Bull. Austral. Math. Soc. 49A10
Vol. 39 (1989) [239–247]

AN EXISTENCE THEOREM FOR AN OPTIMAL CONTROL PROBLEM IN BANACH SPACES

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In this paper we prove the existence of an optimal admissible state-control pair for a nonlinear distributed parameter system, with control constraints of feedback type and with an integral cost criterion. An example is also worked in detail.

1. Introduction

The purpose of this note is to establish the existence of optimal solutions for a large class of nonlinear, distributed parameter control systems.

The optimal control problem is the following:

$$(\star)\qquad \begin{aligned} &J(x,u) = \int_0^b L(t,x(t),u(t))dt \to \inf \\ &[\text{where}\quad \dot{x}(t) + A(t,x(t)),u(t) = 0 \qquad \text{a.e.}] \\ &\qquad x(0) = x_0 \\ &\qquad u(t) \in U(t,x(t)) \qquad \text{a.e.} \end{aligned}$$

We prove that under very mild hypotheses, this infinite dimensional optimisation problem has a solution. Our existence results extends significantly earlier ones by Vidyasagar [14], Joshi [8] and Papageorgiou [10]. In these papers either the system was finite dimensional, or the hypotheses on the data were more restrictive.

The mathematical setting is the following. Let H be a separable Hilbert space and X a subspace of H carrying the structure of a separable, reflexive Banach space. We assume that the embedding of X in H is continuous, dense and compact. By identifying H with its dual, we have $X \hookrightarrow H \hookrightarrow X^*$ and all embeddings are continuous, dense and compact. By $(\cdot,\cdot)$ we will denote the duality brackets for the pair (X, X^*), and by $\langle\cdot,\cdot\rangle$ the inner product in H. The two are compatible in the sense that if $x \in X \subseteq H$ and $h \in H \subseteq X^*$, $(x,h) = \langle x,h\rangle$. Let Y be another separable Banach space (the control space). Finally the time horizon is finite, that is $T = [0,b]$. Then $A \in L(Y,X^*) = \{$bounded linear operators from Y into $X^*\}$ and $U : T \times X \to 2^Y \setminus \{\emptyset\}$ is a multifunction representing the control constraint set. Note that the control constraints are of feedback type. More detailed hypotheses on the data of $(\star)$ will be made in Section 3.

Received 30 May, 1988
Research supported by N.S.F. Grant D.M.S.-8802688. This work was done while Professor Papageorgiou was visiting the University of Thessaloniki, Greece.

2. Preliminaries

Let (Ω, Σ) be a measurable space and Z be a separable Banach space. By $P_{f(c)}(Z)$ we will denote the set of nonempty, closed, (convex) subsets of Z. A multifunction $F\colon \Omega \to P_f(Z)$ is said to be measurable if, for all $z \in Z$,

$$\omega \to d(z, F(\omega)) = \inf\{\|z - x\| : x \in F(\omega)\}$$

is measurable. If there is a complete σ-finite measure $\mu(\cdot)$ on (Ω, Σ), then the above definition of measurability is equivalent to saying that $GrF = \{(\omega, x) \in \Omega \times Z : x \in F(\omega)\} \in \Sigma \times B(Z)$, $B(Z)$ being the Borel σ-field of Z (graph measurability of $F(\cdot)$). In general however, while measurability implies graph measurability, the converse is not true. By S_F^1 we will denote the set of selectors of $F(\cdot)$ that belong to the Lebesgue-Bochner space $L^1(Z)$; that is $S_F^1 = \{f \in L^1(Z)\colon f(\omega) \in F(\omega)\mu - \text{a.e.}\}$. It is easy to check using Aumann's selection theorem, that S_F^1 is nonempty if $\omega \to |F(\omega)| = \sup\{\|x\| : x \in F(\omega)\}$ belongs to L_+^1.

