ON THE THEORY OF THE GALILEAN SATELLITES OF JUPITER

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Abstract. In this communication the main equations for the variables: radius vector, longitude, P and Q (variables built from Laplace's perihelium first integral) are given in closed form. These equations are used for deriving the equations of a second-order theory. At this order, the equations for P and Q, are separated and they are integrodifferential linear equations. The equations for the radius vector and for the longitudes, give, after integration, perturbations which are purely trigonometric. The solution shows the features observed in the motion of Jupiter's Galilean satellites. The results are discussed, and extended to include the space variables.

1. Introduction

As it has been pointed out by Kovalevsky (1962) the tables of the four great satellites of Jupiter (Sampson, 1910) do not allow, nowadays, a precise prediction of the phenomena involving these bodies. The errors of such predictions are of the order of one minute, and, in many occasions they have gone up to several minutes.

Since the nominal time unity of Sampson's tables is the UT, it may be expected that a shift of the time scale be among the main sources of errors. Indeed, the analysis of all photographic observations made in the past 40 years, carried out by Rodrigues (1970) and by Ferraz-Mello and Paula (1973), has shown that this shift is increasing two times faster than the difference ET − UT. The best modern observations, made by D. Pascu at McCormick, allow the following estimate for the shift of the Sampson's tables time scale (ST), for the mean epoch 1968.2:

$$\mathrm{ST} - \mathrm{UT} = 1.0 \pm 0.9 \text{ min.}$$

The standard deviation is much larger than should be expected from the observations themselves. Indeed, after correcting the time scale, the standard deviation for the $(O - C)$ of the mutual distances is $0''.2$, while its expected value for the focal length of the telescope employed is $0''.1$. So, these deviations are almost entirely due to tables' errors.

If the evolution of precise measurements in the Solar System from radar astronomy, and the increasing need of better ephemeris for astrodynamics are considered, it is clear that a better theory will be necessary in the near future. So, efforts have been made, mainly at the Bureau des Longitudes (Paris), to derive a new theory for the Galilean satellites of Jupiter.

This theory must take into account the main features of the Galilean system:

(a) The ratio of the semimajor axis (0.2 to 0.6) and the masses of the satellites with respect to the primary (10^{-4}) are characteristics of a planetary problem with strong interactions.

Y. Kozai (ed.), The Stability of the Solar System and of Small Stellar Systems, 167–184. *All Rights Reserved.*

(b) The periods of the satellites are very short and prevent the use of the classical methods of planetary theory, as well as purely numerical theories.

(c) The standard deviations of the best observations already made (0"03), when translated in terms of the Jovicentric longitudes of the satellites, give rise to very high values (50", 34", 21" and 13", respectively).

These features allow us to characterize the problem of the motion of Jupiter's Galilean satellites as a problem of research of absolute orbits, i.e., orbits of low precision, but, which remain valid for very long time intervals. The first attempts to construct absolute planetary theories are due to H. Gyldén and, after him, to G. W. Hill (see Brouwer, 1959). The most important results obtained hereto are those by Brumberg (1970) and by Sagnier (1973a, b), the latter in intimate relation with Jupiter's satellites; in connection with these works the researches by Krasinsky (1968, 1969) providing a very strong mathematical tool for the integration of a certain kind of systems of linear differential equations with periodic coefficients and providing an existence theorem for quasi-periodic solutions of the first kind (*première sore*) in the planar N-body problem, should also be mentioned.

This paper deals with the problem of the construction of second-order absolute orbits for the Galilean system of satellites. The method is the same already used for deriving the equations of a first-order theory (Ferraz-Mello, 1966; Hagihara, 1972). The main ideas for deriving the second-order theory are those shortly described in Ferraz-Mello (1969a, b).

2. The Equations

Let a Jovicentric system of moving axes be considered. Following a suggestion by De Sitter (1918) the angular velocity of this frame is taken so that the mean motions of the three inner satellites are exactly commensurable. Such a choice is possible since the absolute mean motions of these satellites are such that

$$n_1 - 3n_2 + 2n_3 = 0. \tag{1}$$

If ν_1, ν_2 and ν_3 are the Eulerian mean motions of these satellites, it follows that

$$n_i = \nu_i + N, \tag{2}$$

where N is the angular velocity of rotation of the equatorial axes, and so

$$n_1 - 2n_2 = \nu_1 - 2\nu_2 - N = 0^\circ 739\,507\,42 \text{ days}^{-1},$$
$$n_2 - 2n_3 = \nu_2 - 2\nu_3 - N = 0^\circ 739\,507\,42 \text{ days}^{-1},$$

