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Pseudolocality for the Ricci Flow and Applications

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Abstract. Perelman established a differential Li–Yau–Hamilton (LYH) type inequality for fundamental solutions of the conjugate heat equation corresponding to the Ricci flow on compact manifolds. As an application of the LYH inequality, Perelman proved a pseudolocality result for the Ricci flow on compact manifolds. In this article we provide the details for the proofs of these results in the case of a complete noncompact Riemannian manifold. Using these results we prove that under certain conditions, a finite time singularity of the Ricci flow must form within a compact set. The conditions are satisfied by asymptotically flat manifolds. We also prove a long time existence result for the Kähler–Ricci flow on complete nonnegatively curved Kähler manifolds.

1 Introduction

In this article we consider the Ricci flow

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} \tag{1.1}$$

on a complete noncompact Riemannian manifold (M, g) and the heat equation and conjugate heat equation

$$\frac{\partial u}{\partial t} - \Delta^t u = 0 \tag{1.2}$$

$$\frac{\partial u}{\partial t} + \Delta^t u - Ru = 0 \tag{1.3}$$

where Δ^t denotes the Laplacian operator with respect to a solution $g(t)$ to (1.1), and $R(t)$ is the scalar curvature of $g(t)$. Notice that if $g(t)$ is defined on an interval $[0, T]$ and we let $\tau = T - t$, then (1.3) defines a strictly parabolic equation on M with respect to $\tau \in [0, T]$.

The conjugate heat equation corresponding to the Ricci flow was considered in [24], and there Perelman established a differential Li–Yau–Hamilton (LYH) type inequality for its fundamental solutions ([24, Corollary 9.3]) on compact manifolds. The proof was sketched in [24], and a detailed proof was given by Ni in [21]. As an application of the LYH inequality, Perelman proved a pseudolocality result for the

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Ricci flow on compact manifolds ([24, Theorem 10.1]), which basically states that regions of large amounts of curvature cannot instantly affect almost Euclidean regions under the Ricci flow. For more details of the proof, see [7, 14].

In this article we verify these results, including the LYH inequality and pseudolocality, in the case of complete noncompact Riemannian manifolds. We basically follow the original steps described in [24] as well as those in [7, 14, 21].

Our motivation to generalize Perelman's results mentioned above is to study long time existence of Ricci flow and Kähler–Ricci flow on complete noncompact manifold. Using the result on pseudolocality, we obtain the following.

Theorem 1.1 *Let (M^n, g) be a complete noncompact Riemannian manifold with injectivity radius bounded away from zero such that $|Rm|(x) \to 0$ as $x \to \infty$. Let $(M, g(t))$ be the corresponding maximal solution to the Ricci flow* (9.1) *on $M \times [0, T)$. Then either $T = \infty$ or there exists some compact $S \subset M$ with the property that $|Rm(x, t)|$ is bounded on $(M \setminus S) \times [0, T)$.*

The conditions are satisfied if M is an asymptotically flat manifold for example. As a corollary to Theorem 1.1 we also have the following.

Corollary 1.2 *Suppose $T < \infty$ in Theorem* 1.1. *Then $Rm(x, T) \to 0$ as $x \to \infty$ in the sense that, given any $\epsilon > 0$, we may choose a compact set S such that $|Rm(x, t)| \le \epsilon$ for all $(x, t) \in (M \setminus S) \times [0, T)$.*

Combining Theorem 1.1 with the results in [23], we have the following result on the long time existence of Kähler–Ricci on complete noncompact Kähler manifolds with nonnegative holomorphic bisectional curvature.

Theorem 1.3 *Let (M^n, g_0) be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature with injectivity radius bounded away from zero such that $|Rm|(x) \to 0$ as $x \to \infty$. If the holomorphic bisectional curvature is positive at some point, then the Kähler–Ricci flow*

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} \tag{1.4}$$

with initial data g_0 has a long time solution $g(t)$ on $M \times [0, \infty)$.

The Kähler–Ricci flow is an important tool to study uniformization of complete noncompact Kähler manifolds with nonnegative holomorphic bisectional curvature, see [1, 2, 27] for example. In [27] (see also [23]), it was proved that if (M^n, g_0) is a complete noncompact Kähler manifold with nonnegative and bounded holomorphic bisectional curvature, and if the scalar curvature satisfies

$$\frac{1}{V_x(r)} \int_{B_x(r)} R \le \frac{C}{1 + r^\theta} \tag{1.5}$$

for some $C, \theta > 0$ for all x and r, then (1.4) has a long time solution. By the result in [22], (1.5) is true for $\theta = 1$, at least for simply connected M or if the holomorphic bisectional curvature is positive somewhere, for some constant C, which may depend on x. It is unclear whether (1.5) is true in general with C being independent of x except for the case of maximal volume growth, see [20].

In order to prove the LYH type differential inequality for the fundamental solution of (1.2), we need to obtain estimates for the fundamental solution together with some gradient estimates for positive solutions of (1.2) and (1.3). In case the manifold is compact, results have been obtained by Zhang, Kuang–Zhang [15,29] and Ni [21]. Some estimates are also obtained for complete manifolds with nonnegative Ricci curvature by Ni [19]. We consider the case that the manifold is complete, noncompact, and has bounded curvature. The results may have independent interest.

The paper is organized as follows. In every section, our results are obtained on a complete noncompact Riemannian manifold. In Sections 2–4 we establish some basic estimates for positive solutions of the conjugate heat equation associated to a general evolution (2.1) of a Riemannian metric. In Sections 5 and 6 we establish estimates for fundamental solutions of this conjugate heat equation. In Section 7 we apply our previous estimates to establish the LYH inequality for the fundamental solution of the conjugate heat equation associated with the Ricci flow (1.1). Our steps in this section basically follow the steps in [21]. [1] In Section 8 we establish pseudolocality for the Ricci flow (1.1) on complete noncompact Riemannian manifolds. In particular, we show that [24, Theorem 10.1] holds in the noncompact case. In Section 9 we prove Theorems 1.1 and 1.3.

2 An Integral Estimate

Let $\{g(t) \mid t \in [0,T]\}$ be a smooth family of complete Riemannian metrics on M^n such that $g(t)$ satisfies:

$$\frac{\partial}{\partial t} g_{ij}(x,t) = 2h_{ij}(x,t) \tag{2.1}$$

on $M \times [0,T]$, where $h_{ij}(x,t)$ is a smooth family of symmetric tensors.

Consider the equation

$$\frac{\partial u}{\partial t} - \Delta^t u + qu = 0, \tag{2.2}$$

where Δ^t denotes the Laplacian operator with respect to $g(t)$ and q is a smooth function on $M \times [0,T]$.

Let us make the following assumptions:

(A1) $\|h\|, \|\nabla^t h\|$ are uniformly bound on spacetime, where the norm is taken with respect to $g(t)$.

(A2) The sectional curvatures of the metrics $g(t)$ are uniformly bounded on spacetime.

(A3) $|q|, \|\nabla^t q\|$, and $|\Delta^t q|$ are uniformly bounded on spacetime.

Let $H(t)$ be the trace of $h_{ij}(t)$ with respect to $g(t)$.

[1] Our proof does not use the reduced distance $L(y,\tau)$ associated with the Ricci flow, introduced in [24].

Definition 2.1 Let f be a positive function on $(0,T]$. f is said to be regular with the constants $\gamma > 1$ and $A \geq 1$, if

(i) f is increasing, and
(ii) $\frac{f(s)}{f(s/\gamma)} \leq A\frac{f(t)}{f(t/\gamma)}$ for any $0 < s \leq t \leq T$.

Lemma 2.2 *Let Ω be a relatively compact domain of M with smooth boundary and let K be a compact set with $K \subset\subset \Omega$. Let u be any solution to the problem*

$$\begin{cases} u_t - \Delta^t u + qu = 0, \text{ in } \Omega \times [0,T] \\ u\big|_{\partial\Omega\times[0,T]} = 0 \\ \operatorname{supp} u(\cdot,0) \subset K. \end{cases}$$

Let f be a regular function with the constants γ and A. Suppose

$$\int_\Omega u^2 dV_t \leq \frac{1}{f(t)}$$

for any $t > 0$. Then there is a positive constant C depending only on γ, the uniform upper bound for $|q|$ and $|H|$, and a positive constant D depending only on T, γ and the uniform upper bound for $\|h\|$, such that

$$\int_\Omega u^2(x,t)e^{\frac{r^2(x,K)}{Dt}} dV_t \leq \frac{4A}{f(t/\gamma)}e^{Ct}$$

for any $t > 0$, where $r(x,K)$ denotes the distance between x and K with respect to the initial metric.

Proof The proof is almost the same as the proof of Grigor'yan [10, Theorem 2.1].

Let $C_2 > 0$ be a constant such that $|q| + 1/2|H| \leq C_2$, and let $v = e^{-C_2 t}u$. Then v satisfies

$$v_t - \Delta^t v + (C_2 + q)v = 0, \tag{2.3}$$

and

$$\int_\Omega v^2 dV_t = e^{-2C_2 t}\int_\Omega u^2 dV_t \leq \frac{1}{f(t)e^{2C_2 t}} := \frac{1}{\tilde{f}(t)},$$

where $\tilde{f}(t) = f(t)e^{2C_2 t}$ is regular with constants γ and A.

Now for any $R > 0$, define

$$d(x) = \begin{cases} R - r(x,K) & x \in K^R \\ 0 & x \notin K^R, \end{cases}$$

where K^R means the R-neighborhood of K with respect to the initial metric. Then $|\nabla^t d| \leq C_1$ uniformly on spacetime, where C_1 depends on T and the upper bound of $\|h\|$. Then if we let $\xi(x, s-t) = \frac{d^2(x)}{2C_1^2(t-s)}$ for $s > T$ fixed and $0 < t \leq T < s$, we have

$$\frac{\partial}{\partial t}\xi + \frac{1}{2}|\nabla^t \xi|^2 = -\frac{d^2}{2C_1^2(t-s)^2} + \frac{1}{2}\frac{d^2\|\nabla^t d\|^2}{C_1^4(t-s)^2} \leq 0, \tag{2.4}$$

which combines with (2.3) and (2.1) to give

$$\frac{d}{dt}\int_\Omega v^2 e^\xi \, dV_t \leq 0.$$

where we have used the fact that $v = 0$ on $\partial\Omega$, $2C_2 + 2q + H \geq 0$ and (2.4).

It now follows from STEP1 and STEP2 of the proof of Griyor'yan [10, Theorem 2.1], that there exists a positive constant $D > 0$ depending on C_1 and γ, such that

$$\int_\Omega v^2 e^{\frac{r^2(x,K)}{Dt}} dV_t \leq \frac{4A}{\tilde{f}(t/\gamma)}.$$

Thus,

$$\int_\Omega u^2 e^{\frac{r^2(x,K)}{Dt}} dV_t \leq \frac{4A}{f(t/\gamma)} e^{2C_2(1-\frac{1}{\gamma})t}. \qquad \blacksquare$$

3 A Mean Value Inequality

In this section, we prove the following lemma, which will be used to estimate the fundamental solution of (2.2) in Section 5.

Lemma 3.1 *Let u be a positive sub-solution of equation* (2.2) *on* $\Omega \times [0, T]$, *where* Ω *is a domain in* M. *Moreover, suppose that there is a complete Riemannian metric* $\tilde{g}$ *on* M *with Ricci curvature bounded from below by* $-k$ *with* $k \geq 0$, *such that*

$$\frac{1}{C_0}\tilde{g} \leq g(0) \leq C_0\tilde{g}$$

in Ω *for some* $C_0 > 0$. *Let* $\tilde{Q}_r(x,t) := \tilde{B}_x(r) \times (t - r^2, t]$ *whenever it is well defined, where* $\tilde{B}_x(r)$ *means the ball of radius* r *with respect to* $\tilde{g}$. *Then for any* $(x,t) \in \Omega \times (0, T]$ *and* $r > 0$ *such that* $\tilde{Q}_{2r}(x,t) \subset\subset \Omega \times [0, T]$,

$$\sup_{\tilde{Q}_r(x,t)} u \leq \frac{Ce^{At+B\sqrt{k}r}}{r^2\tilde{V}_x(r)} \int_{\tilde{Q}_{2r}(x,t)} u \, d\tilde{V} \, ds$$

where A *depends only on the upper bounds for* $|q|$ *and* $|H|$ *on* Ω, B *depends only on* n, *and* C *depends only on* C_0, n, T *and the uniform upper bound for* $|h|$ *on* Ω. *The notations* $\tilde{V}_x(r)$ *and* $d\tilde{V}$ *denote the volumes with respect to* $\tilde{g}$.