Let E, H be Hausdorff topological spaces and $G\colon E \to 2^H \backslash \{\emptyset\}$ be a multifunction. We say that $G(\cdot)$ is *upper semicontinuous (u.s.c.)* if for all $V \subseteq H$ open, the set $G^+(V) = \{y \in E\colon G(y) \subseteq V\}$ is open in E.

Let W be a separable metric space and $B(W)$ its Borel σ-field. By $M(W)$ we will denote the space of bounded measures on $(W, B(W))$, and by $M_+^1(W)$ the probability measures on W. A *transition probability* is a function $\lambda\colon \Omega \times B(W) \to [0,1]$ such that, for all $A \in B(W)$, $\omega \to \lambda(\omega, A)$ is Σ-measurable while for every $\omega \in \Omega$, $\lambda(\omega, \cdot) \in M_+^1(W)$. When W is compact the above definition of transition probability is equivalent to saying that λ is $\left(\Sigma, B\left(M_+^1(W)\right)\right)$-measurable. Recall that on $M_+^1(W)$ we can define the weak (narrow) topology, which in this case is compact and metrisable (see Dellacherie and Meyer [5]). Finally, note that when W is a compact metric space, then $M(W) = [C(W)]^*$ (Riesz representation theorem) and so $M(W)$ is a separable dual Banach space, hence has the Radon-Nikodym property (R.N.P.). Therefore Theorem 1, p.98 of Diestel and Uhl [4] tells us that $L^\infty(M(W)) = [L^1(C(W))]^*$.

Finally, we recall that if W is a metric space, a function $f\colon \Omega \times W \to R$ is a Caratheodory function if for all $x \in W$ $\omega \to f(\omega, x)$ is measurable and for all $\omega \in \Omega$, $x \to f(\omega, x)$ is continuous on W. If W is compact and $|f(\omega, x)| \leqq \phi(\omega)$ $\mu-$a.e. with $\phi(\cdot) \in L_+^1$, then by associating with every $f(\cdot, \cdot)$ the map $(\omega \to f(\omega, \cdot))$, we see that the space of L^1-bounded Caratheodory functions can be identified with $L^1(C(W))$.

3. An existence theorem

Let $T = [0, b]$ and let (X, H, X^*), Y be the spaces introduced in Section 1. We will make the following hypotheses concerning the data of our optimal control problem $(\star)$:

H(A): $A\colon T \times X \to L(Y, X^*)$ is a map such that:

https://doi.org/10.1017/S0004972700002719 Published online by Cambridge University Press

(a) $t \to A(t,x)$ is measurable;

(b) $(x,u) \to A(t,x)u$ is sequentially continuous from $X_w \times W$ into X_w for all $t \in T$, where X_w is the Banach space X with the weak topology and W is a nonempty w-compact convex set in Y with the weak topology;

(c) $x \to A(t,x)u$ is monotone for all $u \in W$;

(d) $\|A(t,x)u\|_* \leqq c(1+\|x\|)$ a.e. for all $u \in W$ and with $c > 0$;

(e) $(A(t,x)u, x) \geqq c' \|x\|^2$ a.e. for all $u \in W$ and with $c' > 0$.

H(L): $L: T \times H \times Y \to \bar{R} = R \cup \{+\infty\}$ is an integrand such that

(a) $(t,x,u) \to L(t,x,u)$ is measurable;

(b) $(x,u) \to L(t,x,u)$ is sequentially l.s.c. from $H \times Y_w$ into $\bar{R}$, convex in u;

(c) $L(t,x,u) \geqq \phi(t)$ a.e. for all $x \in H$ and all $u \in U(t,x)$.

H(U): $U: T \times H \to P_{fc}(Y)$ is a multifunction such that:

(a) $(t,x) \to U(t,x)$ is measurable;

(b) $x \to U(t,x)$ is u.s.c. from H into Y_w;

(c) $U(t,x) \subseteq W$ a.e. with W being the nonempty, w-compact, convex subset of Y introduced in hypothesis H(A)(b).