i.e.,

$$N = -0^\circ 739\,507\,42 \text{ days}^{-1}.$$

For the plane variables, let Hill's normalized variables

$$u_j = (x_j + iy_j)/a_j, \qquad s_j = (x_j - iy_j)/a_j \tag{3}$$

be introduced. The normalization factors a_j are the mean distances from the satellites to the planet. The heights over the fundamental plane are also normalized by this factor:

$$Z_j = z_j/a_j.$$

Let also a new independent variable,

$$\zeta = \exp i\nu_3 t, \tag{4}$$

and the operator

$$D = \zeta \, \mathrm{d}/\mathrm{d}\zeta, \tag{5}$$

be introduced.

In the computations it is wise to take into account that the motions will not depart too much from coplanar circular uniform motions, whose angular velocities are the observed mean motions and whose radii are the mean distances a_j. The zeroth-order solution for each satellite is given by

$$u_j^0 = \sigma_j \zeta^{g_j}, \qquad s_j^0 = \sigma_j^* \zeta^{-g_j}, \quad (\sigma_j \sigma_j^* = 1), \tag{6}$$

where

$$g_j = \nu_j/\nu_3, \tag{7}$$

and $\sigma_j = \exp i\theta_{0j}$ gives the position of the jth satellite at the time origin. Let then the variables U_j and S_j be introduced through

$$\begin{aligned} u_j &= \sigma_j \zeta^{g_j}(1 + U_j), \\ s_j &= \sigma_j^* \zeta^{-g_j}(1 + S_j). \end{aligned} \tag{8}$$

The equations of motion are, then,

$$\begin{aligned} (D + \kappa_j)^2 \, U_j + \kappa_j^2 &= \lambda_j (a_j/r_j)^3 \, (1 + U_j) + \mathscr{R}_j, \\ (D - \kappa_j)^2 \, S_j + \kappa_j^2 &= \lambda_j (a_j/r_j)^3 \, (1 + S_j) \mathscr{T}_j, \\ D^2 Z_j &= \lambda_j (a_j/r_j)^3 \, Z_j + \mathscr{V}_j, \end{aligned} \tag{9}$$

where

$$\lambda_j = G m_0 (1 + m_j)/\nu_3^2 a_j^3, \tag{10}$$

m_j are the masses of the satellites with respect to the mass of Jupiter (m_0), G is the constant of gravitation, r_j are the vector radii of the satellites,

$$\kappa_j = g_j + m, \tag{11}$$

and

$$m = N/\nu_3 = -0.01448391. \tag{12}$$

$\mathscr{R}_j$, $\mathscr{T}_j$ and $\mathscr{V}_j$ are the disturbing forces for these variables. If only the mutual interac-

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tions are considered, then

$$\mathscr{R}_j=\sum_{i\neq j}\frac{Gm_i\sigma_j^*\zeta^{-g_j}}{v_3^2}\left\{\frac{u_j-\alpha_{ij}u_i}{r_{ji}^3}+\frac{\alpha_{ij}u_i}{r_i^3}\right\},$$
$$\mathscr{T}_j=\sum_{i\neq j}\frac{Gm_i\sigma_j\zeta^{g_j}}{v_3^2}\left\{\frac{s_j-\alpha_{ij}s_i}{r_{ji}^3}+\frac{\alpha_{ij}s_i}{r_i^3}\right\},\tag{13}$$
$$\mathscr{V}_j=\sum_{i\neq j}\frac{Gm_i}{v_3^2}\left\{\frac{Z_j-\alpha_{ij}Z_i}{r_{ji}^3}+\frac{\alpha_{ij}Z_i}{r_i^3}\right\},$$

where r_{ji} are the mutual distances, and

$$\alpha_{ij}=a_i/a_j.\tag{14}$$

It must be observed that the planar equations are conjugated throughout the transformation,

$$U_j\to S_j,\qquad S_j\to U_j,\qquad t\to -t\quad(\zeta\to\zeta^{-1},D\to -D).$$

This property is very useful for checking the calculations throughout this work, and will be used in this paper in order to avoid duplicated derivations.

