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ON A GLOBAL UPPER BOUND FOR JESSEN'S INEQUALITY

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Abstract

In two recent papers a global upper bound is derived for Jensen's inequality for weighted finite sums. In this paper we generalize this result on positive normalized functionals.

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1. Introduction and preliminaries

Let $\widetilde{x} = \{x_i\}$ be a finite sequence of real numbers from the fixed closed interval $I = [a, b]$, $a < b$, and $\widetilde{p} = \{p_i\}$, with $\sum p_i = 1$ a sequence of positive weights associated with $\widetilde{x}$. If we have a convex function $f : I \to \mathbb{R}$, from Jensen's inequality we have

$$0 \le \sum p_i f(x_i) - f\left(\sum p_i x_i\right).$$

The following was proved in [6].

THEOREM 1.1. *Let $\widetilde{x}$, $\widetilde{p}$ be as above. Then, if f is convex on $I = [a, b]$, we have that*

$$\sum p_i f(x_i) - f\left(\sum p_i x_i\right) \le S_f(a, b), \tag{1.1}$$

where

$$S_f(a, b) := f(a) + f(b) - 2f\left(\frac{a+b}{2}\right).$$

However, this fact can be derived from the following two theorems published earlier in [4, Page 50].

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THEOREM 1.2. *Let $\widetilde{x}$, $\widetilde{p}$ be as above. Then, if f is convex on $I = [a, b]$, we have that*

$$f\left(a + b - \sum p_i x_i\right) \leq f(a) + f(b) - \sum p_i f(x_i). \tag{1.2}$$

THEOREM 1.3. *Let $\widetilde{x}$, $\widetilde{p}$ be as above. Then, if f is convex on $I = [a, b]$, we have that*

$$\begin{aligned} f\left(a + b - \sum p_i x_i\right) &\geq 2f\left(\frac{a+b}{2}\right) - f\left(\sum p_i x_i\right) \\ &\geq 2f\left(\frac{a+b}{2}\right) - \sum p_i f(x_i). \end{aligned} \tag{1.3}$$

Combining (1.2) and (1.3) it is clear that we also have (1.1).

The purpose of this paper is to generalize the above results for normalized positive functionals.

Let E be a nonempty set and L be a linear class of real-valued functions $f : E \to \mathbb{R}$ having the properties

$$(af + bg) \in L \quad \forall a, b \in \mathbb{R} \tag{L_1}$$

$$\text{if } 1 \in L, \text{ that is, } f(t) = 1L \quad \forall t \in E, \quad \text{then} \quad f \in L. \tag{L_2}$$

We also consider positive linear functionals $A : L \to \mathbb{R}$. That is, we assume that

$$A(af + bg) = aA(f) + bA(g) \in L \quad \forall\, f, g \in L,\ a, b \in \mathbb{R}, \tag{A_1}$$

$$\text{if } f \in L,\ f(t) \geq 0 \quad \text{on } E \quad \text{then} \quad A(f) \geq 0 \quad (A \text{ is positive}). \tag{A_2}$$

If $A(1) = 1$, we say that A is a normalized functional. The following generalization of the Jensen's inequality for convex functions is known (see [5, Page 47]).

THEOREM 1.4. *Let L satisfy L_1 and L_2 on a nonempty set E, and assume that Φ is continuous convex function on an interval $I \subset \mathbb{R}$. If A is a normalized linear positive functional, then for all $g \in L$ such that $\Phi(g) \in L$ we have $A(g) \in I$ and*

$$\Phi(A(g)) \leq A(\Phi(g)). \tag{1.4}$$

Also, the proof of the following theorem can be found in [5, Page 98].

THEOREM 1.5. *Let Φ be convex on $I = [a, b]$, $(-\infty < a < b < \infty)$; let L satisfy conditions L_1 and L_2 and let A be a positive normalized functional on L. Then for every $g \in L$ such that $\Phi(g) \in L$ (so that $a \leq g(t) \leq b$), we have*

$$A(\Phi(g)) \leq \frac{b - A(g)}{b - a}\Phi(a) + \frac{A(g) - a}{b - a}\Phi(b). \tag{1.5}$$

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2. Main results

THEOREM 2.1. *Let L satisfy L_1 and L_2 and let Φ be a convex function on $I = [a, b]$. Then for any positive normalized linear functional A on L and for any $g \in L$ such that $\Phi(g) \in L$ we have*

$$A(\Phi(g)) - \Phi(A(g)) \leq \Phi(a) + \Phi(b) - 2\Phi\left(\frac{a+b}{2}\right). \tag{2.1}$$

If Φ is concave, the inequality in (2.1) is reversed.