By hypothesis H(A) and using Theorem 4.2 of Barbu [**3**, p.167], we know that given $u(\cdot) \in S^1_W = \{v \in L^1(Y): v(t) \in W \text{ a.e.}\}$, the evolution equation $\dot{x}(t) + A(t,x(t))u(t) = 0$, $x(0) = x_0$ has a unique solution belonging in $W(T) = \{x(\cdot) \in L^2(T,X): \dot{x}(\cdot) \in L^2(T,X^*)\} \subseteq C(T,H)$ (see Lions [**9**]).

By an *admissible pair* for the system ($\star$), we mean a control function $u(\cdot) \in L^1(Y)$ and a state function $x(\cdot) \in C(T,H)$ such that $x(\cdot)$ is the solution of the evolution equation with $u(\cdot)$ as the control and $u(t) \in U(t,x(t))$ a.e. We will denote the set of admissible state-control pairs by $P(x_0)$ and in order for our problem to have content, we will make the following hypothesis:

H_a : $P(x_0)$ is nonempty and, for some $(x,u) \in P(x_0)$, $J(x,u) < \infty$ (that is, system ($\star$) has admissible state-control pairs which have finite cost).

THEOREM. *If hypotheses H(A), H(L), H(U) and H_a hold, and $x_0 \in X$, then there exists $(x,u) \in P(x_0)$ such that*

$$J(x,u) = \inf\{J(x',u'): (x',u') \in P(x_0)\} = m$$

(that is, problem ($\star$) has an optimal solution).

PROOF: Let $\{(x_n,u_n)\}_{n\geq 1}$ be a minimising sequence of admissible state-control pairs; that is, $(x_n,u_n) \in P(x_0)$ and $J(x_n,u_n) \downarrow m$.

From [13], we know that $\{x_n\}_{n\geq 1} \subseteq C(T, X_w)$ and is relatively sequentially compact. Also $\{u_n\}_{n\geq 1} \subseteq S^1_W$ and by Proposition 3.1 of [11], S^1_W is w-compact in $L^1(Y)$ and so by the Eberlein-Smulian theorem is weakly sequentially compact. So, by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $C(T, X_w)$ and $u_n \xrightarrow{w} u$ in $L^1(Y)$. Our claim is that $(x, u) \in P(x_0)$.

First note that from Theorem 3.1 of [12] we have:

$$u(t) \in \overline{\text{conv}}\ w - \overline{\lim}\{u_n(t)\}_{n\geq 1} \subseteq \overline{\text{conv}}\, w - \overline{\lim}\, U(t, x_n(t)) \text{ a.e.}$$

But note that $x_n(t) \xrightarrow{w} x(t)$ in X, and, since X embeds compactly into H, we have $x(t) \xrightarrow{s} x(t)$ in H. Now by hypothesis H(U)(b), $U(t, \cdot)$ is u.s.c. from H into Y_w. So we have $w - \overline{\lim}\, U(t, x_n(t)) \subseteq U(t, x(t))$ a.e. (see Delahaye and Denel [4]). Thus $u(t) \in \overline{\text{conv}}\ U(t, x(t)) = U(t, x(t))$ a.e.

Next let $\{\delta_{u_n(\cdot)}(\cdot)\}_{n\geq 1}$ be the Dirac transition measures associated to the functions $\{u_n(\cdot)\}_{n\geq 1}$. Note that since Y is separable, W with the weak topology is metrisable and compact. From now, this will be the topology on W. Note that $\{\delta_{u_n(\cdot)}(\cdot)\}_{n\geq 1}$ is a bounded subset of $L^\infty(M(W)) = [L^1(C(W))]^*$ and $L^1(C(W))$ is separable. So Theorem 1 of Dunford and Schwartz [7] tells us that $\overline{\{\delta_{u_n(\cdot)}(\cdot)\}_{n\geq 1}}^{w^*}$ is w^*-sequentially compact. So by passing to a subsequence if necessary, we may assume that $\delta_{u_n} \xrightarrow{w^*} \lambda$ in $L^\infty(M(W))$. Then for every $z \in X$, we have:

