F^r(M)$ consisting of diffeomorphisms which are C^r equivariantly isotopic to the identity (through elements of $F^r(M)$). Set $\Gamma_0^r(M) = F_0^r(M) \cap \Gamma^r(M)$.

The main aim of this section is to investigate the group $F^r(M)/\Gamma^r(M)$ and, in particular, find conditions which allow us to assert that an element of $F^r(M)$ actually lies in $\Gamma^r(M)$. First, however, we shall prove a useful technical lemma and then give some examples.

LEMMA 2. *Let G be a finite group and f be a homeomorphism of the compact G-manifold M (we do not assume f is equivariant). Suppose $f(x) \in G(x)$ for all $x \in M$. Fix $z \in M_N$ and suppose $f(z) = gz$, some $g \in G$. Then $f(x) = gx$ for all $x \in M$.*

Proof. Let $X = \{x \in M : f(x) = gx\}$. Obviously X is a closed, non-empty, subset of M . Since G is finite and f is continuous $f(x) = gx$ for all x in some open neighbourhood of z contained in M_N . Therefore $X \cap M_N$ is open and closed in M_N and so $X \cap M_N$ is a union of connected components of M_N . If M_N is connected it follows that $X \supset M_N$ and so, since M_N is dense in M , $X = M$. Suppose M_N is not connected and denote a connected component of M_N contained in X by M_N^a. Set $M_N^b = f(M_N^a)$. By considering f^{-1} we see that M_N^b is also a connected component of M_N. Since M_N is not connected there exist orbit types M_P of codimension one in M. Denote the union of the codimension 1 orbit types by P. Let $y \in P \cap \overline{M_N^a}$ and denote the other connected component of M_N whose closure contains y by M_N^c. Locally, at y, G_y acts as reflections in P. Let $r \in G_y$ denote such an element which acts as a local reflection in P. Choose a slice U for the G-action at y. Since $y \in X$ and f is continuous we may find $x \in M_N^c \cap U$ such that $f(x) \in g(U)$. Either $f(x) = gx$ or $f(x) = grx$. But if $f(x) = grx$ we must have $f(x) \in M_N^b$, contrary to the bijectivity of f. Hence $f(x) = gx$ and so $x \in X$. Therefore $M_N^c \subset X$. It follows that $X \supset M_N$ and so $X = M$. □

EXAMPLE 1. Let G be a finite group acting smoothly on the compact manifold M. Denote the principal orbit type of M by M_N. Since G is finite G_x is independent of $x \in M_N$ and we set $H = G_x$, $x \in M_N$. Note that H is a normal subgroup of G. For $g \in G$ we let $[G, g]$ denote the set $\{h^{-1}g^{-1}hg : h \in G\}$. Define $\tilde{P} = \{g \in G : [G, g] \subset H\}$. It is easily verified that $\tilde{P}$ is a subgroup of G containing H as a normal subgroup. Set $P = \tilde{P}/H$. We claim that $F^r(M) \approx P$, $0 \le r \le \infty$, and that $F^r(M) = \Gamma^r(M)$, $0 \le r \le \infty$. Suppose $\alpha \in \tilde{P}$. Define $f_\alpha \in F^\infty(M)$ by $f_\alpha(x) = \alpha x$, $x \in M$. Since $\alpha \in \tilde{P}$, it is clear that f_α is equivariant. Moreover, since $H \subset G_x$ for all $x \in M$, f_α depends only on the class of α in P. Hence we have defined a map of P into $F^\infty(M)$ which is clearly a group homomorphism. Next suppose $f \in \Gamma^0(M)$. Fix $z \in M_N$. Then $f(z) = gz$ for some $g \in G$. Since G is finite and f is continuous, Lemma 2 implies that $f(x) = gx$ for all $x \in M$. Since f is equivariant it follows easily that $g \in \tilde{P}$ and so we have constructed a map of $\Gamma^0(M)$ into P. This map is obviously the inverse of the homomorphism constructed above and so $F^\infty(M) = \Gamma^0(M) \approx P$, proving our assertions.

EXAMPLE 2. Let $S^1 = [0, 2\pi]/0 = 2\pi$ act on $\mathbb{C}$ in the standard way as multiplication by $e^{i\theta}$. Suppose $f \in F^r(\mathbb{C})$. In case $r = \infty$, Schwartz' theorem on smooth invariants [8] allows us to write $f(z) = g(|z|^2)z$ for some smooth map $g : \mathbb{R} \to \mathbb{C}$. Since $|g(|z|^2)| = 1$ for all $z \in \mathbb{C}$, we may define $\chi \in S^\infty(\mathbb{C})$ by $\chi(z) = g(|z|^2)$ and then $f = \hat{\chi}$. Thus we have proved that $F^r(\mathbb{C}) = \Gamma^r(\mathbb{C})$ in case $r = \infty$. This result is definitely false if $r < \infty$. For example, if we define

$$\begin{aligned} f(z) &= \exp(i/|z|^2)z, \quad z \neq 0, \\ &= 0, \quad z = 0, \end{aligned}$$

we see that $f \in F^0(\mathbb{C})$ but that f is not C^1. Clearly $f \notin \Gamma^0(\mathbb{C})$. Similar examples show that $F^r(\mathbb{C}) \neq \Gamma^r(\mathbb{C})$, $0 \leq r < \infty$.

