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A DENDROLOGICAL PROOF OF THE SCOTT CONJECTURE FOR AUTOMORPHISMS OF FREE GROUPS

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Let α be an automorphism of a free group of rank n. The Scott conjecture, proved by Bestvina-Handel, asserts that the fixed subgroup of α has rank at most n. We give a short alternative proof of this result using **R**-trees.

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Introduction

In this note we give a short alternative proof of the following much celebrated result of Bestvina-Handel [**2**], conjectured earlier by Scott:

Theorem 1. *For every automorphism α of a free group F_n of rank $n \geq 1$ the fixed subgroup* $\text{Fix}\,\alpha = \{w \in F_n \mid \alpha(w) = w\}$ *has rank* $\text{rk Fix}\alpha \leq n$.

After a preliminary version of this note was written, other alternative proofs of this result have been given in [**10**] and in [**9**]. A stronger inequality, which also takes into account infinite fixed words, is given in [**6**] (see the remark at the end of this paper).

The essential ingredients in our proof are the following two theorems. The first one, proved by the third author in [**8**], is a fixed point theorem for the action of outer automorphisms of F_n on the compactification of Culler-Vogtmann's outer space. It may be stated as follows.

Theorem 2 [8]. *Given any automorphism α of F_n, there exists a nontrivial minimal action of F_n on an* **R***-tree T, with trivial edge stabilizers, whose length function $\ell : F_n \to \mathbf{R}^+$ satisfies $\ell \circ \alpha = \lambda \ell$ for some $\lambda \geq 1$. If $\lambda = 1$, then T may be taken to be a simplicial tree.*

Since the paper [**8**] was written, short proofs for the existence of a fixed point have been obtained. As none of them is presently published, we give one in an Appendix, establishing triviality of edge stabilizers (a key point in our approach).

The second theorem we use here is due to the first two authors [**7**]. It is an inequality about branch points in an **R**-tree equipped with a small action of F_n. We state the special case that will be needed.

Theorem 3 [7]. *Let T be an* **R**-*tree with a nontrivial, minimal F_n-action whose edge stabilizers are all trivial. Given $p_1, \dots, p_m \in T$ belonging to distinct orbits, the isotropy subgroups* $\operatorname{Stab} p_i$ *satisfy the inequality*

$$\sum_{i=1}^{m}\left(\operatorname{rk} \operatorname{Stab} p_i - \frac{1}{2}\right) \leq n-1.$$

In particular, $\operatorname{rk}\operatorname{Stab} p \leq n-1$ *for every $p \in T$.*

This follows directly from Theorem III.2 of [7]: since edge stabilizers are trivial, we have $v_1(p) > 0$ for every $p \in T$.

Our proof of the Scott conjecture consists of two parts. In Part 1, we try to prove Theorem 1 by induction on n, using Theorem 2 and analyzing the action on T thanks to Theorem 3. We succeed in all cases but one. In the last case we obtain

$$\operatorname{rk}\operatorname{Fix}\alpha = \operatorname{rk}\operatorname{Fix}\alpha^a + \operatorname{rk}\operatorname{Fix}(i_u \circ \alpha^a),$$

where α^a is some automorphism of F_{n-1} and i_u is conjugation by $u \in F_{n-1} : (i_u \circ \alpha^a)(g) = u\alpha^a(g)u^{-1}$.

This makes it necessary to study several automorphisms simultaneously. Say that two automorphisms α, β of F_n are *similar* if there exists $c \in F_n$ such that $\beta = i_c \circ \alpha \circ (i_c)^{-1}$. In other words, one has $\beta(g) = c\alpha(c^{-1}gc)c^{-1}$ for every $g \in F_n$. Notice that similar automorphisms induce the same outer automorphism and have conjugate fixed subgroups.

In Part 2 we extend the analysis performed in Part 1, so as to prove by induction on n the following strengthening of Theorem 1:

Theorem 1′. *Let $\alpha_0, \dots, \alpha_k$ be automorphisms of F_n representing the same outer automorphism φ and belonging to distinct similarity classes. Then*

$$\sum_{i=0}^{k}(\operatorname{rk}\operatorname{Fix}\alpha_i - 1) \leq n-1.$$

(This theorem is only superficially stronger than Theorem 1. It follows by applying Theorem 1 to the automorphism of $F_n * F_k$ equal to α_0 on F_n and sending the i-th generator t_i of F_k to $t_i u_i$ where $\alpha_i = i_{u_i} \circ \alpha_0$.)