PROOF. From inequality (1.5) we have

$$A(\Phi(g)) - \Phi(A(g)) \leq \frac{b - A(g)}{b-a}\Phi(a) + \frac{A(g) - a}{b-a}\Phi(b) - \Phi(A(g)). \tag{2.2}$$

Now, using (2.2) we deduce (2.1) showing that

$$\frac{b - A(g)}{b-a}\Phi(a) + \frac{A(g) - a}{b-a}\Phi(b) - \Phi(A(g)) \leq \Phi(a) + \Phi(b) - 2\Phi\left(\frac{a+b}{2}\right). \tag{2.3}$$

It is easy to see that (2.3) is equivalent to

$$\Phi(a)\left(1 - \frac{b - A(g)}{b-a}\right) + \Phi(b)\left(1 - \frac{A(g) - a}{b-a}\right) + \Phi(A(g)) \geq 2\Phi\left(\frac{a+b}{2}\right). \tag{2.4}$$

Applying Jensen's inequality to the left-hand side of (2.4) we obtain

$$\begin{aligned}&\frac{1}{2}\left[\Phi(a)\left(1 - \frac{b - A(g)}{b-a}\right) + \Phi(b)\left(1 - \frac{A(g) - a}{b-a}\right) + \Phi(A(g))\right]\\ &\quad\geq \Phi\left(\frac{a+b}{2} + \frac{1}{2}\left[A(g) - \frac{b - A(g)}{b-a}a - \frac{A(g) - a}{b-a}b\right]\right) = \Phi\left(\frac{a+b}{2}\right).\end{aligned}$$

The last equality proves inequality (2.4) which is equivalent to (2.1).

The concave case can be proved by the same arguments using the fact that $-\Phi$ is a convex function. □

The following theorem is an extension of Theorem 1.2.

THEOREM 2.2. *Let Φ be convex on $I = [a, b]$, $(-\infty < a < b < \infty)$; let L satisfy conditions L_1 and L_2 and let A be a positive normalized functional on L. Then for every $g \in L$ such that $\Phi(g) \in L$ (so that $a \leq g(t) \leq b$), we have*

$$\Phi(a + b - A(g)) \leq \Phi(a) + \Phi(b) - A(\Phi(g)).$$

PROOF. For a proof of this result see [1, Page 2]. □

The next theorem is an extension of Theorem 1.3.

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THEOREM 2.3. *Let Φ be convex on $I = [a, b]$, $(-\infty < a < b < \infty)$; let L satisfy conditions L_1 and L_2 and let A be a positive normalized functional on L. Then for every $g \in L$ such that $\Phi(g) \in L$ (so that $a \le g(t) \le b$), we have*

$$\Phi(a + b - A(g)) \ge 2\Phi\left(\frac{a+b}{2}\right) - \Phi(A(g)).$$

PROOF. From the reversed Jensen's inequality [5, Page 83] we have

$$\Phi\left(\frac{px + qy}{p+q}\right) \ge \frac{p\Phi(x) + q\Phi(y)}{p+q} \quad \text{for } q < 0,\ p > 0,\ p + q > 0. \tag{2.5}$$

Putting $p = 2$, $q = -1$, $x = (a + b)/2$ and $y = A(g)$ in (2.5) we obtain the desired result. □

Let us observe that with the combination of Theorems 2.2 and 2.3 we can obtain an alternative proof of Theorem 2.1, just by eliminating the expression $\Phi(a + b - A(g))$.

Now we show that we can improve the upper bound for Jensen's inequality.