EXAMPLE 3. Parametrize the torus $T^2 = S^1 \times S^1$ by $(\theta, \psi) \in [0, 2\pi] \times [0, 2\pi]$ and take the S^1-action on T^2 defined by

$$e^{i\sigma}(\theta, \psi) = (\theta, \psi+2\sigma), \bmod 2\pi .$$

Define $f : T^2 \to T^2$ by

$$f(\theta, \psi) = (\theta, \psi+\theta), \bmod 2\pi .$$

Clearly $f \in F^\infty(T^2)$ but $f \notin F_0^\infty(T^2)$. Notice that we cannot write $f = \hat{\chi}$ for some $\chi \in S^\infty(T^2)$ since $f(\theta, \psi) = e^{i\theta/2}(\theta, \psi)$. In fact it is not hard to verify that $F^\infty(T^2)/\Gamma^\infty(T^2) \cong \mathbb{Z}_2$ (the isotropy group of the S^1-action) and $\Gamma^\infty(T^2)/\Gamma_0^\infty(T^2) \cong \mathbb{Z}$ (the first cohomology group of the orbit space).

EXAMPLE 4. Regard $SU(2)$ as the group of complex matrices

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C},$$

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subject to $|a|^2 + |b|^2 = 1$, and take the corresponding matrix representation of $SU(2)$ on $\mathbb{C}^2$. Let $A \in \mathrm{End}(\mathbb{C}^2)$ be scalar multiplication by $e^{i\theta}$, $\theta \in (0, 2\pi)$. Certainly A is an $SU(2)$-equivariant map; indeed $A \in F^\infty(\mathbb{C}^2)$. Let us try to find a skew $SU(2)$-equivariant map $\chi : \mathbb{C}^2 \to SU(2)$ such that $A = \hat{\chi}$. If we fix $z = (z_1, z_2) \neq 0$ and solve

$$A(z) = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}(z)$$

for a and b we find

$$a(z) = \left(e^{i\theta}|z_1|^2 + e^{-i\theta}|z_2|^2\right)/\|z\|^2 ,$$

$$b(z) = \left(z_1\overline{z}_2\left(e^{i\theta} - e^{-i\theta}\right)\right)/\|z\|^2 ,$$

where $\|z\|^2 = |z_1|^2 + |z_2|^2$. Suppose $\theta \neq \pi$. If we define

$$\chi(z) = \begin{bmatrix} a(z) & b(z) \\ -\overline{b(z)} & \overline{a(z)} \end{bmatrix}, \quad z \neq 0 ,$$

we see that the resulting uniquely determined skew $SU(2)$-equivariant map χ does not extend continuously to $\mathbb{C}^2$. Consequently $F^\infty(\mathbb{C}^2)/\Gamma^\infty(\mathbb{C}^2)$ is infinite. Let us examine what happens if we use the polar blowing-up construction described in Field [4]. Recall that the polar blowing-up of $\mathbb{C}^2$ at 0 is the principal $SU(2)$-manifold $S^3 \times \mathbb{R}$ together with the projection $\pi : S^3 \times \mathbb{R} \to \mathbb{C}^2$ defined by $\pi(u, t) = tu$ (we regard $S^3 \subset \mathbb{C}^2$. Note that here we take $\pi(u, t) = tu$ rather than t^2u as was done in Field [4]). The map A lifts to $\tilde{A} : S^3 \times \mathbb{R} \to S^3 \times \mathbb{R}$ where $\tilde{A}(u, t) = (e^{i\theta}u, t)$. Therefore if $u = (z_1, z_2) \in S^3$ we see that

$$A(u, t) = \left(\begin{bmatrix} a(u) & b(u) \\ -\overline{b(u)} & \overline{a(u)} \end{bmatrix}(u), t\right)$$

where $a(u), b(u)$ are as constructed explicitly above. But a, b

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restrict to smooth maps on S^3 and so we see that $\tilde{A} \in \Gamma^\infty(S^3 \times \mathbb{R})$. If we write $\tilde{A} = \hat{\mu}$, $\mu \in S^\infty(S^3 \times \mathbb{R})$, then we may define a skew $SU(2)$-equivariant map $\chi : \mathbb{C}^2 \to SU(2)$ by

$$\chi(z) = \mu(y) ,$$

where y is any point in $\pi^{-1}(y)$. Although the map χ is not continuous (at 0) nevertheless we do have $A = \hat{\chi}$.