Proof Our proof follows the method of Zhang [29, §5]. For σ in $(1,2]$, let ϕ be a smooth function on $[0,\infty)$, such that: (i) $\phi = 1$ on $[0,r]$, (ii) $\phi = 0$ on $[\sigma r, \infty)$, and (iii) $-\frac{2}{(\sigma-1)r} \leq \phi' \leq 0$. Let η be a smooth function on $[0,\infty)$, such that: (i)

$\eta = 0$ on $[0, t - \sigma^2 r^2]$, (ii) $\eta = 1$ on $[t - r^2, \infty)$, and (iii) $0 \le \eta' \le \frac{2}{(\sigma-1)^2 r^2}$. Then if $\psi(y, s) = \phi(\tilde{r}(x, y))\eta(s)$, where x is fixed and $\tilde{r}(x, y)$ means the distance function of $\tilde{g}$, then $\operatorname{supp} \psi(\cdot, s) \subset \overline{\tilde{B}_x(2r)} \subset \Omega$.

Now consider $v = e^{-C_1 t} u$, where C_1 is some positive constant to be determined. Then $v_t - \Delta^t v + (C_1 + q)v \le 0$, and for any $p \ge 1$,

$$\frac{\partial v^p}{\partial t} - \Delta^t(v^p) + p(C_1 + q)v^p \le 0.$$

We then multiply the above inequality by $v^p \psi^2$, integrate, and proceed as in the proof of [29, Theorem 5.1], to obtain the following, where $w = v^p$ and t' is in $[t - r^2, t]$:

$$\int_{t-\sigma^2 r^2}^{t'} \int_{\tilde{B}_x(\sigma r)} |\nabla^s(\psi w)|^2 dV_s ds + \frac{1}{2} \int_M w^2 \psi^2 dV_{t'} \le \frac{C_2}{(\sigma - 1)^2 r^2} \int_{t-\sigma^2 r^2}^{t'} \int_{\tilde{B}_x(\sigma r)} w^2 dV_s ds$$

where C_2 depends on C_0, T, and the uniform upper bound of $|h|$ on Ω, provided C_1 is large enough, depending only on the upper bounds of $|q|$ and $|H|$ in Ω. So

$$\int_{t-\sigma^2 r^2}^{t} \int_{\tilde{B}_x(\sigma r)} |\tilde{\nabla}(\psi w)|^2 d\tilde{V} ds \le \frac{C_3}{(\sigma - 1)^2 r^2} \int_{t-\sigma^2 r^2}^{t} \int_{\tilde{B}_x(\sigma r)} w^2 d\tilde{V} ds \quad \text{and}$$

$$\max_{t-r^2 \le t' \le t} \int_{\tilde{B}_x(\sigma r)} w^2 \psi^2 d\tilde{V} \le \frac{C_3}{(\sigma - 1)^2 r^2} \int_{t-\sigma^2 r^2}^{t} \int_{\tilde{B}_x(\sigma r)} w^2 d\tilde{V} ds,$$

where C_3 depends only on C_0, T and the uniform upper bound of $|h|$.

Now we can proceed as in the proof of [29, Theorem 5.1] with respect to the metric $\tilde{g}$. Applying the Sobolev inequality in [25] with respect to $\tilde{g}$, the Moser iteration as in [29], and the iteration technique of Li and Schoen [16], we get

$$\sup_{\tilde{Q}_r(x,t)} v \le \frac{C_4 e^{C_8 \sqrt{k} r}}{r^2 \tilde{V}_x(r)} \times \frac{1}{\log^{\frac{n+2}{2}} \gamma} \int_{\tilde{Q}_{\gamma r}(x,t)} v d\tilde{V} ds$$

for any $\gamma \in (1, 2]$, where C_4 depends only on C_0, T the upper bound for $|h|$ and n, and C_8 depends only on n. Now letting $\gamma = 2$, and using the definition of v, it is easy to see that the lemma is true. ■

4 A Li–Yau Type Gradient Estimate

In this section, we derive a Li–Yau type gradient estimate that will also be used in Section 7. We basically follow the proof in [17]. Let $g(t)$ be as in Section 2.

Lemma 4.1 *Let u be a positive solution to equation (2.2). Then for any $\alpha > 1$ and $\epsilon > 0$, there is a constant $C > 0$ depending on α, ϵ, n, T, the uniform upper bounds* *for $|h|, |\nabla h|, |\nabla q|, |\Delta q|, |Rc^t|$, and the bound for the sectional curvature at $t = 0$, such that*

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} - \alpha q \leq C + \frac{(n+\epsilon)\alpha^2}{2t}.$$

Proof In the following ∇ and Δ are understood to be time dependent. For any smooth function f on $M \times [0, T)$, we have the following at a point with normal coordinates with respect to the metric $g(t)$:

$$(\Delta f)_t = \Delta f_t - 2h_{ij} f_{ij} - 2h_{ik;i} f_k + H_i f_i$$
$$(|\nabla f|^2)_t = 2\langle \nabla f_t, \nabla f\rangle - 2h(\nabla f, \nabla f),$$

where repeated indices mean summation.

Let $f = \log u$. Then $\Delta f - f_t = q - |\nabla f|^2$.

For $\alpha > 1$ and $\epsilon > 0$, let $F = t(|\nabla f|^2 - \alpha f_t - \alpha R)$. Then in normal coordinates we compute as in [17] to obtain

$$\begin{aligned}\Delta F = {} & t\Big(2\sum_{ij} f_{ij}^2 - 2\alpha h_{ij} f_{ij}\Big) - 2\langle \nabla F, \nabla f\rangle + F_t - \frac{F}{t} \\ & - 2t(\alpha - 1)h_{ij} f_i f_j - \alpha t(2h_{ki;k} - H_i) f_i + 2tR_{ij} f_i f_j - \alpha t \Delta q - 2t(\alpha - 1)q_i f_i.\end{aligned}$$

By [27] (see also [28]), there is a smooth function ρ such that

$$\begin{cases} \frac{1}{C_1}\rho(x) & \leq r_0(x) \leq C_1 \rho(x) \\ |\nabla^0 \rho| & \leq C_1 \\ |\nabla^0\nabla^0 \rho| & \leq C_1, \end{cases}$$

where C_1 is a constant depending on n and the bound for $|Rm|$ of $g(0)$. Here $r_0(x)$ is the distance with respect to $g(0)$ from a fixed point o and ∇^0 is the covariant derivative with respect to $g(0)$. By the assumption that $|h|$ and $|\nabla h|$ are uniformly bounded on spacetime, we have

$$\begin{cases} \frac{1}{C_2}\rho(x) & \leq r_t(x) \leq C_2 \rho(x) \\ |\nabla \rho| & \leq C_2 \\ |\nabla^2 \rho| & \leq C_2, \end{cases}$$

where C_2 depends on C_1, T, and the uniformly upper bound for $|h|$ and $|\nabla h|$. Here $r_t(x)$ is the distance with respect to $g(t)$ from o.

Let $\eta \in C^\infty([0,\infty)$ be such that $0 \leq \eta \leq 1$, $\eta = 1$ on $[0, 1]$ and $\eta = 0$ on $[2, \infty)$, $\eta > 0$ on $[0, 2)$, $0 \geq \eta'/\eta^{\frac{1}{2}} \geq -C_3$ and $\eta'' \geq -C_3$ on $[0, \infty)$, where C_3 is some positive absolute constant. For any $R > 0$, let $\phi = \eta(\rho/R)$. Suppose ϕF attains a positive maximum at the point (x_0, t_0), where $0 < t_0 \leq T$. Then at (x_0, t_0), as in [17], we have

$$\begin{aligned} 0 & \geq \Delta(\phi F) \\ & \geq t_0 \phi \frac{2}{n+\epsilon}(|\nabla f|^2 - f_t - q)^2 - C_4 F \phi^{\frac{1}{2}} R^{-1} |\nabla f| - \phi \frac{F}{t_0} \\ & \quad - C_4 t_0 \phi |\nabla f|^2 - C_4 t_0 \phi - C_4 F\left(R^{-1} + R^{-2}\right). \end{aligned}$$

Here and below, C_i's are constants depending only on bounds for h, $|\nabla h|$, $|Rm|$ of $g(0)$, $|\Delta q|$, α, ϵ, n, and an upper bound on T.

Multiply both sides by $t_0\phi$, and let $A = \phi|\nabla f|^2$, $B = \phi(f_t + q)$ so that $\phi F = t_0(A - \alpha B)$. Then for any $\delta > 0$ and $\tau > 0$, as in [17], we have

$$0 \geq \phi F\left[-C_5 t_0(1+\delta^{-1})\left(R^{-1}+R^{-2}\right)-1\right] + \frac{2t_0^2}{n+\epsilon}\left[(A-B)^2 - \delta(A-\alpha B)A - \tau A^2\right] - C_5(1+\tau^{-1})t_0^2. \tag{4.1}$$

Now for $\sigma > 0$,

$$\begin{aligned}&(A-B)^2 - \delta(A-\alpha B)A - \tau A^2\\ &\quad= (1-\sigma)(A-\alpha B)^2 + (\sigma-\delta-\tau)A^2 + [-2\sigma\alpha + 2(\alpha-1)+\delta\alpha]AB\\ &\qquad + (\sigma\alpha^2 + 1 - \alpha^2)B^2\end{aligned}$$

First choose σ such that $\sigma\alpha^2 + 1 - \alpha^2 = 0$. Then $0 < \sigma < 1$. Next choose δ so that $-2\sigma\alpha + 2(\alpha - 1) + \delta\alpha = 0$. Finally, choose $\tau = \frac{1}{2}(\sigma - \delta)$. Then

$$(A-B)^2 - \delta(A-\alpha B)A - \tau A \geq (1-\sigma)(A-\alpha B)^2.$$

Putting this into (4.1), we have

$$\begin{aligned}0 &\geq \phi F\left[-C_5 t_0(1+\delta^{-1})\left(R^{-1}+R^{-2}\right)-1\right] + \frac{2t_0^2}{n+\epsilon}\left[(1-\sigma)(A-\alpha B)^2\right]\\ &\quad - C_5(1+\tau^{-1})t_0^2\\ &= \phi F\left[-C_5 t_0(1+\delta^{-1})\left(R^{-1}+R^{-2}\right)-1\right] + \frac{2}{n+\epsilon}\alpha^{-2}(\phi F)^2\\ &\quad - C_5(1+\tau^{-1})t_0^2\end{aligned}$$

So

$$\phi F \leq C_6 t_0\left(R^{-2}+R^{-1}+1\right) + \frac{n+\epsilon}{2}\alpha^2$$

on $M \times [0, T]$. Since $t_0 \leq T$, we see that if $r(x) \leq \frac{1}{2}C_2 R$, then

$$F(x,T) \leq C_6 T\left(R^{-2}+R^{-1}+1\right) + \frac{n+\epsilon}{2}\alpha^2.$$

Since we can decrease T above without affecting the constants C_i, the result will follow by letting $R \to \infty$. ∎

Corollary 4.2 *Using the same assumptions as in the lemma, the following local version of the gradient estimate is true:*

$$\sup_{B_p(C_2R/2)}\left(\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} - \alpha q\right) \leq \frac{(n+\epsilon)\alpha^2}{2t} + C_6\left(R^{-2}+R^{-1}+1\right),$$

where the constants C_i are as in the proof of the lemma.