THEOREM 2.4. *Let L satisfy L_1 and L_2 and let Φ be a convex function on $I = [a, b]$. Then for any positive normalized linear functional A on L and for any $g \in L$ such that $\Phi(g) \in L$ we have*

$$A(\Phi(g)) - \Phi(A(g)) \le \left\{\frac{1}{2} + \frac{1}{b-a}\left|\frac{a+b}{2} - A(g)\right|\right\} \cdot S_{\Phi}(a, b). \tag{2.6}$$

If Φ is concave, the inequality in (2.6) is reversed.

For the proof of this theorem we need following lemma.

LEMMA 2.5. *For a convex function $f : D_f \to \mathbb{R}$, $x, y \in D_f$, $0 \le p, q \le 1$, $p + q = 1$, we have that*

$$\min\{p, q\}S_f(x, y) \le pf(x) + qf(y) - f(px + qy) \le \max\{p, q\}S_f(x, y).$$

PROOF. For a proof of this result see [6]. □

PROOF OF THEOREM 2.4. Using Theorem 1.5 we have

$$A(\Phi(g)) \le \frac{b - A(g)}{b-a}\Phi(a) + \frac{A(g) - a}{b-a}\Phi(b).$$

Denote

$$p = \frac{b - A(g)}{b-a},$$

so $p \in [0, 1]$ and $A(g) = p \cdot a + (1 - p) \cdot b$.

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Hence, we have

$$\begin{aligned}
&A(\Phi(g)) - \Phi(A(g)) \\
&\quad \le \frac{b - A(g)}{b - a}\Phi(a) + \frac{A(g) - a}{b - a}\Phi(b) - \Phi(A(g)) \\
&\quad = p\Phi(a) + (1 - p)\Phi(b) - \Phi(p \cdot a + (1 - p) \cdot b) \\
&\quad \le \max\{p, 1 - p\} S_\Phi(a, b) = \left\{\frac{1}{2} + \frac{1}{b - a}\left|\frac{a + b}{2} - A(g)\right|\right\} S_\Phi(a, b).
\end{aligned}$$

The third line follows from Lemma 2.5. At the end, if Φ is concave, then $-\Phi$ is convex, so that the conclusion follows. □

We can also restate Theorem 2.4 in the following form.

THEOREM 2.6. *Let L satisfy L_1 and L_2, let Φ be a convex function on $I = [a, b]$, and let A be a positive linear functional on L. Suppose that $k \in L$, $k \geq 0$ on E and $A(k) > 0$. Then for any $g_1 \in L$ such that $kg_1 \in L$ and $k\Phi(g_1) \in L$ we have*

$$\frac{A(k\Phi(g_1))}{A(k)} - \Phi\left(\frac{A(kg_1)}{A(k)}\right) \le \left\{\frac{1}{2} + \frac{1}{b - a}\left|\frac{a + b}{2} - \frac{A(kg_1)}{A(k)}\right|\right\} \cdot S_\Phi(a, b). \tag{2.7}$$

If Φ is concave, the inequality in (2.7) is reversed.

In [6] we can find a refinement of the inequality given in (1.1) introducing the *characteristic $c(f)$*:

$$c(f) := \sup \frac{\sum p_i f(x_i) - f\left(\sum p_i x_i\right)}{S_f(a, b)},$$

where the supremum is taken over all $\widetilde{p}, \widetilde{x} \in [a, b]$, $a, b \in D_f$. Hence, we have

$$\sum p_i f(x_i) - f\left(\sum p_i x_i\right) \le c(f) S_f(a, b).$$

The refinement of the bound is described by the next theorem (see [4]).

THEOREM 2.7. *For any convex function f,*

$$\frac{1}{2} \le c(f) \le 1.$$

In our new terms the characteristic for a convex function Φ is described by

$$C(\Phi) = \sup_{A, g} \frac{A(\Phi(g)) - \Phi(A(g))}{S_\Phi(a, b)}, \tag{2.8}$$

where the supremum is taken over all positive normalized linear functionals A on L and over all $g \in L$.

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Here, we give a proof of Theorem 2.7 in our new terms.

First, it is obvious that $C(\Phi) \le 1$.