THEOREM 1. *Let G be a compact Lie group acting smoothly on the compact differential manifold M. Suppose that all G-orbits are of the same dimension. Then*

(1) $\Gamma_0^r(M) = F_0^r(M)$, $0 \le r \le \infty$,

(2) $F^r(M)/\Gamma^r(M)$ *is independent of* r, $0 \le r \le \infty$,

(3) $F^r(M) \cong ISF^r(M) \times \Gamma_0^r(M)$,

where $ISF^r(M)$ denotes the group of isotopy classes of C^r equivariant diffeomorphisms in $F^r(M)$ (isotopies through elements of $F^r(M)$).

The proof of this result will be broken into a number of lemmas.

LEMMA A. *Given any open neighbourhood U of the identity in G and $0 \le r \le \infty$, there exists an open neighbourhood N of the identity in $F^r(M)$ such that if $f \in N$ there exists a C^r map $\gamma : M \to G$ such that for all $x \in M$ we have $f(x) = \gamma(x)x$.*

Proof. Using slices we may reduce the proof of Lemma A to the case when $M = G \times_H D$, where D is a closed disc, centre 0 in an H-representation V and all H-orbits in V have dimension zero. The proof of this special case may be found in Field [6, Lemma B].

REMARK. If the dimension of G-orbits varies, Lemma A will only be valid for $r \ge r_0$, where r_0 is some positive integer less than the number of distinct G-orbit types of M. This may easily be seen using the blowing-up techniques of Field [4] together with the method of proof of Lemma A. In general we cannot take $r_0 = 0$ - the S^1-action of Example 3 provides a suitable counterexample.

LEMMA B. *There exists a neighbourhood* N *of the identity in* $F^r(M)$, $0 \le r \le \infty$, *such that if* $f \in N$ *there exists* $\chi \in S^r(M)$ *such that* $f = \hat{\chi}$.

Proof (*cf.* Field [6, Lemma C]). As in Section 1 we take the semi-direct product structure on $G \times G$ and a $G \times G$-equivariant riemannian metric d on G . Fix an equivariant riemannian metric on M . Let $x \in M$ and set $G_x = H$. Then $G(x)$ has an open tubular neighbourhood W smoothly equivariantly diffeomorphic to $G \times_H V$ where V is the orthogonal complement of $T_x G(x)$ in $T_x M$. In future we regard W as identified with $G \times_H V$. Since all G-orbits are assumed to have the same dimension every orbit of the action of H on V is finite. Let H^0 denote the identity component of H . If $y \in V$ then $H^0 \subset H_y \subset H$ and so there exist only finitely many distinct isotropy groups for the action of H on V say $H = K_1, \ldots, K_p$, where K_p denotes the principal isotropy group of the action of H on V . As in Section 1 we may find for $1 \le j \le P$ a closed neighbourhood U_j of the identity in G such that for $g \in U_j$ there exists a unique minimising geodesic between e and gK_j . Clearly we may assume $U_1 = \ldots = U_p = U$ and that U is a closed d-disc neighbourhood of the identity in G . For $g \in U$, $d(e, gK_j) = d(e, k_j(g))$ for a unique $k_j(g) \in gK_j \cap U$. Since the groups K_j have common identity component H^0 it is clear that we may require U chosen sufficiently small so that $k_1 = \ldots = k_p = k$, say. The map $k : U \to U$ is clearly smooth. By Lemma A, there exists a neighbourhood N_W of the identity in $F^r(M)$ and a C^r map $\gamma : M \to U$ such that $f(z) = \gamma(z)z$ for all $f \in N_W$, $z \in M$. We define $\tilde{\chi}_X : V \to G$ by $\chi_W(y) = k(\gamma(y))$, $y \in V$. Certainly $\tilde{\chi}_W$ is C^r and, by Section 1, $\tilde{\chi}_W$ is skew H-equivariant. Extending $\tilde{\chi}_W$ equivariantly to W we have shown that $f|W = \hat{\chi}_W$ for some $\chi_W \in S^r(W)$. Now take a finite cover of M by G-tubular neighbourhoods $W_1, \ldots, W_m$. Set $\chi_{W_i} = \chi_i$, $N_{W_i} = N_i$, $i = 1, \ldots, m$. Let $N = \bigcap_{i=1}^{m} N_i$. Given $y \in W_i \cap W_j$, $f \in N$, suppose $f(y) = gy$. Since $\chi_i(y)$ is characterised as being the unique point in gH_y minimising distance between H_y and gH_y it follows that $\chi_i(y) = \chi_j(y)$. Hence we may define $\chi \in S^r(M)$ by $\chi|W_i = \chi_i$, $i = 1, \ldots, m$. Clearly $f = \hat{\chi}$. □

LEMMA C. *Let* $f \in F_0^r(M)$, $r \geq 0$. *Then there exists* $\chi \in S^r(M)$ *such that* $f = \hat{\chi}$.