In Part 3 we discuss equality in Theorems 1 and 1′.

1. Analyzing a single automorphism

We try to prove Theorem 1 by induction on n, noting that it is trivial for $n = 1$. Let α be an automorphism of F_n, and let T be given by Theorem 2. If $\lambda = 1$, then T is simplicial. In this case we require that F_n act on T without inversions, and we choose T so as to minimize the number of edges of the quotient graph $\Gamma = T/F_n$.

It follows directly (as in [**8**, §4]) from the uniqueness result for non-abelian actions (see [**5**]) that there exists a homothety $H: T \to T$ with stretching factor λ, which commutes with α in the sense that for every $w \in F_n$ one has

$$\alpha(w)H = Hw \tag{1}$$

as maps from T to T.

1.1 *If H has no fixed point, then* rk Fix $\alpha \leq 1$.

Let $w \in \operatorname{Fix} \alpha$, $w \neq 1$. Consider its characteristic set C_w: it is the axis of w if w acts as a hyperbolic isometry, or the unique fixed point of w if w is elliptic (w has only one fixed point because edge stabilizers are trivial). Since H commutes with w, it preserves C_w. This is possible only if $\lambda = 1$ and H, w are hyperbolic isometries with the same axis. This implies rk Fix $\alpha \leq 1$ since otherwise some nontrivial commutator would have a line of fixed points.

1.2 *If H has a fixed point Q, then* Stab Q *is α-invariant.*

From $wQ = Q$ it follows $\alpha(w)Q = \alpha(w)HQ = HwQ = Q$.

We write α^Q for the automorphism of Stab Q induced by α (note that Stab Q is also α^{-1}-invariant).

1.3 *If Q is the only fixed point of H, then* Stab Q *contains* Fix α.

From $\alpha(w) = w$ it follows $HwQ = \alpha(w)HQ = wQ$, so that $wQ = Q$ if H has only one fixed point.

1.4 If H has at most one fixed point, for instance if $\lambda \neq 1$, we get rk Fix $\alpha \leq n-1$ by applying the induction hypothesis to α^Q (recall that Stab Q has rank $\leq n-1$ by Theorem 3).

1.5 From now on we assume that H is an isometry which has more than one fixed point. Recall that T is a simplicial tree, chosen to minimize the number of edges of $\Gamma = T/F_n$. Let $e = [a, b]$ be an edge fixed (pointwise) by H.

1.6 *Under the assumptions of 1.5, the graph Γ has only one edge.*

Let T' be the tree obtained by collapsing each component of the orbit of e to a point. This orbit is preserved by H, so that H induces an isometry H' of T' satisfying the commutation equation (1) with the induced F_n-action on T'. If Γ has more than one edge, the action of F_n on T' is nontrivial. This contradicts the choice of T.

1.7 First assume that Γ is a segment. Then Bass-Serre theory gives a nontrivial decomposition $F_n = \operatorname{Stab} a * \operatorname{Stab} b$. This decomposition is α-invariant by 1.2, so that $\operatorname{Fix} \alpha = \operatorname{Fix} \alpha^a * \operatorname{Fix} \alpha^b$. The result follows again by induction.

1.8 Finally, we assume that Γ is a loop. Then $F_n = (\operatorname{Stab} a) * \langle t \rangle$, where t is any element such that $t(a) = b$. Note that $\alpha(t)a = \alpha(t)Ha = Hta = b$, so that $\alpha(t) = tu$ with $u \in \operatorname{Stab} a$. If t may be chosen with $\alpha(t) = t$, we have an α-invariant decomposition as before.

1.9 Otherwise $\alpha(t) = tu$ for some $u \in \operatorname{Stab} a$ that cannot be written $u = v\alpha(v^{-1})$ with $v \in \operatorname{Stab} a$. Direct computation, or Bass-Serre theory applied to the action of $\operatorname{Fix} \alpha$ on $\operatorname{Fix} H$, yields $\operatorname{Fix} \alpha = \operatorname{Fix} \alpha^a * t \operatorname{Fix}(i_u \circ \alpha^a)t^{-1}$, so that the induction breaks down here. This method only shows $\operatorname{rk} \operatorname{Fix} \alpha \leq 2^{n-1}$.