3. The Area Integral and Its Application

The integral of the areas in the two-body problem may be used to introduce a new pair of variables for each satellite, and these variables are intimately related to the proper oscillations perpendicular to the orbit. Let this integral be considered in the form

$$\mathbf{c}=\mathbf{r}\times\mathbf{v},\tag{15}$$

where $\mathbf{r}$ is the relative position of the second body and $\mathbf{v}$ its relative velocity in an inertial frame. The vector $\mathbf{c}$ is a constant vector directed along the positive normal to the orbital plane.

Let now $\mathbf{k}$ be the unit vector of the z-axis of the Eulerian frame defined in Section 2, and K_0 the projection of $\mathbf{c}$ along this axis. It follows that

$$K_0=(\mathbf{r}\times\mathbf{v})\cdot\mathbf{k},$$

or

$$K_0=(\mathbf{r}\times\mathbf{v}_e)\cdot\mathbf{k}+mv_3[\mathbf{r}\times(\mathbf{k}\times\mathbf{r})]\cdot\mathbf{k},$$

where

$$\mathbf{v}_e=\mathbf{v}-mv_3\mathbf{k}\times\mathbf{r}$$

is the velocity vector with respect to the Eulerian system of reference. With the normalized Hill's variables, it writes

$$K_0=v_3a^2(\kappa+\tfrac{1}{2}K_2),\tag{16}$$

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where

$$K_2=(1+S)\cdot DU-(1+U)\cdot DS+2\kappa(1+U)(1+S)-2\kappa. \tag{17}$$

We have now the following proposition: K_2 is a second-order quantity with respect to the eccentricity. Indeed, the above calculations were made in the frame of the problem of the two bodies where, to the first order, we must have the classical result $K_1=\kappa\nu_3 a^2$ $(=na^2)$.

The integral of the areas is then used to define two new variables. Let K and H be respectively defined by

$$\begin{aligned} K&=\frac{\sigma^*\zeta^{-\kappa}}{K_1}(c_x+ic_y),\\ H&=\frac{\sigma\zeta^{\kappa}}{K_1}(c_x-ic_y), \end{aligned} \tag{18}$$

where c_x and c_y are the projections of $\mathbf{c}$ over inertial axes in the fundamental plane of reference. Easy calculations lead to

$$K=\frac{ia^2\zeta^{-\kappa}}{K_1}\{-\dot{Z}\zeta^{\kappa}(1+U)+Z[\zeta^{\kappa}(1+U)]^{\cdot}\},$$

$$H=\frac{ia^2\zeta^{\kappa}}{K_1}\{-\dot{Z}\zeta^{-\kappa}(1+S)+Z[\zeta^{-\kappa}(1+S)]^{\cdot}\},$$

and, by introducing the operator D, to

$$\begin{aligned} \kappa K&=\quad(1+U)\cdot DZ-Z[DU+\kappa(1+U)],\\ \kappa H&=-(1+S)\cdot DZ+Z[DS-\kappa(1+S)]. \end{aligned} \tag{19}$$

The pair of variables introduced in this way will serve to describe the proper oscillations along the z-axis and it is close to the Poincaré's variables $I\exp-i(l-\Omega)$ and $I\exp i(l-\Omega)$. They will be introduced in the system as follows: We have for each satellite one equation like

$$D^2Z=\psi(Z).$$

We normalize this equation by introducing $W=DZ$. Then we have

$$DW=\psi(Z), \qquad DZ=W,$$

which are transformed, by introducing the relations

$$K=K(Z,W), \qquad H=H(Z,W),$$

into two equations for the new variables.

4. Laplace's First Integral and Its Application

For each satellite, a second couple of variables will be introduced through functional relations which allow us to transform the remaining equations to Weierstrass' normal form. Laplace's first integral for the two-body problem is the best suitable to suggest the functional relations to be introduced. This integral gives the invariance of the apsidal line and of the eccentricity in this problem. It writes

$$\mathbf{p} = -\mathbf{r}/r - (G\mu)^{-1}(\mathbf{r}\times\mathbf{v})\times\mathbf{v}, \tag{20}$$

where μ is the sum of the masses and $\mathbf{p}$ is a constant vector directed to the pericenter.