Remark 4.3 Note that the constants in the local gradient estimates depend only on local data and the local behavior of the function ρ.

As in [17], we have the following.

Corollary 4.4 *Let u be a positive solution of equation* (2.2)*. Then, for any $\alpha > 1$ and $\epsilon > 0$, there are $C_1 > 0$ depending on T and the upper bound for $|h|$, and $C_2 > 0$ depending on α, ϵ, n, T, the upper bounds for $|h|, |\nabla^t h|, |q|, |\nabla^t q|, |\Delta^t q|, |Rc^t|$, and the curvature bound for the initial metric, such that*

$$u(x_1, t_1) \le u(x_2, t_2)\Big(\frac{t_2}{t_1}\Big)^{\frac{(n+\epsilon)\alpha}{2}} \exp\Big(\frac{C_1 \alpha r^2(x_1, x_2)}{t_2 - t_1} + C_2(t_2 - t_1)\Big)$$

for any $x_1, x_2 \in M$ and $0 < t_1 < t_2 \le T$.

We also have the following.

Corollary 4.5 *Let u be a positive solution of equation* (2.2)*. Then there is a positive constant C depending only on n, T, the upper bounds for $|h|, |\nabla^t h|, |\nabla^t q|, |\Delta^t q|, |Rc^t|$, and the curvature bound of $g(0)$, such that for any $x \in M$ and $0 < s < t \le T$,*

$$u(x, s) \le \frac{C}{V_x(\sqrt{t-s})}\Big(\frac{t}{s}\Big)^{n+1} \int_{B_x(\sqrt{t-s})} u(y, t)\, dV(y).$$

Remark 4.6 Since $g(t)$ is uniformly equivalent to $g(0)$, by volume comparison, we can see that in the corollary, the geodesic ball and its volume can be chosen with respect to any $g(t)$, perhaps with a different constant.

5 Upper and Lower Estimates of the Fundamental Solutions

In the following, we will apply the last three sections to get upper and lower estimates for the fundamental solutions of equation (2.2). We always assume (A1)–(A3) in Section 2 are true.

Let $Z(x, t; y, s)$, $0 \le s < t \le T$ be the fundamental solution of equation (2.2):

$$\frac{\partial}{\partial t} u - \Delta^t u + qu = 0.$$

That is to say,

$$\begin{cases} \frac{\partial}{\partial t} Z(x, t; y, s) - \Delta^t_x Z(x, t; y, s) + q(x) Z(x, t; y, s) = 0 \\ \lim_{t \to s} Z(x, t; y, s) = \delta_y. \end{cases}$$

The fundamental solution exists and is positive, see for example [11].

Then $Z(x, t; y, s)$ is also the fundamental solution of the conjugate equation. That is,

$$\begin{cases} -\frac{\partial}{\partial s} Z(x, t; y, s) - \Delta^s_y Z(x, t; y, s) + (q(y) - H(y)) Z(x, t; y, s) = 0 \\ \lim_{s \to t} Z(x, t; y, s) = \delta_x. \end{cases}$$

The fundamental solution $\mathcal{Z}(x,t;y,s)$ can be obtained as follows.

Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \subset\subset M$ be an exhaustion of M by relatively compact domains with smooth boundary in M. Let $\mathcal{Z}_k(x,t;y,s)$ be the corresponding fundamental solution on Ω_k with zero Dirichlet boundary condition. Then $\mathcal{Z}_k$ is an increasing sequence by maximum principle and $\mathcal{Z}$ is the limit of $\mathcal{Z}_k$ as $k \to \infty$. Moreover, we have the following smooth convergences:

$$\mathcal{Z}_k(\,\cdot\,,\,\cdot\,;y,s) \to \mathcal{Z}(\,\cdot\,,\,\cdot\,;y,s)$$

uniformly on any compact subset of $M \times (s,T]$ and

$$\mathcal{Z}_k(x,t;\,\cdot\,,\,\cdot\,) \to \mathcal{Z}(x,t;\,\cdot\,,\,\cdot\,)$$

uniformly on any compact subset of $M \times [0,t)$.

Lemma 5.1 *There is a positive constant C depending only on T and the upper bounds for $|q|$ and $|H|$, such that*

$$\int_M \mathcal{Z}(x,t;y,s)dV_t(x) \le C \tag{5.1}$$

for any $0 < s < t \le T$. Moreover, if $q = H$, then

$$\int_M \mathcal{Z}(x,t;y,s)dV_t(x) = 1$$

for any $0 < s < t \le T$.

Proof With the above notations, let

$$I_k(t) = \int_{\Omega_k} \mathcal{Z}_k(x,t;y,s)dV_t(x).$$

Then

$$\begin{aligned}\frac{d}{dt}I_k(t) &= \frac{d}{dt}\int_{\Omega_k} \mathcal{Z}_k(x,t;y,s)dV_t(x) = \int_{\Omega_k}(\Delta_x^t \mathcal{Z}_k - q\mathcal{Z}_k)dV_t(x) + \int_{\Omega_k} H\mathcal{Z}_k dV_t(x)\\ &= \int_{\Omega_k}(H-q)\mathcal{Z}_k dV_t(x) + \int_{\partial\Omega_k} \frac{\partial Z_k}{\partial \vec{n}_t} dS_t(x) \le C_1 I_k(t)\end{aligned}$$

since $\mathcal{Z}_k \ge 0$ on $\Omega_k \times (s,T]$ and it is 0 on $\partial\Omega_k \times (s,T]$, where C_1 depends on the uniform upper bounds of $|q|$ and $|H|$. Since $I_k(s) = 1$, we have $I_k(t) \le e^{C_1(t-s)} \le e^{C_1 T}$. By letting $k \to \infty$, we get the first inequality (5.1).

Suppose $q = H$. Let $\phi = \eta(\rho/R)$ be the same as in the proof of Lemma 4.1. For any t_1, t_2 with $s < t_1 < t_2 \le T$, we have

$$\begin{aligned}\Big|\int_M \phi Z dV_{t_2}(x) - \int_M \phi Z dV_{t_1}(x)\Big| &= \Big|\int_{t_1}^{t_2}\int_M \phi(Z_t + HZ)dV_t(x)dt\Big| \\ &= \Big|\int_{t_1}^{t_2}\int_M \phi\Delta^t Z dV_t(x)dt\Big| \\ &= \Big|\int_{t_1}^{t_2}\int_M Z\Delta^t\phi dV_t(x)dt\Big| \\ &\le \frac{C_2(t_2-t_1)}{R}\cdot\max_{t_1\le t\le t_2}\int_M Z dV_t \le \frac{C_3}{R},\end{aligned}$$

where C_2, C_3 are independent of R and (5.1) has been used. Let $R \to \infty$; we get

$$\int_M Z dV_{t_2}(x) = \int_M Z dV_{t_1}(x)$$

for any $s < t_1 < t_2 \le T$. Note that $\lim_{t\to s^+}\int_M Z dV_t(x) = 1$. The second conclusion in the lemma follows. ■

Corollary 5.2 *There is a positive constant C depending on T and the upper bounds of $|q|$ and $|H|$, such that*

$$\int_M Z(x,t;y,s)dV_s(y) \le C$$

for any $s \in [0,t)$.

Proof Since $Z(x,t;y,s)$ is also the fundamental solution of the conjugate equation, the proof is similar to the proof of Lemma 5.1. ■

Lemma 5.3 *There is a positive constant C depending only on T, n, the lower bound for the Ricci curvature of the initial metric and the upper bounds for $|q|$ and $|h|$, such that*

$$Z(x,t;y,s) \le \frac{C}{V_x(\sqrt{t-s})} \quad \textit{and} \quad Z(x,t;y,s) \le \frac{C}{V_y(\sqrt{t-s})}.$$

Proof Applying the mean value inequality in Lemma 3.1 to

$$u(y,s) = Z(x,t;y,t-s)$$

with $r = \frac{\sqrt{s}}{2}$ and using volume comparison we have

$$\begin{aligned}Z(x,t;y,t-s) = u(y,s) &\le \frac{C_1 e^{A_1 s + B_1 r}}{r^2 V_y(r)}\int_0^s\int_M u dV_t ds \\ &\le \frac{C_2 e^{A_1 s + B_2\sqrt{s}}}{V_y(\sqrt{s}/2)} \le \frac{C_3 e^{A_1 T + B_3\sqrt{T}}}{V_y(\sqrt{s})}.\end{aligned}$$

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Here C_1, C_2, C_3 depend only on n, T, and the upper bounds for q and $|h|$, A_1 depends only the upper bounds for $|q|$ and $|H|$, and B_i's depend only on n and the lower bound for the Ricci curvature of $g(0)$. So we get the second inequality in the lemma.

Applying a similar method to $u(x,t) = \mathcal{Z}(x, t+s; y, s)$ with $r = \frac{\sqrt{s}}{2}$, we get the first inequality. ∎

Lemma 5.4 *There exist positive constants C and D with C depending only on T, n, the lower bound for the Ricci curvature of the initial metric, and the upper bounds for $|q|$ and $|h|$, and D depending only on T and the upper bound for $|h|$, such that for $0 \le s < t \le T$,*

$$\int_M \mathcal{Z}^2(x,t;y,s) e^{\frac{r^2(x,y)}{D(t-s)}} dV_t(x) \le \frac{C}{V_y(\sqrt{t-s})} \quad \textit{and}$$

$$\int_M \mathcal{Z}^2(x,t;y,s) e^{\frac{r^2(x,y)}{D(t-s)}} dV_s(y) \le \frac{C}{V_x(\sqrt{t-s})}.$$

Proof We only prove the first inequality. The proof of the second one is similar. By Lemma 5.3 and the fact that $\mathcal{Z}_k$ increasing to $\mathcal{Z}$,

$$\int_{\Omega_k} \mathcal{Z}_k^2(x,t;y,s) dV_t(x) \le \frac{C_1}{V_y(\sqrt{t-s})} \int_{\Omega_k} \mathcal{Z}_k(x,t;y,s) dV_t(x) \le \frac{C_2}{V_y(\sqrt{t-s})}$$

for $0 \le s < t \le T$, where C_1 and C_2 depend on T, n, the lower bound for the Ricci curvature of the initial metric, and the upper bounds for $|q|$ and $|h|$. Now fix $t > s$ and consider the function $u(x,\tau) = Z_k(x, \tau + s; y, s)$, $0 < \tau \le t - s$.

Let $f(\tau) = V_y(\sqrt{\tau})$. Then for $0 < \tau_1 < \tau_2 \le T$,

$$\frac{f(\tau_1)}{f(\tau_1/4)} = \frac{V_y(\sqrt{\tau_1})}{V_y(\sqrt{\tau_1}/2)} \le \frac{V_k(\sqrt{\tau_1})}{V_k(\sqrt{\tau_1}/2)} \le \frac{V_k(\sqrt{T})}{V_k(\sqrt{T}/2)} \le A \frac{f(\tau_2)}{f(\tau_2/4)},$$

where $V_k(r)$ denotes the volume of the ball of radius r in the space form with Ricci curvature $-k$ ($-k$ is the lower bound for the Ricci curvature of the initial metric) and $A = \frac{V_k(\sqrt{T})}{V_k(\sqrt{T}/2)}$. So f is regular with the constants A and $\gamma = 4$. By Lemma 2.2,

$$\int_{\Omega_k} \mathcal{Z}_k^2(x,t;y,s) e^{\frac{r^2(x,y)}{D(t-s)}} dV_t(x) \le \frac{C}{V_y(\sqrt{t-s})},$$

where D depends on T and the uniformly upper bound for $|h|$, and C depends on T, n, the lower bound for the Ricci curvature of the initial metric, and the uniformly upper bounds for $|q|$ and $|h|$.