To show $C(\Phi) \ge 1/2$, we first define the positive, normalized functional A_1 by

$$A_1(g) = p_0 g(x) + (1 - p_0) g(y),$$

where x, y are some points in the starting set E and $0 < p_0 < 1$. Finally,

$$\begin{aligned} C(\Phi) = \sup_{A,g} \frac{A(\Phi(g)) - \Phi(A(g))}{S_\Phi(a, b)} &\ge \sup_{p_0,x,y} \frac{A_1(\Phi(g)) - \Phi(A_1(g))}{S_\Phi(a, b)} \\ &= \sup_{p_0,x,y} \frac{p_0\Phi(g(x)) + (1 - p_0)\Phi(g(y)) - \Phi(p_0 g(x) + (1 - p_0) g(y))}{S_\Phi(x, y)} \\ &\ge \sup_{p_0} [\min\{p_0, 1 - p_0\}] = \frac{1}{2} \end{aligned}$$

by Lemma 2.5.

3. The Hadamard inequality

Let us note that from (2.6) we have in the case $A(g) = (a + b)/2$ that

$$\begin{aligned} \Phi\left(\frac{a+b}{2}\right) \le A(\Phi(g)) \le \Phi\left(\frac{a+b}{2}\right) + \frac{1}{2} S_\Phi(a, b), \\ \Phi\left(\frac{a+b}{2}\right) \le A(\Phi(g)) \le \frac{\Phi(a) + \Phi(b)}{2}, \end{aligned} \tag{3.1}$$

which is a generalization of the well-known Hadamard inequality (see [5, Page 146]).

In what follows we denote by e_i ($i \in \mathbb{N}$) the function $e_i : [a, b] \to \mathbb{R}$ defined by

$$e_i(x) = x^i, \quad x \in [a, b].$$

Let $A : C[a, b] \to \mathbb{R}$ be a linear positive functional and let a_i be defined by

$$a_i := A(e_i), \quad i \in \mathbb{N}.$$

In what follows we assume that $a_0 = 1$. For such a functional, Jessen's inequality is well known and it states that for any convex function Φ we have

$$A(\Phi) \ge \Phi(a_1) \quad \text{and} \quad A(\Phi) \le \frac{b - a_1}{b - a}\Phi(a) + \frac{a_1 - a}{b - a}\Phi(b).$$

The following result was obtained by Lupaş in [3].

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THEOREM 3.1 (Lupaş [3]). *Let* $A : C[a, b] \to \mathbb{R}$ *be a positive linear functional with* $A(e_0) = 1$. *Then, for any convex function* $\Phi \in C[a, b]$, *there exist distinct points* $\xi_1, \xi_2 \in [a, b]$ *such that*

$$A(\Phi) - \Phi(a_1) = (a_2 - a_1^2)\left[\xi_1, \frac{\xi_1 + \xi_2}{2}, \xi_2; \Phi\right],$$

where the divided difference of a function Φ *on the nodes* $x_1, \ldots, x_k$ *is denoted by* $[x_1, \ldots, x_k; \Phi]$.

THEOREM 3.2. *For any convex function* $\Phi \in C[a, b]$ *the following inequality holds:*

$$A(\Phi) - \Phi(a_1) \le \left[\frac{1}{2} + \frac{1}{b-a}\left|\frac{a+b}{2} - a_1\right|\right] S_\Phi(a, b).$$

PROOF. Set $g = e_1$ in Theorem 2.4. □

REMARK 3.3. In fact, Theorems 2.4 and 3.2 are equivalent. Indeed, let $B : L \to \mathbb{R}$ defined by

$$B(\Phi) = A(\Phi \circ g),$$

where A is a positive normalized linear functional and $g \in L$ such that $\Phi \circ g \in L$. It follows from Theorem 3.2 that for any convex function $\Phi : [a, b] \to \mathbb{R}$ we have

$$B(\Phi) - B(e_1) \le \left[\frac{1}{2} + \frac{1}{b-a}\left|\frac{a+b}{2} - a_1\right|\right] S_\Phi(a, b).$$

Since $B(e_1) = A(g)$ we obtain Theorem 2.4.