$$\begin{aligned} 0 &= (z, x_n(t) - x_0) + \left(z, \int_0^t A(s, x_n(s))u_n(s)\mathrm{d}s\right) \\ &= (z, x_n(t) - x_0) + \left(z, \int_0^t \int_W A(s, x_n(s))v\delta_{u_n(s)}(\mathrm{d}v)\mathrm{d}s\right) \\ &= (z, x_n(t) - x_0) + \int_0^t \int_W (z, A(s, x_n(s))v)\delta_{u_n(s)}(\mathrm{d}v)\mathrm{d}s. \end{aligned}$$

Recalling that W is compact and using hypothesis H(A) (b), we have:

$$\sup |(z, A(s, x_n(s))v - A(s, x(s))v)| = |(z, A(z, x_n(s))v_n - A(s, x(s))v_n)|, \quad v_n \in W.$$

By passing to a subsequence if necessary, we may assume that $v_n \xrightarrow{w} v$ in Y. Then by hypothesis H(A) (b) we have:

$$|(z, A(s, x_n(s))v_n - A(s, x(s))v_n)| \to 0 \text{ as } n \to \infty.$$

So if we set $\hat{A}_n(z)(s, v) = (z, A(s, x_n(s))v)$ and $\hat{A}(z)(s, v) = (z, A(s, x(s))v)$ then $\hat{A}_n(z)(\cdot, \cdot)$ and $\hat{A}(z)(\cdot, \cdot)$ are Caratheodory functions on $T \times W$ and we just saw that

$\hat{A}_n(z)(t,\cdot) \to \hat{A}(z)(t,\cdot)$ in $C(W)$, for all $t \in T$. Invoking the dominated convergence theorem we get $\hat{A}_n(z)(\cdot,\cdot) \xrightarrow{s} \hat{A}(z)(\cdot,\cdot)$ in $L^1(T, C(W))$. So we deduce that:

$$\int_0^t \int_W \hat{A}_n(z)(t,v)\delta_{u_n(s)}(dv)ds = \int_0^t (z, A(s, x_n(s)))u_n(s)ds$$
$$\to \int_0^t \int_W \hat{A}(z)(t,v)\lambda(s)(dv)ds = \int_0^t \int_W (z, A(s,x(s))v)\lambda(s)(dv)ds.$$

So we have that for all $z \in X$:

$$\left(z, x(t) - x_0 + \int_0^t \int_W A(s,x(s))v\lambda(s)(dv)ds\right) = 0.$$

Since $X \hookrightarrow X^*$ densely, we get that:

$$x(t) - x_0 + \int_0^t \int_W A(s,x(s))v\lambda(s)(dv)ds = 0$$
$$\Rightarrow \dot{x}(t) + \int_W A(s,x(s))v\lambda(s)(dv) = 0 \text{ a.e., } x(0) = x_0.$$

Recalling that $u_n \xrightarrow{w} u$ in $L^1(Y)$ and $\delta_{u_n} \xrightarrow{w^*} \lambda$ in $L^\infty(M(W))$, we have for all $A \in B(T)$:

$$\int_A u_n(s)ds = \int_A \int_W v\delta_{u_n(s)}(dv)ds \to \int_A u(s)ds = \int_A \int_W v\lambda(s)(dv)ds.$$

Since $A \in B(T)$ was arbitrary, we get that:

$$u(t) = \int_W v\lambda(t)(dv) \text{ a.e.}$$

So

$$\int_W A(t,x(t))v\lambda(t)(dv) = A(t,x(t)) \int_W v\lambda(t)(dv)$$
$$= A(t,x(t))u(t) \text{ a.e.}$$

Therefore, we conclude that $\dot{x}(t) + A(t,x(t))u(t) = 0$ a.e., $x(0) = x_0$, and we already know that $u(t) \in U(t,x(t))$ a.e. Hence (x,u) is an admissible state-control pair for $(\star)$.