Decomposing $\mathbf{p}$ along a couple of inertial axes in the fundamental plane of reference, we write

$$P = (p_x + ip_y)\,\sigma^*\zeta^{-\kappa},$$
$$Q = (p_x - ip_y)\,\sigma\zeta^{\kappa},$$

then, to the third order in the orbital eccentricity,

$$\begin{aligned} P &= -\frac{a}{r}(1+U) + \frac{1}{\kappa}\left(1+\frac{K_2}{2\kappa}\right)[DU + \kappa(1+U)] - \frac{1}{\kappa}K\cdot DZ,\\ Q &= -\frac{a}{r}(1+S) - \frac{1}{\kappa}\left(1+\frac{K_2}{2\kappa}\right)[DS - \kappa(1+S)] + \frac{1}{\kappa}H\cdot DZ, \end{aligned} \tag{21}$$

where K, H and K_2 are that defined in Section 3. It should be emphasized that all computations in the derivation of P and Q are made in the frame of the two-body problem. Indeed, the aim of these derivations is only to give rise to the functional relations we want. So, for example, Kepler's third law: $G = \kappa^2 v_3^2 a^3$ may be used without any constraint.

In order to solve these equations with respect to DU and DS let them be used to derive a new equation for K_2 where DU and DS are replaced by P and Q. From Equations (21), it follows:

$$\begin{aligned} (1+S)P + (1+U)Q = &-2\frac{a}{r}(1+U)(1+S) +\\ &+\frac{1}{\kappa}\left(1+\frac{K_2}{2\kappa}\right)[(1+S)DU - (1+U)DS + 2\kappa(1+U)(1+S)] -\\ &-\frac{1}{\kappa}DZ[(1+S)K - (1+U)H] + 0(4\text{th}); \end{aligned}$$

and then, taking into account that

$$(1+S)\cdot DU - (1+U)\cdot DS + 2\kappa(1+U)(1+S) = 2\kappa + K_2,$$

and that K_2 is a second-order quantity in the problem of the two bodies, and solving with respect to K_2, we have

$$K_2 = -A + \tfrac{1}{2}DZ[(1+S)K - (1+U)H] + 0(4\text{th}), \tag{22}$$

where

$$A=\kappa\left[1-\frac{a}{r}(1+U)(1+S)\right]-\tfrac{1}{2}\kappa[(1+S)P+(1+U)Q]. \tag{23}$$

It follows, to the third order,

$$\begin{aligned} DU&=\kappa P+\kappa\left(\frac{a}{r}-1\right)(1+U)+\tfrac{1}{2}A(1+U)+W_2(1+U),\\ DS&=-\kappa Q-\kappa\left(\frac{a}{r}-1\right)(1+S)-\tfrac{1}{2}A(1+S)-W_2^*(1+S), \end{aligned} \tag{24}$$

where

$$\begin{aligned} W_2&=-\tfrac{1}{4}DZ[(1+S)K-(1+U)H]+(1+S)K\cdot DZ,\\ W_2^*&=-\tfrac{1}{4}DZ[(1+S)K-(1+U)H]-(1+U)H\cdot DZ. \end{aligned} \tag{25}$$

Equations (24) will be taken as defining our new parameters P and Q. They will be introduced in the system as follows: We have for each satellite one pair of equations like

$$D^2U=\psi_1(U,S), \qquad D^2S=\psi_2(U,S).$$

We normalize these equations by introducing the functional relations

$$DU=\varphi_1(U,S,P,Q), \qquad DS=\varphi_2(U,S,P,Q).$$

From their derivatives and the original equations, we have

$$DP=\varphi_3(U,S,P,Q), \qquad DQ=\varphi_4(U,S,P,Q).$$

And this completes the transformation of our system of equations in a system of first-order equations.

The reasons for the choice of the functional relations given by Equations (24) may be discussed. Since this choice is arbitrary it would be possible in this theory to use the same functional relations already used in the first-order theory. The new choice corresponds to have P and Q close to the Poincaré's variables $e\exp-i(l-\varpi)$ and $e\exp i(l-\varpi)$ to the second order in the elliptical parameters; this fact is of an utmost importance for it warrants linear equations for P and Q in the second-order theory. We would also ask about the possibility of introducing a third variable by taking the space component of p, which conjugates itself with Z in the same way as P and Q conjugate themselves with U and S. The difficulty for such a procedure, from an algebraic point of view, lies in the fact that the space component of $\mathbf{p}$ is a third-order quantity. On the other hand such possibility would not lead to easy interpretations of the variables as describing proper oscillations.