2. Analyzing several automorphisms

Let φ be an outer automorphism of F_n. Let T and λ be as in the beginning of Part 1, so that $\ell \circ \varphi = \lambda \ell$. If $\lambda = 1$, recall that T is chosen so as to minimize the number of edges of Γ.

For any $\alpha \in \operatorname{Aut}(F_n)$ representing φ, there is a homothety $H_\alpha : T \to T$ such that $\alpha(w)H_\alpha = H_\alpha w \ (w \in F_n)$.

2.1 If $\beta = i_h \circ \alpha$ is another representative of φ, we may (and will) take $H_\beta = hH_\alpha$. Since F_n acts on T with trivial edge stabilizers, no nondegenerate arc $e \subset T$ may be fixed pointwise by both H_α and H_β if $h \neq 1$.

2.2 If $\beta = i_c \circ \alpha \circ (i_c)^{-1}$ is similar to α, we get $H_\beta = c\alpha(c^{-1})H_\alpha = cH_\alpha c^{-1}$. In particular $\operatorname{Fix} H_\beta = c \operatorname{Fix} H_\alpha$.

2.3 Let $\alpha_0, \ldots, \alpha_k$ be representatives of φ belonging to distinct similarity classes. Denoting $r(\alpha) = \operatorname{rk} \operatorname{Fix} \alpha - 1$ we now prove the following inequality by induction on n:

$$\sum_{i=0}^{k} r(\alpha_i) \leq n - 1.$$

It is clear if $n = 1$, so we assume $n \geq 2$. By 1.1 we may assume that each $H_i = H_{\alpha_i}$ has at least one fixed point.

2.4 Suppose $Q \in T$ is fixed by both H_i and $H_j \, (i \neq j)$. Let α_i^Q and α_j^Q be the automorphisms of $\operatorname{Stab} Q$ induced by α_i and α_j (see 1.2).

If $\operatorname{rk} \operatorname{Stab} Q \geq 2$, *then* α_i^Q, α_j^Q *represent the same outer automorphism of* $\operatorname{Stab} Q$ *and belong to distinct similarity classes in* $\operatorname{Aut}(\operatorname{Stab} Q)$.

If $\alpha_j = i_h \circ \alpha_i$, we have $h \in \operatorname{Stab} Q$ because $H_j = hH_i$ (see 2.1). Now suppose there exists $v \in \operatorname{Stab} Q$ such that $\alpha_j(g) = v\alpha_i(v^{-1}gv)v^{-1}$ for all $g \in \operatorname{Stab} Q$. Then

$$h\alpha_i(g)h^{-1} = \alpha_j(g) = v\alpha_i(v^{-1})\alpha_i(g)\alpha_i(v)v^{-1}$$

for $g \in \operatorname{Stab} Q$. Since $\operatorname{Stab} Q$ has rank ≥ 2, we deduce $h = v\alpha_i(v^{-1})$ so that $\alpha_j(g) = v\alpha_i(v^{-1}gv)v^{-1}$ holds for every $g \in F_n$. This is a contradiction since α_i and α_j are not similar.

2.5 First assume that each H_i has exactly one fixed point Q_i (e.g., if $\lambda \neq 1$). Replacing each α_i by a similar automorphism, 2.2 lets us assume that for $i \neq j$ either $Q_i = Q_j$ or Q_i, Q_j belong to different orbits. Let $\mathcal{Q} \subset T$ be the set of all points Q_i and $\pi : \{0, \ldots, k\} \to \mathcal{Q}$ the map taking i to Q_i.

We then write

$$\sum_{i=0}^{k} r(\alpha_i) = \sum_{i=0}^{k} r(\alpha_i^{Q_i}) = \sum_{Q \in \mathcal{Q}} \sum_{i \in \pi^{-1}(Q)} r(\alpha_i^{Q_i}) \leq \sum_{Q \in \mathcal{Q}} (\operatorname{rk} \operatorname{Stab} Q - 1) < n - 1.$$

The first equality comes from 1.3. The first inequality is clear for points Q with $\operatorname{rk} \operatorname{Stab} Q \leq 1$. For other points it follows from the induction hypothesis thanks to 2.4. The second inequality is Theorem 3 since different points of $\mathcal{Q}$ are in different F_n-orbits.