From Lemma 2 of Balder [2] we know that we can find $L_m : T \times H \times W \to R$ Caratheodory integrands such that $L_m(t,x,v) \uparrow L(t,x,v)$ as $m \to \infty$. Then, as before, we can show that for every $m \geqq 1$, we have:

$$\int_0^b \int_W L_m(t, x_n(t), v)\delta_{u_n(s)}(dv)ds \to \int_0^b \int_W L_m(t, x(s), v)\lambda(s)(dv)ds \text{ as } n \to \infty.$$

On the other hand, from the monotone convergence theorem, we have:

$$\int_0^b \int_W L_m(t, x(s), v)\lambda(s)(dv)ds \to \int_0^b \int_W (t, x(s), v)\lambda(s)(dv)ds \text{ as } m \to \infty.$$

Hence by a diagonalisation process, we have:

$$\int_0^b L_{k(n)}(s, x_n(s), u_n(s))ds \to \int_0^b \int_W L(t, x(s), v)\lambda(s)(dv)ds.$$

But recall that $\{(x_n, u_n)\}_{n \geqq 1}$ is a minimising sequence of admissible state-control pairs. So

$$\begin{aligned} \lim \int_0^b L_{k(n)}(s, x_n(s), u_n(s))ds &= \int_0^b \int_w L(t, x(s), v)\lambda(s)(dv)ds \\ &\leqq \lim J(x_n, u_n) = m. \end{aligned}$$

On the other hand by hypothesis H(L) (b) $L(t, x, \cdot)$ is convex. So through Jensen's inequality, we get that:

$$\begin{aligned} \int_0^b \int_W L(s, x(s), v)\lambda(s)(dv)ds &\geqq \int_0^b L\Big(s, x(s), \int_W v\lambda(s)(dv)\Big)ds \\ &= \int_0^b L(s, x(s), u(s))ds = J(x, u). \end{aligned}$$

Therefore $J(x, u) \leqq m$. Since $(x, u) \in P(x_0)$, we deduce that $J(x, u) = m$ and so (x, u) is the desired optimal state-control pair for problem $(\star)$. ∎

4. An example

In this section we present a nonlinear distributed parameter control system on which our result applies.

Let W be an open domain in $\mathbb{R}^n$ with smooth boundary $\partial W = \Gamma$ and let $T = [0, b]$. On $T \times W$ we consider the following nonlinear control system:

$$(\star\star) \quad \begin{aligned} &\frac{\partial x(t,z)}{\partial t} + \sum_{|\alpha| \leqq m-1} (-1)^{|\alpha|} A_\alpha(t, z, k(x(t,z)))u(t,z) = 0, \text{ where} \\ &k(x) = \{D^\alpha x : \ |\alpha| \leqq m - 1\}; \\ &[D^\beta x = 0 \text{ on } T \times \Gamma, \quad |\beta| \leqq m - 2 \text{ and } 2m > n;] \\ &x(0, z) = x_0(z) \text{ on } \{0\} \times W; \\ &|u(t,z)| \leqq \phi(t, x(t,z)) \text{ a.e. with } \phi(t, y) \leqq M. \end{aligned}$$

https://doi.org/10.1017/S0004972700002719 Published online by Cambridge University Press

We will make the following hypotheses:

(1) $(t,z) \to A(t,z,r)$ is measurable ($r = (r_0, \dots, r_{m-1})$);
(2) $r \to A(t,z,r)$ is continuous;
(3) $|A(t,z,r)v| \leqq g(z) + c \cdot |r|$ for all $v \in R$ with $g \in L^2(W)$;
(4) $\sum_{|\alpha| \leqq m-1} (A_\alpha(t,z,r)v - A_\alpha(t,z,r')v, r_\alpha - r'_\alpha) \geqq 0$ for all $v \in R$;
(5) $\sum_{|\alpha| \leqq m-1} A_\alpha(t,x,r)r_\alpha \geqq c' |r|^2$.