COROLLARY 3.4. *Let* A *be a normalized linear positive functional. If* $A(\Phi) = A(\Phi(a + b - \cdot))$ *for every* $\Phi \in C[a, b]$, *then for any convex function* $\Phi \in C[a, b]$ *we have*

$$\Phi\left(\frac{a+b}{2}\right) \le A(\Phi) \le \frac{\Phi(a) + \Phi(b)}{2}. \tag{3.2}$$

PROOF. We have $A(a + b - e_1) = A(e_1)$, which implies that $A(e_1) = (a + b)/2$. Therefore, from (3.1) we obtain (3.2). □

REMARK 3.5. Let $\Phi : [a, b] \to \mathbb{R}$ be a convex function and $p : [a, b] \to \mathbb{R}$ be a nonnegative integrable function which is symmetric with respect to the point $(a + b)/2$, that is, $p(x) = p(a + b - x)$. If we consider the normalized linear positive functional

$$A(\Phi) = \frac{\int_a^b p(x)\Phi(x)\,dx}{\int_a^b p(x)\,dx}$$

in (3.2), we obtain

$$\Phi\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \le \int_a^b p(x)\Phi(x)\,dx \le \frac{\Phi(a) + \Phi(b)}{2}\int_a^b p(x)\,dx$$

which is a well-known inequality due to Fejér [2].

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3.1. A functional specific characteristic number In what follows the characteristic number defined in (2.8) is specialized to a particular normalized linear positive functional. Let $\Phi : D \to \mathbb{R}$ be a convex function which is not an affine function and let A be a positive linear functional defined on a linear set of functions $\mathcal{F}$, with domain D. We assume $[a, b] \subset D$ and denote by $\chi_{[a,b]}$ the characteristic function of the interval $[a, b]$. We further assume that for any $a < b$, the condition

$$A(\chi_{[a,b]}) > 0$$

is satisfied.

If Φ is not an affine function and Φ is continuous and convex, we define the number $C_A(\Phi)$ by

$$C_A(\Phi) := \sup \frac{A^{[a,b]}(\Phi) - \Phi\big(a_1^{[a,b]}\big)}{\Phi(a) + \Phi(b) - 2\Phi((a+b)/2)},$$

where

$$A^{[a,b]}(\Phi) := \frac{A(\chi_{[a,b]}\Phi)}{A(\chi_{[a,b]})}, \quad a_1^{[a,b]} := A^{[a,b]}(e_1)$$

and the supremum is taken over all values a, b, $a < b$, $[a, b] \subset D$. From the definition of $C_A(\Phi)$ we have

$$A^{[a,b]}(\Phi) - \Phi\big(a_1^{[a,b]}\big) \le C_A(\Phi)\left(\Phi(a) + \Phi(b) - 2\Phi\left(\frac{a+b}{2}\right)\right).$$

From Theorem 3.2 it follows that

$$C_A(\Phi) \le \frac{1}{2} + \sup_{a<b} \frac{1}{b-a}\left|\frac{a+b}{2} - a_1^{[a,b]}\right|.$$

Now let us consider the functional $A : C[a, b] \to \mathbb{R}$ given by

$$A(\Phi) = \frac{1}{b-a}\int_a^b \Phi(x)\, dx.$$

From Theorem 3.1 we obtain

$$A(\Phi) - \Phi\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{12}\left[\xi_1, \frac{\xi_1+\xi_2}{2}, \xi_2; \Phi\right],$$

which gives

$$\frac{A(\Phi) - \Phi((a+b)/2)}{\Phi(a) + \Phi(b) - 2\Phi((a+b)/2)} = \frac{1}{6}\frac{[\xi_1, (\xi_1+\xi_2)/2, \xi_2; \Phi]}{[a, (a+b)/2, b; \Phi]}.$$

Let $b = a + h$. Then

$$C_A(\Phi) \ge \lim_{h\to 0} \frac{A(\Phi) - \Phi((a+b)/2)}{\Phi(a) + \Phi(b) - 2\Phi((a+b)/2)}.$$

If $\Phi \in C^2(D)$, then

$$\lim_{h\to 0} \frac{A(\Phi) - \Phi((a+b)/2)}{\Phi(a) + \Phi(b) - 2\Phi((a+b)/2)} = \frac{1}{6} = C_A(e_2).$$

We summarize the result obtained as follows.