Proof. Follows easily from Lemma B and we omit details (see the proof of Lemma D in Field [*6*]). □

Proof of Theorem 1. Suppose $f \in F_0^r(M)$. Then by Lemma C there exists $\chi \in S^r(M)$ such that $f = \hat{\chi}$. Hence $F_0^r(M) \subset \Gamma_0^r(M)$. Since the reverse inclusion is obvious we have proved statement (1) of the theorem. Statement (3) follows by observing that if $f, f' \in F^r(M)$ determine the same element of $ISF^r(M)$ then $f'f^{-1} \in F_0^r(M)$. Statement (2) follows from (3) once we have shown that the group $ISF^r(M)$ is independent of r . For this it is enough to show that every element of $ISF^r(M)$ can be represented by a smooth equivariant diffeomorphism. The proof of this assertion is easily accomplished by localising using an equivariant partition of unity, applying Wassermann's approximation theorem [*10*] and using the fact that G-orbits all have the same dimension. We omit the tedious details. □

We now analyse the structure of the groups $F^r(M)$, $\Gamma^r(M)$ a little more closely. We continue to assume that all G-orbits have the same dimension.

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LEMMA 3. *Let* $f \in F^r(M)$ *and suppose that for some* $z \in M_N$, $f(z) \in N(G_z)^0(z)$. *Then* $f(x) \in N(G_x)^0(x)$ *for all* $x \in M$.

Proof. Using Lemma 2 it follows easily that $f(x) \in N(G_x)^0(x)$ for all $x \in M_N$. The result now follows from Lemma E of Field [6]. □

For $r \geq 0$, set $\hat{F}^r(M) = \left\{f \in F^r(M) : f(x) \in N(G_x)^0(x), \text{ all } x \in M\right\}$ and $\hat{\Gamma}^r(M) = \Gamma^r(M) \cap \hat{F}^r(M)$. It is easily verified that $\hat{F}^r(M)$, $\hat{\Gamma}^r(M)$ are normal subgroups of $F^r(M)$, $\Gamma^r(M)$ respectively.

PROPOSITION 1. *The group* $F^r(M)/\hat{F}^r(M)$ *is finite and independent of* $r \geq 0$. *It has order less than or equal to the order of* $N(H)/H.N(H)^0$ *where* H *denotes any principal isotropy group of the action of* G *on* M. *In particular, if* G *is abelian*

$$F^r(M)/\hat{F}^r(M) \approx G/K.G^0 ,$$

where K *denotes the principal isotropy group of the action of* G *on* M.

Proof. First observe that if $z \in M$ then $N(G_z)(z) \cong N(G_z)/G_z$. Moreover, since $\left(N(G_z)/G_z\right)^0 \approx N(G_z)^0/G_z \cap N(G_z)^0$, we have $N(G_z)^0(z) \cong \left(N(G_z)/G_z\right)^0$. Suppose $f, f' \in F^r(M)$. If there exists $z \in M_N$ such that $f(z)$, $f'(z)$ belong to the same connected component of $N(G_z)(z)$ then $f'f^{-1}(z) \in N(G_z)^0(z)$ and so, by Lemma 3, $f'f^{-1}(x) \in N(G_x)^0(x)$ for all $x \in M$. It follows immediately that $F^r(M)/\hat{F}^r(M)$ is finite with order bounded by the order of $\left(N(H)/H\right)/\left(N(H)/H\right)^0$, where H denotes any principal isotropy group of the action of G on M. But

$$\left(N(H)/H\right)/\left(N(H)/H\right)^0 \approx \left(N(H)/H\right)/\left(N(H)^0/H \cap N(H)^0\right) \approx \left(N(H)/H\right)/\left(H.N(H)^0/H\right)$$
$$\approx N(H)/H.N(H)^0 .$$

The assertion about the independence of r of the quotient group follows using Wassermann's approximation theorem as in the proof of Theorem 1. We leave the remaining statement to the reader. □

Let H denote a principal isotropy subgroup of the action of G on M and M^H denote the fixed point set of H. Let $M^{\langle H\rangle}$ denote the set of points in M with isotropy group H and $\tilde{M}^H$ denote the closure of $M^{\langle H\rangle}$. Observe that M^H, $\tilde{M}^H$ have the structure of compact $N(H)$-manifolds and that $\tilde{M}^H$ is a union of connected components of M^H (see Schwartz [9, Theorem 11.6]). We let $S^r(\tilde{M}^H)$ denote the group of C^r skew $N(H)$-equivariant maps of $\tilde{M}^H$ into $N(H)/H$.