5. The Central Problem

We call central problem in the theory of the four great satellites of Jupiter the restricted problem in which the satellites and the planet lie in a fixed plane under the action of their mutual attractions, disregarding the effects arising from their shapes or from external bodies. This restricted problem has the main difficulties of the general problem and its solution shows the main features of the observed motions. Indeed, aside the characteristics already discussed in Section 1, some others may be considered:

(a) *The quasi-resonances.* The sidereal mean motions of the satellites are such that $n_1 - 2n_2$ and $n_2 - 2n_3$ are small and will give rise to small divisors in the integration step. For example, in Laplace's theory (Tisserand, 1896), the longitude of Io has the inequality

$$\delta v_1 = \frac{m_2 n_1 f(\alpha_{12})}{n_1 - 2n_2 + g(J_2)} \sin(2l_1 - 2l_2). \tag{26}$$

Since its period is close to the period of the satellite it is named induced equation of the centre, and its half amplitude is the forced eccentricity. For the four satellites we have respectively,

Proper eccentricity	Forced eccentricity
0.000 01	0.004 12
0.000 13	0.009 43
0.001 39	0.000 63
0.007 36	–

These values show that the first two satellites depart from uniform circular motions more owning to perturbations than they do owning to proper oscillations. In the choice of the criteria for defining the small quantities of the theory, this fact is determinant.

(b) *The proper oscillations.* In reason of the strong interactions we cannot consider each orbit as having its own free oscillation (equation of the centre). The strong interactions do not allow to take as intermediate solutions those arising from separated integrations. The intermediate orbit must arise from the integration of the system formed by the four pairs of variables P_i and Q_i, simultaneously. So we will have four proper oscillations which will be apparent in the orbit of each satellite. The free oscillations in the longitudes of the satellites will have the form

$$\delta v_j = 2 \sum_i M_{ji} \sin(l_j - \varpi_i). \tag{27}$$

The fact that the system oscillates as a whole leads to the necessity of having the P_i and Q_i close to the Poincaré's variables at least to the second order in the elliptical parameters for the sake of having linear equations for these quantities in the second-order theory.

(c) *The libration.* The Galilean resonance, which arises from

$$n_1 - 3n_2 + 2n_3 = 0, \tag{28}$$

will give rise to libration's inequalities in the longitudes of the first three satellites.

The best results indicate for their half amplitudes respectively 8″.7, 24″ and 2″.3. These values must be compared with those giving the standard deviations of the best observations (Section 1), and such comparison allows to disregard a deeper study of this phenomenon when deriving a theory for ephemeris purposes, notwithstanding its very high mathematical interest. The complete modern treatment of this phenomenon has been made by Sagnier (1973b), who succeeded in deriving formal quasi-periodic solutions of the second kind for the central problem including the Galilean libration.

6. The Equations of the Central Problem

The equations of the central problem are those given in Section 2, when restricted to the plane variables U_j and S_j and to the disturbing functions arising from the mutual interactions. The functional relations which are to be introduced are

$$\begin{aligned} DU_j &= \kappa_j P_j + \kappa_j(1/C_j - 1)(1+U_j) + \tfrac{1}{2}A_j(1+U_j), \\ DS_j &= -\kappa_j Q_j - \kappa_j(1/C_j - 1)(1+S_j) - \tfrac{1}{2}A_j(1+S_j), \end{aligned} \tag{29}$$

where

$$C_j = [(1+U_j)(1+S_j)]^{1/2}, \tag{30}$$

and

$$A_j = \kappa_j(1-C_j) - \tfrac{1}{2}\kappa_j[(1+S_j)P_j + (1+U_j)Q_j]. \tag{31}$$

The conjugacy of all equations, already mentioned in Section 1 is preserved; the equations are invariant with respect to the transformation

$$U_j \to S_j, \qquad S_j \to U_j, \qquad P_j \to Q_j, \qquad Q_j \to P_j, \qquad t \to -t. \tag{32}$$