Letting $k \to \infty$, we get the first inequality. ∎

Theorem 5.5 *There exist positive constants C and D with C depending only on T, n, the lower bound for the Ricci curvature of the initial metric, and the upper bounds for $|q|$ and $|h|$, and D depending only on T and the upper bound for $|h|$, such that for $0 \le s < t \le T$,*

$$\mathcal{Z}(x,t;y,s) \le \frac{C}{V_x^{\frac{1}{2}}(\sqrt{t-s}) V_y^{\frac{1}{2}}(\sqrt{t-s})} \times e^{-\frac{r^2(x,y)}{D(t-s)}}.$$

Proof By the triangle inequality, we have

$$r^2(x,\zeta) + r^2(\zeta,y) - \frac{r^2(x,y)}{2} \geq 0.$$

Let D be as in Lemma 5.4 and let $\tau = (s+t)/2$. Then by the semigroup property and Lemma 5.4,

$$\begin{aligned}
Z(x,t;y,s) &= \int_M Z(x,t;\zeta,\tau)Z(\zeta,\tau;y,s)dV_\tau(\zeta) \\
&\leq \int_M Z(x,t;\zeta,\tau)Z(\zeta,\tau;y,s)e^{\frac{r^2(x,\zeta)}{2D(t-s)}+\frac{r^2(\zeta,y)}{2D(t-s)}-\frac{r^2(x,y)}{4D(t-s)}}dV_\tau(\zeta) \\
&\leq e^{-\frac{r^2(x,y)}{4D(t-s)}}\Big(\int_M Z^2(x,t;\zeta,\tau)e^{\frac{r^2(x,\zeta)}{D(t-s)}}dV_\tau(\zeta)\Big)^{1/2} \\
&\quad \times \Big(\int_M Z^2(\zeta,\tau;y,s)e^{\frac{r^2(\zeta,y)}{D(t-s)}}dV_\tau(\zeta)\Big)^{1/2} \\
&\leq \frac{C}{V_x^{\frac{1}{2}}(\sqrt{t-s})V_y^{\frac{1}{2}}(\sqrt{t-s})} \times e^{-\frac{r^2(x,y)}{4D(t-s)}}.
\end{aligned}$$

■

Using volume comparison as in [17], we have the following.

Corollary 5.6 *There exist positive constants C and D with C depending only T, n, the lower bound for the Ricci curvature of the initial metric, and the upper bounds for $|q|$ and $|h|$, and D depending only on T and the upper bound for $|h|$, such that*

$$Z(x,t;y,s) \leq \frac{C}{V_x(\sqrt{t-s})}e^{-\frac{r^2(x,y)}{D(t-s)}} \quad \textit{and} \quad Z(x,t;y,s) \leq \frac{C}{V_y(\sqrt{t-s})}e^{-\frac{r^2(x,y)}{D(t-s)}}$$

for any $0 < s < t < T$.

Next we want to obtain lower estimates of the fundamental solution. We will proceed as in [8].

Lemma 5.7 *There is a positive constant c depending only on T and the upper bounds of $|q|$ and $|H|$, such that*

$$\int_M Z(x,t;y,s)dV_t(x) \geq c \quad \textit{and} \quad \int_M Z(x,t;y,s)dV_s(y) \geq c$$

for any $0 < s < t < T$.

Proof We only prove the first inequality. The proof of the second one is similar. Let $\phi = \eta(\rho/R)$ be the same function as in the proof of Lemma 4.1. As in the proof of Lemma 5.1, for any $t_1 < t < t_2$ in (s,T),

$$\frac{d}{dt}\int_M \phi Z dV_t \geq -\frac{C_2}{R} - C_1\int_M \phi Z dV_t$$

where C_1 depends on the upper bounds for $|q|$ and $|H|$, C_2 is independent of R, and Lemma 5.1 has been used in the last inequality. So

$$\int_M \phi \mathcal{Z} dV_t \geq e^{-C_1(t-s)} \Big\{ e^{C_1(t_1-s)} \int_M \phi \mathcal{Z} dV_{t_1} - \frac{C_2(1-e^{C_1(t-s)})}{C_1 R} \Big\}$$

for any $T > t > t_1 > s$. Letting $R \to \infty$ and $t_1 \to s^+$, we get

$$\int_M \mathcal{Z} dV_t \geq e^{-C_1(t-s)} \geq e^{-C_1 T}$$

for any $t \in (s, T)$. ■

Lemma 5.8 *Let c be the constant in Lemma 5.7. Then, there is a constant $A > 1$ depending only on n, T, the lower bound for the Ricci curvature of the initial metric, and the upper bounds for $|q|$ and $|h|$, such that*

$$\int_{B_y(A\sqrt{t-s})} \mathcal{Z}(x,t;y,s) dV_t(x) \geq \frac{c}{2} \quad \text{and} \quad \int_{B_x(A\sqrt{t-s})} \mathcal{Z}(x,t;y,s) dV_s(y) \geq \frac{c}{2}$$

for any $0 < s < t < T$.

Proof We only prove the first inequality; the proof of second one is similar. Let $A > 1$. By the second inequality of Corollary 5.6,

$$\begin{aligned}
\int_{M \setminus B_y(A\sqrt{t-s})} \mathcal{Z}(x,t;y,s) dV_t(x) &\leq \frac{C_1}{V_y(\sqrt{t-s})} \int_{M \setminus B_y(A\sqrt{t-s})} e^{-\frac{r^2(x,y)}{D(t-s)}} dV(x) \\
&= \frac{C_1}{V_y(\sqrt{t-s})} \int_{A\sqrt{t-s}}^{\infty} e^{-\frac{r^2}{D(t-s)}} dV_y(r) \\
&\leq C_1 \int_{A\sqrt{t-s}}^{\infty} \frac{V_y(r)}{V_y(\sqrt{t-s})} \times e^{-\frac{r^2}{D(t-s)}} d\Big(\frac{2r}{D(t-s)}\Big) \\
&\leq \frac{C_1}{D} \int_{A\sqrt{t-s}}^{\infty} \Big(\frac{r}{\sqrt{t-s}}\Big)^n e^{-\frac{r^2}{D(t-s)} + C_2 \frac{r}{\sqrt{t-s}}} d\Big(\frac{r^2}{t-s}\Big),
\end{aligned}$$

where C_2 depends on n, T, and the lower bound for the Ricci curvature of the initial metric.

Choose $A \geq 2C_2 D + 1$, then

$$\begin{aligned}
\int_{M \setminus B_y(A\sqrt{t-s})} \mathcal{Z}(x,t;y,s) dV_t(x) &\leq \frac{C_1}{D} \int_{A\sqrt{t-s}}^{\infty} \Big(\frac{r}{\sqrt{t-s}}\Big)^n e^{-\frac{r^2}{2D(t-s)}} d\Big(\frac{r^2}{t-s}\Big) \\
&= 2C_3 \int_{2DA^2}^{\infty} x^{\frac{n}{2}} e^{-x^2} dx \leq \frac{c}{2},
\end{aligned}$$

provided A is large enough depending only on n, T, the lower bound for the Ricci curvature of the initial metric, and the upper bounds for $|q|$ and $|h|$. This complete the proof of the first inequality. ■

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By the lemma, volume comparison, and the Harnack inequality in Corollary 4.4, we have the following.

Lemma 5.9 *There is a constant $c > 0$ depending on n, T, and the upper bounds of $|h|, |\nabla^t h|, |q|, |\nabla^t q|, |\Delta^t q|, |Rc^t|$, such that*

$$\mathcal{Z}(x,t;x,s) \geq \frac{c}{V_x(\sqrt{t-s})} \quad \text{and} \quad \mathcal{Z}(x,t;y,s) \geq \frac{c}{V_y(\sqrt{t-s})}$$

for any x, y and $0 < s < t < T$.

Proposition 5.10 *There exist positive constants c and d with c depending on n, T, and the upper bounds for $|h|, |\nabla^t h|, |q|, |\nabla^t q|, |\Delta^t q|, |Rc^t|$, and d depending only on T and the upper bound for $|h|$, such that*

$$\mathcal{Z}(x,t;y,s) \geq \frac{c}{V_x(\sqrt{t-s})} \times e^{-\frac{r^2(x,y)}{d(t-s)}} \quad \text{and} \quad \mathcal{Z}(x,t;y,s) \geq \frac{c}{V_y(\sqrt{t-s})} \times e^{-\frac{r^2(x,y)}{d(t-s)}}.$$

for any $0 < s < t < T$.

Proof We only prove the second inequality; the proof of the first one is similar. Let $\tau = \frac{t-s}{2}$. By Corollary 4.4 and Lemma 5.9,

$$\mathcal{Z}(x,t;y,s) \geq c_1 \mathcal{Z}(y,t-\tau;y,s) e^{-\frac{r^2(x,y)}{c_2\tau}} \geq \frac{c_3}{V_y(\sqrt{\tau})} e^{-\frac{r^2(x,y)}{c_2\tau}} \geq \frac{c_4}{V_y(\sqrt{t-s})} e^{-\frac{r^2(x,y)}{c_2(t-s)}}. \ \blacksquare$$

Corollary 5.11 *For the same positive constants c and d as in Proposition 5.10,*

$$\mathcal{Z}(x,t;y,s) \geq \frac{c}{V_x^{\frac{1}{2}}(\sqrt{t-s}) V_y^{\frac{1}{2}}(\sqrt{t-s})} \times e^{-\frac{r^2(x,y)}{d(t-s)}}$$

for any $0 < s < t < T$.

Proof The result follows from the inequalities in Proposition 5.10 and integrating as in the proof of Theorem 5.5. ■

6 More Gradient Estimates

In this section we want to obtain more gradient estimates, which are generalizations of the gradient estimates in [29] and [18] to complete noncompact manifolds. The estimates will be used in later sections.

Let $u > 0$ be a solution of the equation

$$\Delta^t u - u_t = 0 \tag{6.1}$$

on $M \times [0,T]$ corresponding to the Ricci flow $\frac{\partial}{\partial t} g = -2Ric$ or a solution of

$$\Delta^\tau u - u_\tau - Ru = 0 \tag{6.2}$$

on $M \times [0,T]$ corresponding to the backward Ricci flow $\frac{\partial}{\partial \tau} g = 2Ric$, where R is the scalar curvature, $\tau = T - t$.

Let us make the following assumption:

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(a1) $g(t)$ is complete and $|\nabla^k Rm|$ are uniformly bounded on spacetime by c_k for all k.

Lemma 6.1 *Let $u > 0$ be a solution of* (6.1) *or* (6.2) *such that $u \leq A$ for all t. There is a constant C depending only on those constants c_k for $k = 0, 1$ in the assumption* (a1), n, T, and A such that $|\nabla u|(x, t) \leq C/t$ for all $x \in M \times (0, T]$.

Before we prove the lemma, let us modify a maximum principle in [9].

Lemma 6.2 *Suppose $g(t)$ is a smooth family of complete metrics defined on M, $0 \leq t \leq T$ with Ricci curvature bounded from below and $|\frac{\partial}{\partial t} g| \leq C$ on $M \times [0, T]$. Suppose $f(x, t)$ is a smooth function defined on $M \times [0, T]$ such that*

$$\Delta^t f - \frac{\partial}{\partial t} f \geq 0 \tag{6.3}$$

and

$$\int_0^T \int_M \exp(-a r_t^2(o, x)) f^2(x, t) dV_t < \infty \tag{6.4}$$

for some $a > 0$. If $f(x, 0) \leq 0$ for all $x \in M$, then $f \leq 0$ on $M \times [0, T]$.

Proof In [9], condition (6.4) with f being replaced by $|\nabla f|$ is assumed. From their proof, it is easy to see that the result is still true if f is replaced by $|\nabla f_+|$, where $f_+ = \max\{f, 0\}$.