Consider the Dirichlet form corresponding to the differential operator in divergence form in our problem ($\star\star$). So we have:

$$p_u(t,x,y) = \sum_{|\alpha| \leqq m-1} \int_W A_\alpha\big(t,z,k(x(z))u(z)D_y^\alpha(z)\big)\mathrm{d}z.$$

Then for $t \in T$, let $A_u(t)\colon W_0^{m,2}(W) \to W^{-m,2}(W)$ be defined by $(A_u(t)x,y) = p_u(t,x,y)$.

Since $u \to A_u(t)x$ is continuous, linear on $L^2(W)$, we can write $A(t,x)u$.

We will show that $(x,u) \to A(t,x)u$ is sequentially weakly continuous from $W_0^{m,2}(W) \times L^2(W)$ into $W^{-m,2}(W)$. So let $x_n \xrightarrow{w} x$ in $W_0^{m,2}(W)$ and $u_n \xrightarrow{w} u$ in $L^2(W)$. Since $W_0^{m,2}(W) \hookrightarrow W_0^{m-1,2}(W)$ compactly (see Adams [**1**]), we get that $x_n \xrightarrow{s} x$ in $W_0^{m-1,2}(W)$. Also hypotheses (1), (2), (3) and Krasnoselski's theorem tell us that the Nemitsky operator $\hat{A}_\alpha$ corresponding to A_α is continuous from $W_0^{m-1,2}(W)$ into $L^2(W)$. So we have $\hat{A}_\alpha(t,x_n) \xrightarrow{s} \hat{A}_\alpha(t,x)$ in $L^2(W)$ and since $u_n \xrightarrow{w} u$ in $L^2(W)$, we get that $\hat{A}_\alpha(t,x_n)u_n \xrightarrow{w} \hat{A}_\alpha(t,x)u$ in $L^1(W)$. Recall that since $2m > n$, from the Sobolev embedding theorem we have that $W_0^{m,2}(W) \hookrightarrow C(\bar{W})$ and so

$$\int_W A_\alpha(t,z,x_n(z))u_n(z)v(z)\mathrm{d}z \to \int_W A_\alpha(t,z,x(z))u(z)v(z)\mathrm{d}z.$$

Therefore $(x,u) \to A(t,x)u$ is sequentially weakly continuous as claimed.

Furthermore, by the Pettis measurability theorem, (see Diestel and Uhl [**6**]), it is easy to check that $t \to A(t,x)$ is measurable. Also from hypothesis (3), (4) and (5) we have that for $u \in K = \{v \in L^2(W)\colon \|v\|_2 \leqq M\lambda(W)^{\frac{1}{2}}\}$, $x \to A(t,x)u$ is monotone,

$$\|A(t,x)u\|_{W^{-m,2}(W)} \leqq C_1\left(1 + \|x\|_{W_0^{m,2}}\right)$$

and

$$(A(t,x)u,x)_{W_0^{-m,2}} \geqq C_2 \|x\|^2_{W_0^{m,2}}.$$

Set $U(t,x) = \{v \in L^2(W)\colon |v(z)| \leqq \phi(t,x(z)) \text{ a.e.}\}$. If we assume that $\phi(t,\cdot)$ is continuous, we can easily check that $(t,x) \to U(t,x)$ is measurable, while $x \to U(t,x)$

is an u.s.c. set valued mapping from $L^2(W)$ into $L^2(W)_w$. Furthermore $U(t,x) \subseteq K$ and the latter is weakly compact in $L^2(W)$.