PROPOSITION 2. *Let M be a compact G-manifold and suppose that all G-orbits have the same dimension. Then for $r \geq 0$ we have canonical isomorphisms*

$$F^r(M) \approx F^r(\tilde{M}^H) \approx S^r(\tilde{M}^H) ,$$

where $F^r(\tilde{M}^H)$ denotes the set of C^r $N(H)$-equivariant diffeomorphisms of $\tilde{M}^H$ covering the identity map of the orbit space.

Proof. Suppose $\mu \in S^r(\tilde{M}^H)$. Then $\mu(nx) = n\mu(x)n^{-1}$ for all $x \in \tilde{M}^H$, $n \in N(H)$. Define $\tilde{f}_\mu : \tilde{M}^H \to \tilde{M}^H$ by $\tilde{f}_\mu(x) = \mu(x)x$. Certainly $\tilde{f}_\mu$ is C^r. Moreover, $\tilde{f}_\mu$ is $N(H)$-equivariant since for $x \in \tilde{M}^H$, $n \in N(H)$ we have

$$\begin{aligned} \tilde{f}_\mu(nx) &= \mu(nx)nx \\ &= n\mu(x)n^{-1}nx = n\mu(x)x \\ &= n\tilde{f}_\mu(x) . \end{aligned}$$

Suppose $gx = hy$, $x, y \in \tilde{M}^H$, $g, h \in G$. Then $h^{-1}g \in N(H)$. Now $g\tilde{f}_\mu(x) = hh^{-1}g\tilde{f}_\mu(x) = h\tilde{f}_\mu(h^{-1}gx) = h\tilde{f}_\mu(y)$. Hence we may extend $\tilde{f}_\mu$ to a C^r equivariant diffeomorphism f_μ of M by setting $f_\mu(gx) = g\tilde{f}_\mu(x)$, $x \in \tilde{M}^H$, $g \in G$. Obviously $f_\mu \in F^r(M)$ and so we have constructed a homomorphism $\mu \mapsto f_\mu$ of $S^r(\tilde{M}^H)$ into $F^r(M)$ which factors through $F^r(\tilde{M}^H)$. Conversely, suppose $f \in F^r(M)$. Denote the restriction of f to $\tilde{M}^H$ by $\tilde{f}$. Thus $\tilde{f} \in F^r(\tilde{M}^H)$. For $x \in M^{\langle H \rangle}$, $\tilde{f}$ determines a unique C^r skew $N(H)$-equivariant map $\gamma_f : M^{\langle H \rangle} \to N(H)/H$ by $\tilde{f}(x) = \gamma_f(x)x$. We claim γ_f extends uniquely to $\tilde{M}^H$ as a C^r skew $N(H)$-equivariant map. Fix $x \in \tilde{M}^H$ and choose a slice S at x. Let $\tilde{f}(x) = gx$, some $g \in N(H)$. Then $g^{-1}\tilde{f}(x) = x$ and $g^{-1}\tilde{f} : \tilde{M}^H \to \tilde{M}^H$ covers the identity map on the orbit space $\tilde{M}^H/N(H)$. Choose a C^∞ local section ω of $N(H)$ over some open neighbourhood U of the identity coset in $N(H)/N(H)_x$. Shrinking S if necessary we may assume that $\omega(U)(S) \supset (g^{-1}\tilde{f})(S)$. Fix $x \in M^{\langle H \rangle} \cap S$. Now $g^{-1}\tilde{f}(z) \in \omega(U)\left(N(H)_x(z)\right)$. If $g^{-1}\tilde{f}(z) \notin \omega(U)(z)$, choose $k \in N(H)_x$ so that $kg^{-1}\tilde{f}(z) \in \omega(U)(z)$. As in the proof of Lemma 2 it then follows that $kg^{-1}\tilde{f}(y) \in \omega(U)(y)$ for all $y \in S$. Hence, by the implicit function theorem, there exists a C^r map $\pi : S \to N(H)$ such that $kg^{-1}\tilde{f}(y) = \pi(y)y$, $y \in S$. Regarding π as a map into $N(H)/H$, we see that $\pi = \gamma_f$ on $M^{\langle H \rangle} \cap S$. Extending π $N(H)$-equivariantly to $N(H)(S)$, we see that $\gamma_f | M^{\langle H \rangle} \cap N(H)S$ extends to a C^r $N(H)$-skew equivariant map from $M(H)S$ to $N(H)/S$. Since $x \in \tilde{M}^H$ was chosen arbitrarily we have shown that γ_f extends to all of $\tilde{M}^H$ as a C^r skew $N(H)$-equivariant map. Clearly $f \mapsto \gamma_f$ is the inverse of the map $\mu \to f_\mu$ constructed above. $\square$