2.6 If some H_i (say H_0) fixes (pointwise) an edge $e = [a, b]$, then $\Gamma = T/F_n$ has only one edge by 1.6. Using 2.1 and 2.2 we see that for $i > 0$ the map H_i has only one fixed point Q_i. By 2.2 we may assume $Q_i = a$ or b (unless Q_i is the midpoint of an edge, but then $\operatorname{Fix} \alpha_i$ is trivial by 1.3).

2.7 First assume that Γ is a segment. Using 1.7 we write $r(\alpha_0) = 1 + r(\alpha_0^a) + r(\alpha_0^b)$. Taking π to be the obvious map from $\{1, \ldots, k\}$ to $\{a, b\}$, we then get as in 2.5:

$$\begin{aligned} \sum_{i=0}^{k} r(\alpha_i) &= 1 + r(\alpha_0^a) + \sum_{i \in \pi^{-1}(a)} r(\alpha_i^a) + r(\alpha_0^b) + \sum_{i \in \pi^{-1}(b)} r(\alpha_i^b) \\ &\leq 1 + \operatorname{rk} \operatorname{Stab} a - 1 + \operatorname{rk} \operatorname{Stab} b - 1 = n - 1. \end{aligned}$$

2.8 If Γ is a loop, we assume $Q_i = a$ for every $i \geq 1$. We study α_0 as in 1.8 and 1.9. In the situation of 1.8 we argue as in 2.7. In the situation of 1.9 we get:

$$\sum_{i=0}^{k} r(\alpha_i) = \sum_{i=0}^{k} r(\alpha_i^a) + r(i_u \circ \alpha_0^a) + 1.$$

There is nothing to prove if $\operatorname{Stab} a$ has rank 1. Otherwise we argue as follows. The automorphisms $\alpha_0^a, \alpha_1^a, \ldots, \alpha_k^a, i_u \circ \alpha_0^a$ represent the same outer automorphism of $\operatorname{Stab} a$. The inductive proof will then be complete if we show that no two of them are similar.

By 2.4 we only need to check that $i_u \circ \alpha_0^a$ is not similar to any of the others. Arguing as in 2.4 we see that it is not similar to α_0^a since otherwise we would be in the situation of 1.8. Now we note that $i_u \circ \alpha_0$ is similar to α_0 in $\mathrm{Aut}(F_n)$ since $i_u \circ \alpha_0 = (i_t)^{-1} \circ \alpha_0 \circ i_t$. It follows that $i_u \circ \alpha_0$ and α_i are not similar for $i \geq 1$. Using 2.4 we see that $i_u \circ \alpha_0^a$ and α_i^a are not similar. □

3. Equality

Suppose equality holds in Theorem 1′. Perform the analysis of Part 2. We have shown that case 2.5 is impossible. In case 2.7 (resp. 2.8), equality holds in $\mathrm{Stab}\, a$ and $\mathrm{Stab}\, b$ (resp. in $\mathrm{Stab}\, a$). Furthermore both α_0^a and α_0^b (resp. α_0^a and $i_u \circ \alpha_0^a$) fix nontrivial elements. An easy induction then yields the following result:

Theorem. *Let $\alpha_0, \dots, \alpha_k$ be automorphisms of F_n representing the same outer automorphism φ and belonging to distinct similarity classes. If $\sum_{i=0}^{k}(\mathrm{rk}\,\mathrm{Fix}\,\alpha_i - 1) = n - 1$ and $n \geq 2$, then some α_i with $\mathrm{rk}\,\mathrm{Fix}\,\alpha_i \geq 2$ fixes an element of a free basis of F_n.*

Corollary [4]. *If $\alpha \in \mathrm{Aut}\,(F_n)$ satisfies $\mathrm{rk}\,\mathrm{Fix}\,F_n = n$, then α fixes an element of a free basis of F_n.*

If $n \geq 2$, one may require in this corollary that the fixed basis element belong to an α-invariant free factor of rank $n - 1$.

Remark. In connection with the paper [3], the third author has conjectured some years ago the following strengthening of Theorem 1′:

$$\sum_{i=0}^{k}\left(\mathrm{rk}\,\mathrm{Fix}\,\alpha_i - 1 + \frac{1}{2}a(\alpha_i)\right) \leq n - 1$$

where $a(\alpha_i)$ denotes the number of *attractive fixed points at* ∞ of α_i (see [3]). A proof of this conjecture has been obtained in joint work with A. Jaeger, see [6]. The paper [6] uses the analysis presented here, but it is much more involved and its main emphasis is on new geometric techniques.