For the functional A defined by

$$A(\Phi) = \frac{1}{b-a} \int_a^b \Phi(x)\, dx$$

and any function $\Phi \in C^2(D)$ we have

$$C_A(e_2) \leq C_A(\Phi) \leq \frac{1}{2}.$$

We conclude the discussion on the characteristic $c_A(\cot)$ by stating the following conjecture.

CONJECTURE 3.6. *Let A be a normalized linear positive functional on a linear class of functions L, $A : L \to \mathbb{R}$ and let $\Phi \in L$ be a convex, nonaffine function. Then the following inequality holds*

$$C_A(e_2) \leq C_A(\Phi) \leq 1.$$

4. Applications to means

In this section we give some applications of Theorem 2.4 to some well-known means.

We start with the generalized mean with respect to the operator A and Ψ:

$$M_\psi(g, A) := \psi^{-1}\{A(\psi(g))\}, \quad g \in L.$$

THEOREM 4.1. *Let L satisfy conditions L_1 and L_2 and let A be a positive normalized functional on L. Let $\chi, \psi : [a, b] \to \mathbb{R}$ be continuous and strictly monotonic functions, such that the function $\Phi = \chi \circ \psi^{-1}$ is convex. If, for every $g \in L$, the functions $\psi(g), \phi(g) \in L$, then $\Phi(A(\psi(g)))$ is well defined and the inequality*

$$\chi(M_\chi(g, A)) - \chi(M_\psi(g, A)) \leq \left\{\frac{1}{2} + \frac{1}{b-a}\left|\frac{a+b}{2} - \psi(M_\psi(g, A))\right|\right\} \cdot S_\Phi(a, b) \tag{4.1}$$

holds. The inequality in (4.1) is reversed if the function Φ is concave.

PROOF. For $g \in L$, we have both $\psi(g), \chi(g) \in L$ by assumption. Hence, $\Phi(\psi(g)) = \chi(g) \in L$. Thus, if Φ is convex, then (4.1) follows from Theorem 2.4 with g replaced by $\psi(g)$. □

The next step is the application to generalized classical means:

$$M^{[r]}(g, A) := \begin{cases} A(g^r)^{1/r}, & r \neq 0; \\ \exp(A(\ln g)), & r = 0. \end{cases}$$

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THEOREM 4.2. *Let L satisfy conditions L_1 and L_2 and let A be a positive normalized functional on L. If, for every $g \in L$, the functions $g^r, \ln g \in L$, $r \neq 0$, then $\Phi(A(\psi(g)))$ is well defined and the inequality*

$$\begin{aligned}\{M^{[s]}(g, A)\}^s &- \{M^{[r]}(g, A)\}^s \\ &\le \left\{\frac{1}{2} + \frac{1}{b-a}\left|\frac{a+b}{2} - \{M^{[r]}(g, A)\}^r\right|\right\} \cdot \left(a^{s/r} + b^{s/r} - 2\left(\frac{a+b}{2}\right)^{s/r}\right),\end{aligned} \tag{4.2}$$

holds for $s > 0$, $s > r$ or $s < 0$, $s < r$. In the case $s > 0$, $s < r$ or $s < 0$, $s > r$ the inequality (4.2) is reversed. Also

$$\begin{aligned}A(\ln g) &- \ln\big(M^{[r]}(g, A)\big) \\ &\le \left\{\frac{1}{2} + \frac{1}{b-a}\left|\frac{a+b}{2} - \{M^{[r]}(g, A)\}^r\right|\right\} \cdot \ln\left(\frac{4ab}{(a+b)^2}\right),\end{aligned} \tag{4.3}$$

for $s = 0$, $r < 0$. In the case $s = 0$, $r > 0$ the inequality (4.3) is reversed. Finally

$$\{M^{[s]}(g, A)\}^s - \big(M^{[0]}(g, A)\big)^s \le \left\{\frac{1}{2} + \frac{1}{b-a}\left|\frac{a+b}{2} - 1\right|\right\} \cdot \big(e^{sa/2} - e^{sb/2}\big)^2, \tag{4.4}$$

for $r = 0$, $s > 0$. In the case $r = 0$, $s < 0$ the inequality (4.4) is reversed.