Continuing with our assumption that H is a principal isotropy group of the action of G on M we note that, in general, $N(H) \neq C(H).H$. It is true, however, that $N(H) \supset C(H).H$ and $N(H)^0 = C(H)^0.H^0$, the latter statement following from Section 1. We let $\alpha : C(H) \to N(H)/H$, $\beta : C(H)^0 \to N(H)^0/H \cap N(H)^0$ denote the associated projection maps and remark that α is generally not onto whilst β is always onto. We have an $N(H)$-action on $C(H)$ (or $C(H)^0$) defined by $c \mapsto ncn^{-1}$, $c \in C(H)$, $n \in N(H)$. We say that a map $\phi : \tilde{M}^H \to C(H)$ (or $C(H)^0$) is $N(H)$-skew equivariant if ϕ is $N(H)$-equivariant relative to the $N(H)$-actions on $\tilde{M}^H$ and $C(H)$ (or $C(H)^0$). Suppose $f \in F^r(M)$. We say that $\gamma_f : \tilde{M}^H \to N(H)/H$ lifts to $C(H)$ if there exists a C^r map $\hat{\gamma}_f : \tilde{M}^H \to C(H)$ such that $\alpha\hat{\gamma}_f = \gamma_f$. Even if γ_f lifts to $\hat{\gamma}_f$ it need not generally be true that $\hat{\gamma}_f$ is $N(H)$-equivariant and so we cannot assert that the lifting of γ_f is a sufficient condition for f to lie in $\Gamma^r(M)$. It is, however, obviously a necessary condition. Suppose now that $f \in \hat{F}^r(M)$. Then $\gamma_f : \tilde{M}^H \to (N(H)/H)^0$. Since $(N(H)/H)^0 \approx N(H)^0/H \cap N(H)^0$ we may regard γ_f as a map into $N(H)^0/H \cap N(H)^0$. We say γ_f lifts to $C(H)^0$ if there exists a C^r-map $\hat{\gamma}_f : \tilde{M}^H \to C(H)^0$ such that $\beta\hat{\gamma}_f = \gamma_f$. Again the lifting of γ_f to $C(H)^0$ is obviously a necessary condition for f to lie in $\hat{\Gamma}^r(M)$.

PROPOSITION 3. *Suppose M is a compact G-manifold with all G-orbits of the same dimension and H is a principal isotropy subgroup of the action of G on M. Assume that $N(H)$ is connected and H is finite. Let $f \in F^r(M)$. Then γ_f lifts to $C(H)^0$ if and only if $f \in \hat{\Gamma}^r(M)$.*

Proof. Let $f \in F^r(M)$ and suppose γ_f lifts to $C(H)^0$. Fix $x \in M^{\langle H\rangle}$. For all $n \in N(H)$ we have $f(nx) = nf(x)$ and so $\hat{\gamma}_f(nx)nx = n\hat{\gamma}_f(x)x$. Therefore $\hat{\gamma}_f(x)^{-1}n^{-1}\hat{\gamma}_f(nx)n \in H$. The connectedness of $N(H)$ together with the finiteness of H now implies that $\hat{\gamma}_f(x)^{-1}n^{-1}\hat{\gamma}_f(nx)n = e$. Hence $\hat{\gamma}_f$ is skew $N(H)$-equivariant. The converse is trivial.

EXAMPLE. Let M be a compact S^1-manifold with all isotropy groups finite. Denote the principal isotropy group of the action by K. Let $f \in F^r(M)$. Then $f \in \Gamma^r(M)$ if and only if $\gamma_f : M \to S^1/K$ lifts to S^1. In particular, $F^r(M)/\Gamma^r(M) \cong K$.

REMARK 1. Using the above ideas it is now easy to show that if $f \in F^r(M)$ is sufficiently C^0-close to the identity then $f \in \Gamma^r(M)$. Indeed, if f is C^0 close to the identity so is γ_f and so γ_f lifts to $C(H)^0$. But now we can lift $\hat{\gamma}_f$ to $c(h)$ and average over $N(H)$ to obtain a skew $N(H)$-equivariant lifting of γ_f.