Observe that by (6.4), there exists $T_i \uparrow T$ such that

$$\int_M \exp(-a r_{T_i}^2(o, x)) f^2(x, T_i) dV_{T_i} < \infty.$$

Then one can obtain their condition by using (6.4) and (6.3) by a cutoff argument on $[0, T_i]$, perhaps with another $a > 0$. ■

Proof of Lemma 6.1 Let us prove the case where u is a solution of (6.2). The other case is similar. Note that we may consider the interval $[\epsilon, T]$ first, and then let $\epsilon \to 0$. Hence we may assume that u is smooth up to $\tau = 0$. For simplicity, we will write t instead of τ, and Δ instead of Δ^t. Here and below, C_i denotes constants depending only on n, T, and c_0, c_1. Let $f = (|\nabla u|^2 + 1)^{\frac{1}{2}}$. Then one may compute that $(\Delta - \partial_t)(tf) \geq -C_1 f$, and

$$(\Delta - \partial_t)(\frac{1}{2}C_1 u^2 - C_2 t - \frac{1}{2}C_1 A^2) = C_1|\nabla u|^2 + C_1 R u^2 + C_2 \geq C_1 f^2 \geq C_1 f,$$

where C_2 is chosen so that $C_1 R u^2 + C_2 - C_1 \geq 0$.

Hence $(\Delta - \partial_t)Q \geq 0$, where

$$Q = tf + \tfrac{1}{2}C_1 u^2 - C_2 t - \tfrac{1}{2}C_1 A^2.$$

Since $Q \leq 0$, at $t = 0$, the result will follow from Lemma 6.2, provided that we can prove

$$\int_0^T \int_M \exp(-a^2 r_t^2) Q^2 dV_t dt < \infty$$

for some $a > 0$. Here $r_t(x)$ is the distance from a fixed point. Since the curvature is bounded, by volume comparison, it is sufficient to prove that

$$\int_0^T \int_M \exp(-a^2 r_t^2)|\nabla u|^2 dV_t dt < \infty \tag{6.5}$$

for some $a > 0$. Since we have

$$\left(\Delta - \frac{\partial}{\partial t}\right)(\exp(-C_3 t)u) \geq 0.$$

Using a cutoff argument and the fact that u is bounded, it is easy to see that (6.5) is true. ∎

Lemma 6.3 *With the same notations and assumptions as before, the following estimates are true:*

Suppose $u > 0$ is a solution of (6.2) *or* (6.1), *and suppose $u \leq A$. Then*

$$t\frac{|\nabla u|^2}{u} \leq C\left[u \log \frac{A}{u} + u\right]$$

for some constant C depending only on T, n, and c_k, $0 \leq k \leq 2$, in the assumption (a1).

Proof We prove the case where u is a solution of (6.2). The other case can be proved similarly. As in [21], for some suitable positive constants C_1, C_2, C_3, if we let

$$\Phi = \varphi\frac{|\nabla u|^2}{u} - \exp(C_1\tau)u\log\left(\frac{A}{u}\right) - C_2\tau u,$$

where $\varphi = \tau/(1 + C_3\tau)$, then $(\Delta^t - \frac{\partial}{\partial\tau})\Phi \geq 0$. Here and below, C_i's will denote constants depending on quantities mentioned in the lemma. Again, we may consider u being a solution in the interval $[\epsilon, T]$ for $\epsilon > 0$ first, and $\tau = 0$ corresponding to $t = \epsilon$, then let $\epsilon \to 0$. Let $p \in M$ be a fixed point, then by Corollary 4.4

$$u(p, \frac{\epsilon}{2}) \leq (C_4 + c)u(x, \tau)\exp((C_4 + c)r^2(p, x))$$

for some c depending only on quantities mentioned in the lemma and ϵ, and $\epsilon \leq \tau \leq T$. By Lemma 6.1 and the fact that u is bounded, we conclude that there is $a > 0$ such that $\exp(-ar_t(p, x)^2)\Phi^2$ is integrable on $M \times [\epsilon, T]$ with respect to $dV_\tau d\tau$. Since $\Phi \leq 0$ initially, we can apply Lemma 6.2 to conclude that the lemma is true for bounded and positive solutions of (6.2). ∎

7 A Li–Yau–Hamilton Type Differential Inequality

Let $(M^n, g(t))$ be a solution of the Ricci flow (1.1) on $M \times [0, T]$ for some $T > 0$. We always assume that M^n is noncompact, $g(t)$ is complete, and (a1) in the previous section is true. Let $\mathcal{Z}(x, t; y, s)$ with $0 \leq s < t \leq T$ be the fundamental solution of

$$\frac{\partial u}{\partial t} - \Delta^t u = 0. \tag{7.1}$$

Let $p \in M$ be fixed and let $u(x,t) = \mathcal{Z}(p,T;x,t) > 0$. Then u is a solution of the conjugate heat equation $-\frac{\partial u}{\partial t} - \Delta^t u + Ru = 0$ on $M \times [0,T]$, where R is the scalar curvature and Δ^t is the Laplacian with respect to $g(t)$. When there is no confusion, we simply denote Δ^t as Δ. Let v be defined by

$$v = \left[\tau(2\Delta f - |\nabla f|^2 + R) + (f - n)\right] u, \tag{7.2}$$

where f is defined by $u = e^{-f}/(4\pi\tau)^{\frac{n}{2}}$ and $\tau = T - t$ (this notation is adopted throughout this section).

Let $h_0 \geq 0$ be a smooth function with compact support and let $0 < t_0 < T$. Let $h(x,t)$ be the solution of (7.1) on $M \times [t_0, T]$ with initial data $h(x,t_0) = h_0(x)$. We want to prove the following.

Theorem 7.1 *With the above notations and assumption* (a1), *we have*

(i) *for any $t_0 < t < T$. $hv(\cdot,t) \in L^1(M, g(t))$,*
(ii) *for any $t_0 < t_1 < t_2 < T$, $\int_M hv dV_{t_1} \leq \int_M hv dV_{t_2}$.*
(iii) $\limsup_{t \to T^-} \int_M hv dV_t \leq 0$.

Lemma 7.2 *Theorem* 7.1(i) *is true.*

Proof For any $T > t > t_0$, by Corollary 5.6 and Lemma 6.3, there are $a, C_1 > 0$ such that for all $x \in M$

$$\Big(\frac{|\nabla u|^2}{u} + u + h + |\nabla h|\Big)(x,t) \leq C_1 \exp(-ar^2(x)). \tag{7.3}$$

Since the curvature is bounded and since $-f = \log u + \frac{n}{2}\log(4\pi\tau)$, $|\nabla f|^2 uh$, fuh, nuh, Ruh are all in L^1, where $\tau = T - t$. Moreover, since

$$\Delta f = -\frac{\Delta u}{u} + |\nabla f|^2$$

in order to prove the lemma, it is sufficient to prove that $h\Delta u$ is in L^1. By Lemma 4.1, letting $\alpha > 1$ be fixed, we have for $T > t > 0$,

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_\tau}{u} - C_2 \leq 0$$

for some $C_2 > 0$. Hence

$$\begin{aligned}\int_M |h\Delta u| &= \int_M |u_\tau + Ru| h \leq \alpha^{-1}\int_M (\alpha u_\tau - \frac{|\nabla u|^2}{u} + C_2)h + C_3 \\ &= \int_M h\Delta u + C_4 < C_5\end{aligned}$$

for some constants $C_3 - C_5$, where we have used (7.3), integration by parts together with a cutoff argument. This completes the proof of the lemma. ■

Remark 7.3 From the proof, it is easy to see that for $0 < \tau_1 < \tau_2 < T$, there is a constant C so that

$$\int_M (|vh| + |v|) dV_t \le C$$

for all $\tau_1 \le \tau \le \tau_2$. Moreover, (7.3) is true for a constant C for all $\tau_1 \le \tau \le \tau_2$.

Lemma 7.4 *Theorem* 7.1(ii) *is true.*

Proof By [24], we have

$$\left(\frac{\partial}{\partial\tau} - \Delta + R\right) v = -2\tau \left|R_{ij} + f_{ij} - \frac{1}{\tau} g_{ij}\right|^2 u, \tag{7.4}$$

where $\tau = T - t$. Let ρ, η, and $\phi(\rho/r)$ be the functions defined in the proof of Lemma 4.1. Fix $0 < t_1 < t_2 < T$. By Remark 7.3 and (7.3) for any $t_1 \le t \le t_2$,

$$\begin{aligned}\frac{d}{d\tau}\int_M \phi hv dV_t &= \int_M \phi(v_\tau h + vh_\tau + vhR) dV_t \le \int_M \phi(h\Delta v - v\Delta h) dV_t \\ &= \int_M \big(v\langle\nabla\phi, \nabla h\rangle - h\langle\nabla\phi, \nabla v\rangle\big)\, dV_t \\ &= \int_M \big(2v\langle\nabla\phi, \nabla h\rangle + hv\Delta\phi\big)\, dV_t \le \frac{C}{r}\end{aligned}$$

for some $C > 0$ for all $\tau_1 \le \tau \le \tau_2$. By integrating from τ_1 to τ_2, and letting $r \to \infty$, the result follows. ■

Next, we want to prove Theorem 7.1(iii). We need several lemmas.

Lemma 7.5 *For any $\alpha > 1$ and $\delta, \epsilon > 0$, there is a constant $C(\alpha, \delta, \epsilon)$ independent of t such that, if $\frac{T}{2} < t < T$, then*

$$(1 - 2\alpha\delta)\int_M \frac{|\nabla u|^2}{u} h dV_t \le C + \frac{(n+\epsilon)\alpha^2}{2\tau}\int_M uh dV_t,$$

where C is a constant independent of t and $\tau = T - t$.

Proof Let α and ϵ be given, then by Lemma 4.1,

$$\frac{|\nabla u|^2}{u} \le \alpha(u_\tau + Ru) + C_1 + \frac{(n+\epsilon)\alpha^2}{2\tau} = \alpha\Delta u + C_1 + \frac{(n+\epsilon)\alpha^2}{2\tau}.$$

Here and below, C_i's are positive constants independent of τ. Let $\phi(\rho/r)$ be as in the

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proof of Lemma 7.4. Then

$$\begin{aligned}\int_M \phi^2 \frac{|\nabla u|^2}{u} hdV_t &\leq \alpha \int_M \phi^2 h\Delta u dV_t + C_1 \int_M \phi^2 uhdV_t + \frac{(n+\epsilon)\alpha^2}{2\tau} \int_M \phi^2 uhdV_t \\ &\leq \alpha \left(-2 \int_M \phi h \langle \nabla u, \nabla \phi \rangle dV_t - \int_M \phi^2 \langle \nabla u, \nabla h \rangle dV_\tau \right) \\ &\quad + C_2 + \frac{(n+\epsilon)\alpha^2}{2\tau} \int_M uhdV_t \\ &\leq 2\alpha\delta \int_M \phi^2 \frac{|\nabla u|^2}{u} hdV_t + \frac{C_3\alpha}{\delta r^2} \int_M hudV_t \\ &\quad + \frac{\alpha}{\delta} \int_M \phi^2 \frac{|\nabla h|^2}{h} udV_t + C_2 + \frac{(n+\epsilon)\alpha^2}{2\tau} \int_M uhdV_t.\end{aligned}$$

Note that $\int_M hudV_t$ is bounded by a constant independent of $\tau \in (0, T/2)$ because h is bounded and $\int_M udV_t$ is bounded independent of τ by Lemma 5.1. Also, $|\nabla h|^2/h$ is bounded independent of $\tau \in (0, T/2)$ by Lemma 6.3. Let $r \to \infty$, we have

$$(1 - 2\alpha\delta) \int_M \frac{|\nabla u|^2}{u} hdV_t \leq C_1(\alpha, \epsilon, \delta) + \frac{(n+\epsilon)\alpha^2}{2\tau} \int_M uhdV_t. \qquad \blacksquare$$

Lemma 7.6 *For any* $\delta > 0$,

$$\int_M (-\Delta u) hdV_t \leq 2\delta \int_M \frac{|\nabla u|^2}{u} hdV_t + C,$$

where C is a constant independent of t, provided $0 < \tau < \frac{T}{2}$.