Appendix

In this appendix we provide a proof of Theorem 2. We follow closely the argument sketched in [1] for irreducible automorphisms, using the existence of (not necessarily stable) train track representatives as given by Theorem 5.12 of [2], but not the main body of Bestvina-Handel's approach (indivisible Nielsen paths, etc.).

Let ϕ be the outer automorphism associated to α. By [2, Theorem 5.12], we may represent ϕ by a relative train track map $f : \tau \to \tau$, where τ is a finite graph without vertices of valence 1 with $\pi_1\tau \simeq F_n$, and f respects a maximal filtration $\tau_0 \subset \tau_1 \subset \ldots \subset \tau_r = \tau$. Furthermore there exists a nonzero length assignment $L(e) \geq 0$ for the edges e of τ, and a constant $\lambda \geq 1$, such that for $p \geq 1$ one has $L(f^p(e)) = \lambda^p L(e)$ for every edge e and $L(f(\gamma)) \leq L(\gamma)$ for every loop γ; here $L(.)$ denotes the length of the shortest representative in the homotopy class (relative to endpoints). The function L comes from the Perron-Frobenius eigenvector of the (irreducible) transition matrix $M(f|_{\tau\backslash\tau_{r-1}})$ and λ is the corresponding eigenvalue, see [2].

If $\lambda = 1$, then the transition matrix is a permutation matrix, and we let T be the simplicial $\mathbf{R}$-tree T_0 obtained by lifting L to the universal covering of τ.

If $\lambda > 1$, we consider the length function ℓ_0 of the action of F_n on T_0, the non-increasing sequence of length functions $\ell_n = \frac{1}{\lambda^n}(\ell \circ \alpha^n)$, and the limit ℓ_∞. This limit satisfies $\ell_\infty \circ \alpha = \lambda \ell_\infty$. By [5] either it is identically 0, or it is the length function of an action of F_n on an $\mathbf{R}$-tree T whose edge stabilizers are cyclic or trivial. There remains to show that ℓ_∞ is not identically 0, and that edge stabilizers are trivial.

Let e be an edge with $L(e) > 0$. Since $L(f^p(e)) = \lambda^p L(e)$ tends to infinity and τ is finite, we can find a reduced loop γ which is a subpath of some $f^p(e)$, with $L(\gamma)$ arbitrarily large. The image $f(\gamma)$ is a loop, which is reduced except possibly at its basepoint: $f(\gamma) = \delta\gamma'\delta^{-1}$, where γ' is a reduced loop. Since f is a homotopy equivalence of a finite graph, there is a bound C, depending only on f, for the simplicial length of δ. Setting $C' = C\max(L(e_i))$ we obtain $\ell(\alpha(\gamma)) = L(\gamma') \geq \lambda L(\gamma) - 2C' = \lambda\ell(\gamma) - 2C'$. Similarly we get $\ell(\alpha(\alpha^q(\gamma))) \geq \lambda\ell(\alpha^q(\gamma)) - 2C'$ for $q \geq 1$, hence $\ell_\infty(\gamma) > 0$ if $L(\gamma)$ is large enough. This proves that ℓ_∞ is not identically 0.

To prove triviality of edge stabilizers, we may assume that the action of F_n on T is minimal. Let H be the homothety introduced in Part 1. Let $c \subset T$ be a segment fixed by some nontrivial $w \in F_n$. Let $s \geq 1$ be the largest integer such that w is an s-th power. Since the length of $H^k(c)$ grows arbitrary large with k and there exists a finite collection of segments whose union meets every orbit, we can find, for sufficiently large k, disjoint non-degenerate subarcs $c_0, \ldots, c_s$ of $H^k(c)$ such that $c_i = v_i c_0$ for some $v_i \in F_n$ $(i = 1, \ldots, s)$.

The element $w' = \alpha^k(w)$ fixes $H^k(c)$. Since $w', v_1, \ldots, v_s$ all have different actions on c_0, there exists i such that v_i and w' do not generate a cyclic subgroup of F_n. Then w' and $v_i^{-1}w'v_i$ generate a free subgroup of rank 2 which fixes c_i pointwise, contrary to the fact that edge stabilizers are cyclic or trivial.

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