REMARK 2. Let $K^r(M) = \{\chi \in S^r(M) : \hat{\chi} = \text{identity}\}$. Then $K^r(M)$ is a closed subgroup of $S^r(M)$, $r \geq 0$. By Proposition 2, $F^r(M) \approx S^r(\tilde{M}^H)$. But $S^r(\tilde{M}^H)$ is the space of C^r $N(H)$-equivariant maps of $\tilde{M}^H$ to $N(H)/H$ where we take the $N(H)$-action on $N(H)/H$ defined by $k \mapsto nkn^{-1}$, $n \in N(H)$, $k \in N(H)/H$. In particular, taking $r = \infty$, we see that $S^r(\tilde{M}^H)$ has the structure of a Fréchet Lie group (see Field [3] or Palais [7] for background on differential structures on spaces of smooth equivariant maps). Using the remarks preceding Proposition 3 together with the argument of the proof of Proposition 3 it is easily verified that $\Gamma^\infty(M)$ has the structure of a closed Fréchet Lie subgroup of $F^\infty(M)$. Since $S^\infty(M)$ may be represented as a space of smooth G-equivariant maps we see also that $S^\infty(M)$ has the structure of a Fréchet Lie group with closed Fréchet Lie subgroup $K^\infty(M)$. It now follows straightforwardly that the sequence

$$1 \to K^\infty(M) \to S^\infty(M) \xrightarrow{p} \Gamma^\infty(M) \to 1 ,$$

$p(\chi) = \hat{\chi}$, is a short exact sequence of Frechet Lie groups. Lemma B implies that p admits local smooth sections.

We now turn to the case where the dimension of G-orbits varies. Recall from Field [4] that associated to any compact G-manifold M we have a compact $G \times (\mathbb{Z}_2)^{N-1}$-manifold $\hat{M}$ (the "resolution" of M) and projection $\pi : \hat{M} \to M$ satisfying:

(1) the action of G on $\hat{M}$ is principal;

(2) π is equivariant in the sense that

$$\pi\big((g, \gamma)x\big) = g\pi(x) \ , \quad g \in G \ , \quad \gamma \in (\mathbb{Z}_2)^{N-1} \ ;$$

(3) for $r \geq N-1$, there exists a continuous homomorphism

$$\phi : \mathrm{Diff}^r_G(M) \to \mathrm{Diff}^{r-N+1}_{G\times(\mathbb{Z}_2)^{N-1}}(M)$$

such that $\pi\phi(f) = f\pi$, $f \in \mathrm{Diff}^r_G(M)$.

THEOREM 2. *Let $f \in F^\infty_0(M)$. Then there exists a skew G-equivariant map $\mu \in S^\infty(\hat{M})$ such that*

(1) $\phi(f) = \hat{\mu}$,

(2) $f(x) = \mu(y)x$, *$x \in M$ and y any point in $\pi^{-1}(x)$.*

Proof. If $f \in F^\infty_0(M)$ then $\phi(f) \in F^\infty_0(\hat{M})$. Since all $G \times (\mathbb{Z}_2)^{N-1}$ orbits have the same dimension, Theorem 1 implies that there exists $\mu \in S^\infty(M)$ such that $\phi(f) = \hat{\mu}$. Now $\pi\phi(f) = f\pi$ and so if $\pi(y) = x$ we have $f(x) = \mu(y)x$. □

REMARK. Theorem 2 may be strengthened. If we let q denote the number of orbit types of M with G-orbits having the same dimension as the principal G-orbit type then we need only blow up M $(N-q)$-times to obtain a $G \times (\mathbb{Z}_2)^{N-q}$-manifold with all G-orbits of the same dimension. Using Property (3) of the resolution we see that the theorem is valid for $f \in F^{N-q+1}_0(M)$ rather than $F^\infty_0(M)$. Of course, we can apply Theorem 1 to each orbit type to find a skew G-equivariant map $\gamma : M \to G$ such that $f = \hat{\chi}$. But this approach tells us nothing about the singularities of χ .

One consequence of Theorem 2 and Example 4 is that we need to modify the definition of G-structural stability for equivariant diffeomorphisms given in Field [5].

DEFINITION. Let $f \in \mathrm{Diff}^r_G(M)$. We say f is G-structurally stable if we can find a neighbourhood N of f (C^r topology) such that for all $j \in N$ there exists an equivariant homeomorphism h of M and skew G-equivariant map $Q : M \to G$ such that

$$Q(f(x))h(f(x)) = jh(x) \text{ , all } x \in M \text{ .}$$

REMARK. In Field [5] we required Q to be continuous. By what we have shown above it is unrealistic to require Q to be continuous unless, for example, all G-orbits have the same dimension. Note, however, that $Q \circ h : M \to M$ is always an equivariant homeomorphism even if Q is not continuous.

We conclude with an analysis of S^1-manifolds with non-empty fixed point set. First an elementary technical lemma whose proof we include for completeness.