Proof Let ϕ be the same as before. As in the proof of Lemma 7.5,

$$\int_M \phi^2 (-\Delta u) hdV_\tau \leq 2\delta \int_M \phi^2 \frac{|\nabla u|^2}{u} hdV_t + \frac{C}{\delta r^2} \int_M hudV_t + C,$$

where C is independent of τ. Since $h\Delta u$ is in $L^1(M, g(t))$ by the proof of Lemma 7.2, the result follows by letting $R \to \infty$. $\blacksquare$

Lemma 7.7 $\limsup_{t \to T^-} \int_M \tau h(2\Delta f - |\nabla f|^2 + R) udV_t \leq \frac{n}{2} h(x, T)$.

Proof Let $\alpha > 1$, $\delta, \epsilon > 0$ be constants to be chosen later. By Lemmas 7.5 and 7.6,

$$\begin{aligned}\int_M \tau h(2\Delta f - |\nabla f|^2 + R) udV_t &= \int_M \tau h\left(-2\Delta u + \frac{|\nabla u|^2}{u} + Ru\right) dV_t \\ &\leq \tau(1 + 4\delta) \left[\int_M \frac{|\nabla u|^2}{u} hdV_\tau + C_1 \right] + \tau \int_M huRdV_t \\ &\leq \tau \frac{1 + 4\delta}{(1 - 2\alpha\delta)} \left[C_2 + \frac{(n+\epsilon)\alpha^2}{2\tau} \int_M uhdV_\tau \right] + \tau C_3,\end{aligned}$$

where $C_1 - C_3$ are constants independent of τ, provided $0 < \tau < \frac{T}{2}$. Here we have used the fact that h is bounded and $\int_M u dV_t$ is uniformly bounded independent of τ by Lemma 5.1. Choose δ small such that $1 - 2\alpha\delta > 0$. Then

$$\begin{aligned}\limsup_{t\to T^-}\int_M \tau h(2\Delta f - |\nabla f|^2 + R)u dV_\tau &\le \frac{(n+\epsilon)(1+4\delta)\alpha^2}{(1-2\alpha\delta)}\limsup_{t\to T^-}\int_M uhdV_t\\ &= \frac{(n+\epsilon)(1+4\delta)\alpha^2}{2(1-2\alpha\delta)}h(p,T),\end{aligned}$$

where we have used the fact that $u \to \delta_p$ as $t \to T^-$ and that h is smooth and bounded. The result follows by letting $\alpha \to 1, \epsilon, \delta \to 0$. ■

Lemma 7.8 $\limsup_{t\to T^-}\int_M fuhdV_t \le \frac{n}{2}h(x,T)$.

Proof Let $\delta > 0$ be fixed and choose $C_1 > 0$ such that $C_1^{-1}\tau^{\frac{n}{2}} \le V_p^t(\sqrt{\tau}) \le C_2\tau^{\frac{n}{2}}$ for any $\tau \in [0,\delta]$, where $\tau = T - t$.

(i) We claim that, for any $\epsilon > 0$, there is a constant $A > 0$, such that

$$\int_{M\setminus B_p^t(A\sqrt{\tau})} fhu\, dV_t \le \epsilon \text{ for any } \tau \in (0,\delta].$$

By Corollary 5.6, we have

$$u(x,\tau) \le \frac{C_1}{V_p^t(\sqrt{\tau})}e^{-\frac{r_t^2(x,p)}{D\tau}}$$

for some positive constant C_2 and D. Then

$$\begin{aligned}\int_{M\setminus B_p^t(A\sqrt{\tau})} fhu\, dV_t &= \int_{M\setminus B_p^t(A\sqrt{\tau})} 2hu\log\frac{(\frac{1}{4\pi\tau})^{\frac{n}{4}}}{\sqrt{u}}dV_t\\ &\le \int_{M\setminus B_p^t(A\sqrt{\tau})} 2hu\cdot\frac{(\frac{1}{4\pi\tau})^{\frac{n}{4}}}{\sqrt{u}}dV_t\\ &\le \frac{C_3}{\tau^{\frac{n}{4}}}\int_{M\setminus B_p^t(A\sqrt{\tau})}\sqrt{u}dV \le \frac{C_4}{\tau^{\frac{n}{2}}}\int_{M\setminus B_p^t(A\sqrt{\tau})} e^{-\frac{r_t^2(x,p)}{2D\tau}}dV_t(x)\\ &= \frac{C_4}{\tau^{\frac{n}{2}}}\int_{A\sqrt{\tau}}^\infty e^{-\frac{r^2}{2D\tau}}A_p^t(r)dr\\ &= \frac{C_4}{\tau^{\frac{n}{2}}}\left(\left[e^{-\frac{r^2}{2D\tau}}V_p^t(r)\right]_{A\sqrt{\tau}}^\infty + \int_{A\sqrt{\tau}}^\infty V_p^t(r)e^{-\frac{r^2}{2D\tau}}d(\frac{r^2}{2D\tau})\right)\\ &\le C_5\int_{A\sqrt{\tau}}^\infty \tau^{-\frac{n}{2}}\sinh^{n-1}(C_6 r)e^{-\frac{r^2}{2D\tau}}d\left(\frac{r^2}{\tau}\right)\\ &\le C_7\int_{A\sqrt{\tau}}^\infty\left(\frac{r}{\sqrt{\tau}}\right)^n e^{C_6 r}e^{-\frac{r^2}{2D\tau}}d\left(\frac{r^2}{\tau}\right)\end{aligned}$$

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$$
\begin{aligned}
&= C_7 \int_{A^2}^{\infty} \rho^{\frac{n}{2}} e^{-\frac{\rho}{2D}} e^{C_6\sqrt{\rho\tau}}\, d\rho \\
&\le C_7 \int_{A^2}^{\infty} \rho^{\frac{n}{2}} e^{-\frac{\rho}{4D}}\, d\rho \quad (\text{if } A \ge 4DC_6\sqrt{\delta}) \\
&\le \epsilon \quad (\text{if we choose } A \text{ large enough}),
\end{aligned}
$$

where C_1–C_7 are constants independent of A and ϵ. Here we have used the following facts: $\log x \le x$; h and u are positive and h is bounded; $V_p^t(r) \le C(n)\sinh^n(C_6 r)$ by volume comparison; $\sinh(C_6 r)/r \le Ce^{C_6 r}$ for some constant C depending only on C_6.

(ii) By the asymptotic behavior of the heat kernel, see [11] for example, there is an open neighborhood U of p, some positive constants τ_0 and C_6, and a positive function $u_0 \in C^\infty(U \times [0,\tau_0])$ with $u_0(p,0) = 1$, such that

$$
\Big| u - \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-\frac{r_t^2(x,p)}{4\tau}} u_0(x,\tau) \Big| \le C_6 \tau^{1-\frac{n}{2}}
$$

for any $x \in U$ and $\tau \in (0,\tau_0]$. Hence, for any $x \in B_p^t(A\sqrt{\tau})$ when τ is small,

$$
\begin{aligned}
u &\ge \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-\frac{r_t^2(x,p)}{4\tau}} u_0(x,\tau) - C_6 \tau^{1-\frac{n}{2}} \\
&\ge \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-\frac{r_t^2(x,p)}{4\tau}} \left(1 - \frac{C_7\tau}{u_0(x,\tau)} e^{\frac{r_t^2(x,p)}{4\tau}} \right) u_0(x,\tau) \\
&\ge \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-\frac{r_t^2(x,p)}{4\tau}} (1 - C_8\tau) u_0(x,\tau),
\end{aligned}
$$

where all the constants are independent of τ. So

$$
f(x,t) \le \frac{r_t^2(x,p)}{4\tau} - \log(1 - C_8\tau) - \log u_0(x,\tau)
$$

for any $x \in B_p^t(A\sqrt{\tau})$ when τ is small enough.

Hence

$$
\begin{aligned}
&\int_{B_p^t(A\sqrt{\tau})} fhu\, dV_t \\
&\quad \le \int_{B_p^t(A\sqrt{\tau})} \Big(\frac{r_t^2(x,p)}{4\tau} - \log(1 - C_8\tau) - \log u_0(x,\tau) \Big) hudV_t \\
&\quad \le \int_{B_p^t(A\sqrt{\tau})} \frac{r_t^2(x,p)}{4\tau} hudV_t + C_9\tau \\
&\quad = \int_{B_p^t(A\sqrt{\tau})} \frac{r_t^2(x,p)}{4\tau} \Big(\frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-\frac{r_t^2(x,p)}{4\tau}} u_0(x,\tau) + C_6 \tau^{1-\frac{n}{2}} \Big) hdV_t + C_9\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_{B_p^t(A\sqrt{\tau})} \frac{r_t^2(x,p)}{4\tau}\frac{1}{(4\pi\tau)^{\frac{n}{2}}}e^{-\frac{r_t^2(x,p)}{4\tau}}u_0(x,\tau)h dV_t\\
&\quad + C_6\int_{B_p^t(A\sqrt{\tau})}\frac{r_t^2(x,p)}{\tau^{\frac{n}{2}}}h dV_t + C_9\tau\\
&= \frac{1}{(4\pi\tau)^{\frac{n}{2}}}\int_0^{A\sqrt{\tau}}\frac{r^2}{4\tau}e^{-\frac{r^2}{4\tau}}A_p^t(r)\tilde{h}(r,t)dr + C_{10}\tau,
\end{aligned}
$$

where

$$
\tilde{h}(r,t) = \frac{1}{A_p^t(r)}\int_{\partial B_p^t(r)} h u_0 dS_t.
$$

Hence

$$
\int_{B_p^t(A\sqrt{\tau})} f h u\, dV_t \le \frac{1}{2\pi^{\frac{n}{2}}}\int_0^{\frac{A^2}{4}}\rho^{\frac{n}{2}}e^{-\rho}\frac{A_p^t(2\sqrt{\tau\rho})}{(2\sqrt{\tau\rho})^{n-1}}\tilde{h}(2\sqrt{\tau\rho},t)d\rho + C_{10}\tau,
$$

where C_9 and C_{10} are both independent of τ.

Note that

$$
\frac{A_p^t(2\sqrt{\tau\rho})}{(2\sqrt{\tau\rho})^{n-1}} \to \alpha_{n-1} \text{ uniformly for } \rho\in\left[0,\frac{A^2}{4}\right] \text{ as } \tau\to 0^+,
$$

where α_{n-1} means the volume of the standard sphere of dimension $n-1$. Moreover,

$$
\begin{aligned}
\tilde{h}(2\sqrt{\tau\rho},t) &= \frac{1}{A_p^t(2\sqrt{\tau\rho})}\int_{\partial B_p^t(2\sqrt{\tau\rho})} h u_0 dS_t \to h(p,T)u_0(p,0)\\
&= h(p,T) \text{ uniformly for } \rho\in\left[0,\frac{A^2}{4}\right] \text{ as } \tau\to 0^+.
\end{aligned}
$$

So

$$
\begin{aligned}
\limsup_{\tau\to 0^+}\int_{B_p^t(A\sqrt{\tau})} f h u\, dV_t &\le \frac{\alpha_{n-1}h(p,T)}{2\pi^{\frac{n}{2}}}\Gamma\left(\frac{n}{2}+1\right)\\
&= \frac{n\omega_n}{2}\cdot\frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}}h(p,T) = \frac{n}{2}h(p,T)
\end{aligned}
$$

where ω_n means the volume of the unit ball in $\mathbb{R}^n$.

The lemma follows from (i) and (ii). ∎

Proof of Theorem 7.1(iii) The result follows from Lemmas 7.7 and 7.8, the fact that $u\to\delta_p$ as $t\to T^-$, and the fact that h is smooth and bounded. ∎

Corollary 7.9 $v(x,t)\le 0$ *on* $M\times[0,T)$.

8 A Pseudolocality Theorem

In this section, we will extend Perelman's pseudolocality theorem to complete noncompact manifolds. We will prove the following.