We are also given an integral cost functional to be minimised over the admissible state-control pairs of $(\star\star)$. This has the following form:

$$(\star\star') \qquad \int_0^b \int_W L(t,z,x(t,z),u(t,z))\mathrm{d}z\mathrm{d}t.$$

Here $L\colon T \times W \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is an integrand such that:

(6) $(t,z,r,u) \to L(t,z,r,u)$ is l.s.c. and convex in u;

(7) $L(t,z,r,u) \geqq \lambda(t,z)$ a.e. for all $(r,u) \in \mathbf{R}\times\mathbf{R}$ with $\lambda(\cdot,\cdot) \in L^2(T\times W)$.

Let $\tilde{L}\colon T \times L^2(W) \times L^2(W) \to \bar{\mathbf{R}}$ be defined by:

$$\tilde{L}(t,x,u) = \int_W L(t,z,x(z),u(z))\mathrm{d}z.$$

Because of (6) and (7) above, it is easy to check that $\tilde{L}(\cdot,\cdot,\cdot)$ satisfies hypothesis H(L) of Section 3.

So the full optimal control problem with cost criterion $(\star\star)'$ and constraint $(\star\star)$, can have the following abstract formulation:

$$(\star\star\star) \qquad \begin{aligned} &J(x,u) = \int_0^b \tilde{L}(t,x(t),u(t))\mathrm{d}t \to \inf \\ &[\text{s.t. } \dot{x}(t) + A(t,x(t))u(t) = 0 \text{ a.e.}] \\ &x(0) = x_0, \quad u(t) \in U(t,x(t)) \text{ a.e.} \end{aligned}$$

If we take $X = W_0^{m,2}(W)$, $H = L^2(W)$ and $X^* = W^{-m,2}(W)$, we see that $(\star\star\star)$ is equivalent to $(\star)$ and so we can apply the Theorem of Section 3 and get optimal solutions, provided there exist admissible state-control pairs with finite cost and $x_0 \in W_0^{m,2}(W)$.

References

[1] R.A. Adams, *Sobolev Spaces* (Academic Press, New York, 1975).

[2] E. Balder, 'Lower semicontinuity of integral functionals with nonconvex integrands by relaxation-compactification', *SIAM J. Control Optim.* **19** (1981), 533–541.

[3] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces* (Noordhoff Intern. Publishing, Leyden, Netherlands, 1976).

[4] J.P. Delahaye and J. Denel, 'The continuities of the point-to-set maps, definitions and equivalences', *Math. Programming* **10** (1979), 8–12.

[5] J.C. Dellacherie and P.A. Meyer, *Probabilities and Potentials* (North Holland, New York, 1978).

[6] J. Diestel and J. Uhl, 'Vector Measures', in *Math. Surveys Vol 15* (A.M.S., Providence, Rhode Island, 1977).

[7] N. Dunford and J. Schwartz, *Linear Operators* (Wiley, New York, 1958).

[8] M. Joshi, 'On the existence of optimal controls in Banach spaces', *Bull. Austral. Math. Soc.* **27** (1983), 395–401.

[9] J.L. Lions, *Quelques Méthodes de Resolution des Problemes aux Limites Non Lineaires* (Dunod-Gaut Villars, Paris, 1969).

[10] N.S. Papageorgiou, 'Existence of optimal controls for nonlinear systems in Banach spaces', *J. Optim. Theory Appl.* **53** (1987), 541–549.

[11] N.S. Papageorgiou, 'On the theory of Banach space valued multifunctions. Part 1: Integration and and conditional expectation', *J. Multivariate Anal.* **17** (1985), 185–207.

[12] N.S. Papageorgiou, 'Convergence theorems for Banach valued integrable multifunctions', *Internat. J. Math. Math. Sci.* **10** (1987), 433–442.

[13] N.S. Papageorgiou, 'Properties of the relaxed trajectories of evolution equations and optimal control', *SIAM J. Control Optim.* **27** (1989). (in press) .

[14] M. Vidyasagar, 'On the existence of optimal controls', *J. Optim. Thoery Appl* **17** (1975), 273–278.

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