LEMMA 4. *Let* $f : \mathbb{C} \times \mathbb{R}^k \to \mathbb{C}$ *be smooth and vanish at* $(0, 0)$. *Suppose there exists a smooth function* $\alpha(z, t)$ *defined on* $(\mathbb{C} \times \mathbb{R}^k)\backslash\{0, 0\}$ *such that* $f(z, t) \doteq \alpha(z, t)z$, $(z, t) \neq 0$. *Then extends smoothly to* $\mathbb{C} \times \mathbb{R}^k$.

Proof. Fix $N \geq 1$. By Taylor's theorem

$$f(z, t) = \left(\sum_{j=1}^{N} \left(\sum_{i_1=0}^{j} a^j_{i_1 j-i_1}(t) z^{i_1} \bar{z}^{j-i_1}\right)\right) + \sum_{i_1=0}^{N+1} a^{N+1}_{i_1 N+1-i_1}(z, t) z^{i_1} \bar{z}^{N+1-i_1}$$

where the a^j's are smooth $\mathbb{C}$-valued functions. Dividing by z we see that

$$\frac{f(z,t)}{z} = A(z, t) + \sum_{j=1}^{N} \frac{a^j(t)\bar{z}^j}{z} + \frac{a^{N+1}(z,t)\bar{z}^{n+1}}{a} ,$$

where $A, a^1, \ldots, a^{N+1}$ are smooth. We claim $a^1 = \ldots = a^N \equiv 0$. If not, choose p , $1 \leq p \leq N$, to be the smallest integer such that $a^p(t) \not\equiv 0$. Then

$$\frac{f(z,t)}{z} = A(z, t) + \frac{a^p(t)\bar{z}^p}{z} + \frac{\bar{z}^{p+1}}{z} Q(z, t) ,$$

where Q is smooth. Fixing $t \neq 0$, we see that

$$\frac{\partial^{p-1}}{\partial z^{p-1}}\left(\frac{f(z,t)}{z}\right) = (-1)^p(p-1)!\,\frac{a^p(t)\bar{z}^p}{z^p} + \left(\frac{\bar{z}^{p+1}}{z^p}R(z,\ t)+T(z,\ t)\right) ,$$

where R and T are smooth functions on $\mathbb{C} \times \mathbb{R}^k$. Letting $z \to 0$ we see that $a^p(t)\left(\bar{z}^p/z^p\right)$ does not converge unless $a^p(t) = 0$. Since all the other terms do converge to definite limits as $z \to 0$ we see that $a^p(t) = 0$, $t \neq 0$. Since $a^p(t)$ is continuous on $\mathbb{R}^k$ we have therefore shown that $a^p \equiv 0$. Therefore $a^1 = \ldots = a^N \equiv 0$. Consequently, for any $N \geq 1$ we may write

$$\frac{f(z,t)}{z} = A(z,\ t) + \frac{\bar{z}^{N+1}}{z}S(z,\ t) ,$$

where A and S are smooth on $\mathbb{C} \times \mathbb{R}^k$. But the right hand side of this equation is C^{N-1} on $\mathbb{C} \times \mathbb{R}^k$. Since N was chosen arbitrarily it follows that $f(z,\ t)/z$ is smooth on $\mathbb{C} \times \mathbb{R}^k$. □

THEOREM 3. *Let M be a compact S^1-manifold. Then $F_0^\infty(M) = \Gamma_0^\infty(M)$.*

Proof. Let $f \in F_0^\infty(M)$. By Theorems 1 and 2 there exists a skew S^1-equivariant map $\chi : M \to S^1$ such that $f = \hat{\chi}$ and $\hat{\chi}$ is smooth off the fixed point set of the S^1-action. We must show that we can require χ to be smooth on all of M. Let x be a fixed point of the S^1-action on M. Choosing slices at x and $f(x)$ we easily reduce to showing that if $\mathbb{C}^k$ is an S^1-representation with no trivial factors and $f \in F^\infty\left(\mathbb{C}^k\right)$ then $f = \hat{\chi}$ for some $\chi \in S^\infty\left(\mathbb{C}^k\right)$. By Theorems 1 and 2, we may find a smooth skew S^1 equivariant map $\theta : \mathbb{C}^k \times \{0\} \to S^1$ such that

$$f\left(z_1,\ \ldots,\ z_k\right) = \left(\theta(z)^{p_1}z_1,\ \ldots,\ \theta(z)^{p_k}z_k\right) ,\quad z = \left(z_1,\ \ldots,\ z_k\right) \neq 0 .$$

(Here the integers $p_1,\ \ldots,\ p_k$ are just the orders of the irreducible factors of the S^1-representation on $\mathbb{C}^k$.) By Lemma 4, θ extends to a smooth map defined on all of $\mathbb{C}^k$.

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Department of Pure Mathematics,
University of Sydney,
Sydney,
New South Wales 2006,
Australia.