Theorem 8.1 *Let n be fixed. There exist $\delta, \epsilon > 0$ with the following property:*

Suppose $g(x,t)$ is a smooth complete noncompact solution of the Ricci flow (1.1) with bounded curvature on $M^n \times [0,\epsilon^2]$. Suppose at some point $x_0 \in M$, the isomperimetric constant in $B_0(x_0,1)$ is larger than $(1-\delta)c_n$, where c_n is the isoperimetric constant of $\mathbb{R}^n$, and $R(x,0) \geq -1$ for all $x \in B_0(x_0,1)$. Then $|Rm(x,t)| \leq t^{-1} + \epsilon^{-2}$ for $0 < t \leq \epsilon^2$ and $x \in B_t(x_0,\epsilon)$.

By the result in [26], we may assume that the covariant derivatives of the curvature are uniformly bounded in spacetime. The proof is similar to the case for compact manifolds using the estimates obtained in previous sections. See [7, 14, 24]. For the sake of completeness, we will sketch the proof.

Suppose this is not true. Then we can find $(M_i, g_i(t))$, $\delta_i, \epsilon_i > 0$ with $\delta_i, \epsilon_i \to 0$ and $p_i \in M_i$ satisfying the following:

(b1) $g_i(t)$ is a smooth solution of the Ricci flow on $[0,\epsilon_i^2]$ with $|\nabla^k Rm|$ uniformly bounded on $M_i \times [0,\epsilon_i^2]$ for all $k \geq 0$.

(b2) The isomperimetric constant in $B_0^{(i)}(p_i,1)$ is larger than $(1-\delta_i)c_n$.

(b3) There exist $0 < t_i \leq \epsilon_i^2$, and $x_i \in B_{t_i}^{(i)}(p_i,\epsilon_i)$ and $|Rm(x_i,t_i)| \geq t_i^{-1} + \epsilon_i^{-2}$.

Let $A_i = 1/1000n\epsilon_i$. By [24, Claims 1 and 2] (see also [7, 14]), we can find $\bar{x}_i, \bar{t}_i$ with $0 < \bar{t}_i \leq \epsilon_i^2$ and $\bar{x}_i \in B_{\bar{t}_i}^{(i)}(p_i,(2A_i+1)\epsilon_i)$ satisfying the following:

(c1) $Q_i = |Rm(\bar{x}_i,\bar{t}_i)| \geq \frac{1}{\bar{t}_i}$, and if

$$\bar{t}_i - \frac{1}{2}Q_i^{-1} \leq t \leq \bar{t}_i,\ d_{\bar{t}_i}(x,\bar{x}_i) \leq \frac{1}{10}A_i Q_i^{-\frac{1}{2}},$$

then $|Rm(x,t)| \leq 4|Rm(\bar{x}_i,\bar{t}_i)|$.

Consider the rescaled flow: $\widehat{g}_i(t) = Q_i g_i(\bar{t}_i + Q_i^{-1}t)$. Then $\widehat{g}_i$ satisfies the Ricci flow equation on $M_i \times [-\frac{1}{2},0]$ with $|\nabla^k Rm|$ uniformly bounded for all $k \geq 0$. Moreover, the following are true:

(d1) $|Rm(\bar{x}_i,0)| = 1$.

(d2) If $-\frac{1}{2} \leq t \leq 0$, $d_0(x,\bar{x}_i) \leq \frac{1}{10}A_i$, then $|Rm(x,t)| \leq 4$.

Let u_i be the fundamental solution of the conjugate heat equation to the flow $\widehat{g}_i$: $-(u_i)_t - \Delta u_i + R_i u_i = 0$, $\lim_{t\to 0} u = \delta_{\bar{x}_i}$. Let v_i be the corresponding LYH Harnack expression defined in (7.2) with $\tau = -t$. Then $v_i \leq 0$ by Corollary 7.9.

Case 1: Suppose the injectivity radius at $\bar{x}_i$ at $t = 0$ are uniformly bounded from below. Because of (d2) and the Ricci flow equation, we know that the injectivity radius of $\bar{x}_i$ are uniformly bounded from below at $t = -\frac{1}{2}$. Since $A_i \to \infty$, by the compactness result of Ricci flow [7, 12], for any sequence, we can find a subsequence of $\widehat{g}_i$ which converge, still denoted by $\hat{g}_i$. Namely, there is $(M,p,g(t))$ with $g(t)$ being a solution of the Ricci flow on $[-\frac{1}{2},0]$, and an exhaustion U_i of M, diffeomorphisms $\Phi_i : U_i \to M_i$ with the following properties:

(e1) $\Phi_i(p) = \bar{x}_i$.
(e2) $\Phi_i^* \hat{g}_i$ converges in C^∞ sense to g on $M \times (-\frac{1}{2}, 0)$.
(e3) The curvature of $g(t)$ is bounded by 4.
(e4) There exists $0 > t_0 > -\frac{1}{2}$ such that $|Rm(p,t)| \geq \frac{1}{2}$ for all $t > t_0$.

Note that (e4) is a consequence of (d1), (d2), (e2), local derivatives bound for the curvature tensor, and the evolution equation of the curvature tensor.

Lemma 8.2

(f1) *$\Phi_i^* u_i$ subconverge on $M \times (-\frac{1}{2}, 0)$ to a solution of $u_\tau - \Delta u + Ru = 0$.*
(f2) *$u > 0$, and if v is the LYH Harnack expression defined in* (7.2) *corresponding to u, then*

$$v_\tau - \Delta v + Rv = -2\tau |R_{ij} + \nabla_i \nabla_j f - \tfrac{1}{2\tau} g_{ij}|^2,$$

where f is given by $u = e^{-f}/(4\pi\tau)^{\frac{n}{2}}$. Moreover, $v \leq 0$.

Proof We first prove (f1). By Lemma 5.1, $\int_{M_i} u_i dV_\tau^{(i)} = 1$ for all i and τ. Since $u_i > 0$ for $\tau > 0$, by (e2) and the proofs of Corollaries 4.2 and 4.5, we conclude that u_i are locally uniformly bounded. By (e2), it is easy to see that (f1) is true. Note that we can construct the function ρ for $(M, g(t))$ as in the proof of Lemma 4.1 and use this to prove a result similar to Corollary 4.2 for u_i by (e2). See also Remark 4.3.

Next we want to prove (f2). By the proof of Lemma 5.7, for a fixed but small neighborhood U of p, there is $c > 0$ and $\tau > 0$ such that $\int_U u_i dV_t^{(i)} \geq c$ for all i. So $u > 0$ by the fact that $\phi_i^* \widehat{g}_i$ converges to g, (f1), and the maximum principle. The rest of the lemma follows from (7.4). ■

Lemma 8.3 *With the same notation as in Lemma* 8.2*, for any* $0 < \tau_0 < \frac{1}{2}$ *we have*

$$\int_{B_{\tau_0}(p, \sqrt{\tau_0})} v dV_{-\tau_0} < 0.$$

Proof Suppose

$$\int_{B_{\tau_0}(p, \sqrt{\tau_0})} v dV_{-\tau_0} = 0,$$

then $v = 0$ in $B_{\tau_0}(p, \sqrt{\tau_0})$. Let h_0 be a nonnegative smooth function with support in $B_{\tau_0}(p, \sqrt{\tau_0})$ that is positive somewhere. Then for i sufficiently large, we may also consider h_0 to be a smooth function with compact support in M_i. Now solve the forward heat equation with initial data $h_i|_{t=-\tau_0} = h_0$ on $M_i \times [-\tau_0, 0)$. Then since h_0 is bounded, the h_i's are also uniformly bounded in space time. We may thus assume that $h_i \to h$, which solves the heat equation in $(M, g(t))$, and the convergence is uniform on compact sets of $(x, t) \in M \times [-\tau_0, 0)$. By Theorem 7.1(ii), for $0 < \tau < \tau_0$

$$\int_{M_i} v_i(x, -\tau) h_i(x, -\tau) dV_{-\tau}^{(i)} \geq \int_{M_i} v_i(x, -\tau_0) h_0(x) dV_{-\tau_0}^{(i)},$$

where v_i is the LYH Harnack expression for v_i. Since $v_i \leq 0$ by Corollary 7.9 and since h_0 is a fixed function with compact support, letting $i \to \infty$, we can conclude

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for any compact set K in M,

$$\int_K \nu(x,-\tau)h(x,-\tau)dV_{-\tau} \geq \int_M \nu(x,-\tau_0)h_0(x)dV_{-\tau_0} = 0.$$

Since $\nu \leq 0$ and $h > 0$ for $t > -\tau_0$, we have $\nu = 0$ for $0 < \tau < \tau_0$. By (f2) we have

$$R_{ij} + f_{ij} - \frac{1}{2\tau}g_{ij} = 0.$$

for $\tau < \tau_0$. Since the curvature is uniformly bounded, for any $0 < \tau < \tau_0$, $|\nabla f|$ is of at most linear growth. From this one can prove that the vector field $Y_t = (1 - \frac{1}{\tau}t)^{-1}\nabla f(-\tau)$ can be integrated from 0 to t as long as $1 - \frac{1}{\tau}t > 0$, yielding a diffeomorphism ψ_t.

Then the flow $\widetilde{g}(t) = (1 - \frac{1}{\tau_1}t)\psi_t^*(g(-\tau_1))$ is a solution of the Ricci flow on $[0, \tau_1)$ with initial data $g(-\tau_1)$ [6, pp. 22–23]. Now $g(t - \tau)$ for $0 \leq t < \tau$ is also such a solution. Since the curvatures are bounded for both flows, by the uniqueness result of [5], they are the same. However, by (e4) the curvature of $(1 - \frac{1}{\tau}t)\psi_t^*(g(-\tau_1))$ blows up near $t = \tau$, and the curvature of $g(t - \tau)$ are uniformly bounded, so this is impossible. ■

Case 2: Suppose the injectivity radii a_i at $\bar{x}_i$ at $t = 0$ tend to zero. We further rescale the metrics. Let $\hat{\hat{g}}_i(t) = a_i^{-1}\hat{g}_i(a_i t)$. Then $\hat{\hat{g}}_i(t)$ is defined on $[-\frac{1}{2}a_i^{-1}, 0]$ with the following properties:

(g1) $|Rm(\bar{x}_i, 0)| = a_i$.

(g2) If

$$-\tfrac{1}{2}a_i^{-1} \leq t \leq 0,\ d_0(x, \bar{x}_i) \leq \tfrac{1}{10}A_i \cdot a_i^{-\frac{1}{2}},$$

then $|Rm(x,t)| \leq 4a_i$.

(g3) There exists $t_0 < 0$ such that the injectivity radius at $\bar{x}_i$ at time t_0 is less than 2.

We can prove (g3) using the fact that $a_i \to 0$ and the injectivity radius bound in [3].

As before, we can find a limit metric and flow $(M, g(t), p)$ on $(-\infty, 0]$ with the following properties:

(h1) $g(t)$ is flat for all t.

(h2) The injectivity radius of p at time t_0 is less than or equal to 2.

Let ν and f be as before.

Lemma 8.4 *For any $0 < \tau_0 < \infty$ we have $\int_{B_{\tau_0}(p,\sqrt{\tau_0})} \nu dV_{-\tau_0} < 0$.*

Proof As before, if this is not true, then one can prove that $R_{ij} + f_{ij} - \frac{1}{2\tau}g_{ij} = 0$ for $0 < \tau < \tau_0$. Since curvature is bounded, there is $0 < \tau_1 < \tau_0$ such that $f_{ij} \geq g_{ij}$. One can prove that

$$g(t - \tau_1) = \left(1 - \tfrac{1}{\tau_1}t\right)\psi_t^*(g(-\tau_1)). \tag{8.1}$$

The function f is an exhaustion function, see [1], for example. So there is a point such that $\nabla f = 0$. Hence we can find a point q that is a fixed point of ψ_t. Hence the injectivity radius at q with respect to $\psi_t^*(g(-\tau_1))$ is independent of t. Note that M is flat but is not $\mathbb{R}^n$ by (h2), and so the injectivity radius of q is finite. On the other hand, $g(t) = g(-\tau_1)$ because of (h1) and the fact that g satisfies the Ricci flow equation. Hence the injectivity radius of q of $g(t)$ is independent of t. This is impossible as $t \to \tau_1$ by (8.1). ∎

From these, it is easy to see that [24, Claim 3, §10] is also true. Now one can use the sequence $(M_i, g_i(t), P_i)$ to derive a contradiction to a log Sobolev inequality by arguing as in [24], see also [7, 14, 18]. This proves Theorem 8.1 by contradiction.