PROOF. The proof of the theorem follows from application of Theorem 4.1 and the cases given in the lines that follow. Let $\psi(x) = x^r$, $r \neq 0$, $\chi(x) = x^s$, $s \neq 0$, $\psi(x) = \ln x$, $r = 0$, $\chi(x) = \ln x$ and $s = 0$. For $s, r \neq 0$ a function $\Phi = (\chi \circ \psi^{-1})(x) = x^{s/r}$ is convex if $s > 0$, $s > r$ or $s < 0$, $s < r$.

For $s = 0$, $r \neq 0$ a function $\Phi(x) = (\chi \circ \psi^{-1})(x) = (1/r) \ln x$ is convex if $r < 0$.

For $r = 0$, $s \neq 0$ a function $\Phi(x) = (\chi \circ \psi^{-1})(x) = e^{sx}$ is convex if $s > 0$.

Now, using Theorem 4.2 we have proved (4.2). The last part of the theorem follows from concavity of the function. □

The next application is Hölder's inequality of the first type.

THEOREM 4.3. *Let L satisfy L_1 and L_2 and let A be a positive linear functional on L. Let $p > 1$ and q such that $1/p + 1/q = 1$. If $f, g > 0$ on E, f^p, g^q, $fg \in L$, then we have*

$$\begin{aligned}A(fg) &- \big(A(f^p)\big)^{1/p}\big(A(g^q)\big)^{1/q} \\ &\ge \left\{\frac{A(g^q)}{2} + \frac{A(g^q)}{b-a}\left|\frac{a+b}{2} - A\left(\frac{f^p}{g^q}\right)\right|\right\}\left(a^{1/p} + b^{1/p} - 2\left(\frac{a+b}{2}\right)^{1/p}\right).\end{aligned} \tag{4.5}$$

For $0 < p < 1$ the inequality (4.5) is reversed.

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PROOF. Let us note that the function

$$\Phi(x) = \frac{x^s}{s(s-1)}, \quad s \neq 0, 1$$

is a convex function. Now, from (2.7) we obtain

$$\frac{1}{s(s-1)}\left[\frac{A(k(g_1)^s)}{A(k)} - \left(\frac{A(kg_1)}{A(k)}\right)^s\right] \\ \le \left\{\frac{1}{2} + \frac{1}{b-a}\left|\frac{a+b}{2} - \frac{A(kg_1)}{A(k)}\right|\right\} \cdot \frac{1}{s(s-1)}\left(a^s + b^s - 2\left(\frac{a+b}{2}\right)^s\right). \tag{4.6}$$

After we make substitutions, $s = 1/p$, $k = g^q/A(g^q)$ and $g_1 = f^p/g^q$, we obtain the desired inequality. The reverse of inequality in (4.5) for $0 < p < 1$ follows from the inequality given in (4.6). □

The second type of Hölder's inequality is as follows.

THEOREM 4.4. *Let L satisfy L_1 and L_2 and let A be a positive linear functional on L. Let $p \in \mathbb{R} \setminus (0, 1)$ and q such that $1/p + 1/q = 1$. If $f, g > 0$ on E, $f^p, g^q, fg \in L$, then we have*

$$\left((A(f^p))^{1/p}(A(g^q))^{1/q}\right)^p - (A(fg))^p \\ \le \left\{\frac{A(g^q)^p}{2} + \frac{A(g^q)^p}{b-a}\left|\frac{a+b}{2} - A(fg^{1-q})\right|\right\}\left(a^p + b^p - 2\left(\frac{a+b}{2}\right)^p\right). \tag{4.7}$$

For $0 < p < 1$ the inequality (4.7) is reversed.

PROOF. Again, we consider the convex function

$$\Phi(x) = \frac{x^s}{s(s-1)}, \quad s \neq 0, 1$$

for $s = p \in \mathbb{R} \setminus (0, 1)$, and Theorem 2.6. Using (2.7) with the substitutions $k = g^q$, $g_1 = fg^{1-q}$ we obtain the desired result.

The reverse of this inequality follows by the same argument as in the previous theorem. □

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