9 Singularity Formation and Longtime Existence

We now apply the pseudolocality result to describe where singularities to the Ricci flow can form under certain assumptions. More precisely, in Theorem 9.3 we prove that any finite time singularities of the Ricci flow (9.1) under Assumption 9.1 below must form within a compact set. In Theorem 9.5 we apply this result to complete nonnegatively curved Kähler manifolds and prove a long time existence result for the Kähler–Ricci flow.

Consider the Ricci flow

$$\begin{cases} \dfrac{\partial}{\partial t} g &= -2Rc \\ g(0) &= g \end{cases} \tag{9.1}$$

on a complete noncompact Riemannian manifold. Let us make the following assumption.

Assumption 9.1 Let (M, g) be a complete noncompact Riemannian manifold of dimension n such that

(i) $|Rm(x)| \to 0$ as $d(x, p) \to \infty$ for some fixed point p
(ii) the injectivity radius $inj(M, g)$ of (M, g) is bounded from below.

Lemma 9.2 *Let (M, g) be as in Assumption 9.1 and let $(M, g(t))$ be the corresponding maximal solution to the Ricci flow (9.1) on $M \times [0, T)$. For any $0 < \delta < 1$, there exists $r > 0$ with the following property. Given any $t' < T$ there exists $0 < d' < \infty$ such that for any $x \in M$ and $0 < t \le t'$ with $d_t(x, p) \ge d'$, we have*

$$Vol_t(\partial\Omega)^n \ge (1 - \delta) c_n Vol_t(\Omega)^{n-1}$$

for any $\Omega \subset B_t(x, r)$.

Proof Let $0 < \delta < 1$ be given and let $r_0 = \operatorname{inj}(M, g(0))/2$. By conditions (i) and (ii) in Assumption 9.1, we can find some $0 < d_0 < \infty$ such that for any x where $d_0(x, p) \ge d_0$, we have

$$Vol_0(\partial\Omega)^n \ge \left(1 - \tfrac{\delta}{2}\right) c_n Vol_0(\Omega)^{n-1} \tag{9.2}$$

for any $\Omega \subset B_0(x, r_0)$.

By [13, Theorem 18.2], given $t' < T$ and any $\eta > 0$ there exists some compact $S \subset M$ such that

$$|Rm(x,t)| \leq \eta \tag{9.3}$$

for all $(x, t) \in (M \setminus S) \times [0, t']$.

By choosing η sufficiently small we see from (9.1), (9.2), and (9.3) that we may choose some $d' > d_0$ such that for any x where $d_t(x, p) \geq d'$, we have

$$Vol_t(\partial\Omega)^n \geq (1-\delta)c_n Vol_t(\Omega)^{n-1}$$

for any $\Omega \subset B_t(x, \frac{r_0}{2})$. Thus $\frac{r_0}{2}$ satisfies the conclusion of the lemma. ■

Theorem 9.3 *Let (M, g) satisfy Assumption 9.1 and let $(M, g(t))$ be the corresponding maximal solution to the Ricci flow (9.1) on $M \times [0, T)$. Then either $T = \infty$ or there exists some compact set $S \subset M$ with the property that $|Rm(x, t)|$ is uniformly bounded on $(M \setminus S) \times [0, T)$.*

Proof Assume that $T < \infty$ and let $g(t)$ be a *maximal* solution to (9.1) on $M \times [0, T)$. Thus there exist sequences $x_i \in M$ and $t_i \to T$ such that $|Rm(x_i, t_i)| \to \infty$ as $i \to \infty$. We will show that there exists some compact $S \subset M$ such that every such sequence x_i must be contained inside S. S will then clearly satisfy the conclusion of the theorem.

Suppose there is a sequence (x_i, t_i) satisfying the above condition and $d_0(p, x_i) \to \infty$. Let δ, ϵ be as in Theorem 8.1. For such δ, let r be as in Lemma 9.2. By rescaling our solution $g(t)$ in both time and space, we may assume that $r = 1$ (without affecting δ). Now let $t' = T - \epsilon^2$, and choose d' as in Lemma 9.2. Then by [13, Theorem 18.2], we may assume d' sufficiently large so that $R(y, t') \geq= -1$ for all $y \in B_{t'}(x, 1)$ where $d_{t'}(x, p) \geq 0$. We may assume that $\epsilon > 0$ is small enough such that $t' > 0$.

Let $\eta_k \to 0$ and let $\tau_k = t' - \eta_k > 0$ and $g_k(t) = g(\tau_k + t)$. Then $g_k(t)$ is well defined on $[0, \epsilon^2]$. By [13, 26], we know that for each k the curvature tensor of $g_k(t)$ together with its derivatives are uniformly bounded in $[0, \epsilon^2]$. Let i_0 be large enough such that if $i \geq i_0$, then $d_t(x_i, p) \geq d'$ for all $0 < t \leq t'$.

By Theorem 8.1, we have $|Rm_k(x_i, t)| \leq t^{-1} + \epsilon^{-2}$ for all $i \geq i_0$ and $0 \leq t \leq \epsilon^2$. Here Rm_k is the curvature tensor for g_k. Now $Rm(x_i, t_i) = Rm_k(x_i, t_i - \tau_k)$. We have

$$t_i - \tau_k = t_i - t' + \eta_k = t_i - T + \epsilon^2 + \eta_k.$$

Hence for fixed $i \geq i_0$ such that $T - t_i \leq \epsilon^2$, we have $0 \leq t_i - \tau_k \leq \epsilon^2$ for k sufficiently large.

So

$$|Rm(x_i, t_i)| = |Rm_k(x_i, t_i - \tau_k)| \leq (t_i - \tau_k)^{-1} + \epsilon^{-2}.$$

Now letting $k \to \infty$ and then letting $i \to \infty$, we have $\limsup_{i\to\infty} |Rm(x_i, t_i)| \leq 2\epsilon^{-2}$. This contradicts our initial assumption, and thus completes the proof of the theorem. ■

Corollary 9.4 *Suppose $T < \infty$ in Theorem 9.3. Then $Rm(x, T) \to 0$ as $x \to \infty$ in the sense that, given any $\epsilon > 0$, we may choose a compact set S such that $|Rm(x, t)| \leq \epsilon$ for all $(x, t) \in S^c \times [0, T)$.*

Proof Assume the corollary is false. Thus there exits a spacetime sequence (x_k, t_k) such that $d_0(p, x_k) \to \infty$, $t_k \to T$ and $|Rm(x_k, t_k)| \geq C_1$ for some $C_1 > 0$.

1. Fix some small $\epsilon_1 > 0$ to be chosen later and let $s_k = t_k - \epsilon_1$. Then by Theorem 9.3 and [13, Theorems 18.2 and 13.1], we may assume the compact set $S \subset M$ from Theorem 9.3 was chosen sufficiently large so that for some $C_2 > 0$ we have $|\frac{d}{dt} Rm(x_k, t)| \leq C_2$ for all k sufficiently large and $t \in [0, t_k]$.

2. For k sufficiently large, $s_k \in [0, T - \epsilon_1]$. Thus by Theorem 18.2 in [13], given any $\epsilon_2 > 0$ we may assume $S \subset M$ was chosen sufficiently larger still so that $|Rm(x_k, s_k)| \leq \epsilon_2$ for k sufficiently large.

Thus by 1 and 2, for k sufficiently large we have that

$$|Rm(x_k, t_k)| \leq |Rm(x_k, s_k)| + C_2\epsilon_2 \leq \epsilon_2 + C_2\epsilon_1.$$

Note that ϵ_1, ϵ_2 were chosen independently of each other and that C_2 can be chosen independent of these. Thus for sufficiently small choices of ϵ_1 and ϵ_2, we arrive at a contradiction. This completes the proof by contradiction. ■

We now apply Theorem 9.3 to the case of nonnegatively curved Kähler manifolds.

Theorem 9.5 *Let (M, g) be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature, strictly positive at some point. Then if (M, g) satisfies Assumption 9.1, the Kähler–Ricci flow has a long time solution $g(t)$ on $M \times [0, \infty)$.*

Proof Assume the theorem is false and that $g(t)$ is a *maximal* solution on $M \times [0, T)$ for $T < \infty$. Let the set S be as in Theorem 9.3. Now define

$$F(x, t) = \log \frac{\det g(x, t)}{\det g(x, 0)}.$$

By [27], we know that $g(t)$ also has nonnegative holomorphic bisectional curvature and hence $F(x, t) \leq 0$. We will show that $F(x, t)$ is also uniformly bounded from below on $M \times [0, T)$. If this is true, then the curvature is also uniformly bounded on $M \times [0, T)$ by the argument in [27, §7], and it is then easy to show that the solution $g(t)$ can be extended beyond T by [5, 26]. This would contradict the maximality T, and would thus prove the theorem.

Since the curvature $|Rm(x, t)|$ is uniformly bounded in $(M \setminus S) \times [0, T)$ and since

$$-F(x, t) = \int_0^t R(x, \tau) d\tau,$$

where R is the scalar curvature, there exists C_1 such that

$$0 \leq -F(x, t) \leq C_1 \tag{9.4}$$

for all $(x, t) \in (M \setminus S) \times [0, T)$.

Next we want to prove that there exists C_2 such that

$$-F(x, t) \le C_2 \tag{9.5}$$

for all $(x, t) \in S \times [0, T)$.

Let $\tilde{M}$ be the universal cover of M. Then by [22], $\tilde{M} = \tilde{N} \times \tilde{L}$ holomorphically and isometrically, where $\tilde{N}$ is compact and $\tilde{L}$ satisfies

$$\frac{1}{\tilde{V}_{\tilde{o}}(r)} \int_{\tilde{B}_{\tilde{o}}(r)} \tilde{R} \le \frac{C}{1 + \tilde{r}} \tag{9.6}$$

where $\tilde{B}_{\tilde{o}}(r)$ is the geodesic ball in $\tilde{L}$ and $\tilde{V}_{\tilde{o}}(r)$ is its volume. [2] Also $\tilde{R}$ is the scalar curvature of $\tilde{L}$. Since $|Rm(x)| \to 0$ as $x \to \infty$ in M and the holomorphic bisectional curvature is positive at some point, we must have $\tilde{M} = \tilde{L}$.

Suppose there exist sequences $x_i \in S$ and $t_i \to T$ such that $F(x_i, t_i) \to -\infty$. By (9.4), we may assume that

$$m(t_i) := \min_M F(\cdot, t_i) = F(x_i, t_i)$$

provided i is sufficiently large. After we lift the Kähler–Ricci flow to $\tilde{M}$, there is a compact set $\tilde{S}$ in $\tilde{M}$ and $\tilde{x}_i \in \tilde{S}$ such that $F(\tilde{x}_i, t_i) \to -\infty$ and $\min_{\tilde{M}} F(\cdot, t_i) = F(\tilde{x}_i, t_i) = m(t_i)$. Here $F(\tilde{x}, t)$ is the logarithm of the ratio of volume elements for the flow in $\tilde{M}$.

Then by the proof of [23, Corollary 2.1] we have

$$-m(t_i) \le C_3 \int_0^{\sqrt{at_i(1-m(t_i))}} \frac{s}{1+s} ds \le C_3 \sqrt{at_i(1 - m(t_i))}$$

for some positive constants a, C_3, where we have used the fact that $\tilde{S}$ is compact and (9.6). From this we see that (9.5) is true, which together with (9.4) says that $F(x, t)$ is uniformly bounded on $M \times [0, T)$. But this is a contradiction, and thus our theorem must be true. ∎

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