If the technique of utilization of the functional relations already discussed in Section 4 is adopted, we have, after some appropriate differentiations and substitutions, the equations

$$\begin{aligned} DP_j + \kappa_j P_j &= \frac{\lambda_j - \kappa_j^2}{\kappa_j C_j^3}(1+U_j) + \frac{1}{\kappa_j}\mathscr{R}_j - \\ &\quad - L_j - \frac{1}{4\kappa_j}B_j(1+U_j) + \frac{1}{4\kappa_j}\chi_j(1+U_j), \\ DQ_j - \kappa_j Q_j &= -\frac{\lambda_j - \kappa_j^2}{\kappa_j C_j^3}(1+S_j) - \frac{1}{\kappa_j}\mathscr{T}_j + \\ &\quad + L_j^* - \frac{1}{4\kappa_j}B_j(1+S_j) + \frac{1}{4\kappa_j}\chi_j(1+S_j), \end{aligned} \tag{33}$$

where

$$\begin{aligned} L_j = \tfrac{1}{2}A_j P_j - \tfrac{1}{8}A_j(1+U_j)[Q_j(1+U_j) - P_j(1+S_j)] + \\ + \frac{A_j}{C_j^3}(C_j^2 - 1)(1+U_j) + \frac{1}{4\kappa_j}A_j^2(1+U_j). \end{aligned} \tag{34}$$

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L_j^* is its conjugate through the above defined transformation

$$B_j = (1 - \tfrac{1}{2}C_j^2)^{-1} \kappa_j [(1+S_j) L_j - (1+U_j) L_j^*], \tag{35}$$

and

$$\chi_j = (1 - \tfrac{1}{2}C_j^2)^{-1} [(1+S_j) \mathscr{R}_j - (1+U_j) \mathscr{T}_j]. \tag{36}$$

The new equations of the motion are the set formed by Equations (29) and (33) which are normalized, with respect to the variables U_j, S_j, P_j and Q_j.

We must notice that these equations are exact, that is, no approximation has been made in the course of their derivation. The fact that the functional relations (Equations (29)) are themselves approximate relations in the two-body problem, do not matter. Indeed, the two-body problem has been used only to suggest the form of Equations (29), which are exact as they define the P_j and Q_j.

7. First Integrals of the Functional Relations. Poisson Technique

When $P_j = Q_j = 0$, Equations (29) may be written

$$\begin{aligned} D(1+U_j) &= \mathscr{A}_j(1+U_j), \\ D(1+S_j) &= -\mathscr{A}_j(1+S_j), \end{aligned} \tag{37}$$

where

$$\mathscr{A}_j = \mathscr{A}_j(U_j, S_j) = \kappa_j(1/C_j - 1) + \tfrac{1}{2}\kappa_j - \tfrac{1}{2}\kappa_j C_j. \tag{38}$$

Two first integrals may be obtained. Firstly, from Equations (37), it follows:

$$(1+S_j)\cdot D(1+U_j) + (1+U_j)\cdot D(1+S_j) = 0,$$

and then

$$(1+U_j)(1+S_j) = C_j^2 = \text{const}. \tag{39}$$

On account of the meaning of U_j and S_j it is easily seen that this integral accounts for the circularity of the motion when $P_j = Q_j = 0$. From Equations (37) it follows, still, that

$$(1+S_j)\cdot D(1+U_j) - (1+U_j)\cdot D(1+S_j) = 2\mathscr{A}_j(1+U_j)(1+S_j),$$

or, if we put

$$\xi_j = (1+U_j)/(1+S_j), \tag{40}$$

that

$$D\xi_j/\xi_j = 2\mathscr{A}_j. \tag{41}$$

Let it be remarked that, for $P_j = Q_j = 0$, $\mathscr{A}_j$ is a constant. Indeed, from its definition, and from Equations (39), it follows

$$\mathscr{A}_j = -\tfrac{1}{2}\kappa_j + \kappa_j/C_j - \tfrac{1}{2}\kappa_j C_j.$$

So, the integration of Equation (41) may be easily performed and leads to

$$\log \xi_j = \log C'_j + 2\mathscr{A}_j \log \zeta,$$

or

$$\frac{1+U_j}{1+S_j}\,\zeta^{-2\mathscr{A}_j} = C'_j. \tag{42}$$

This integral is related to the uniformity of the motion when $P_j = Q_j = 0$.

These integrals may be extended to the general case ($P_j \neq 0$ and $Q_j \neq 0$), by means of Poisson's method for the variation of the first integrals (see Kurth, 1959). Let Equations (29) be written completely:

$$\begin{aligned} DU_j &= \mathscr{A}_j(1+U_j) + G_j, \\ DS_j &= -\mathscr{A}_j(1+S_j) - H_j, \end{aligned} \tag{43}$$

where G_j and H_j are the terms which vanish when $P_j = Q_j = 0$, and which are to be treated as perturbations. The variational equations of Poisson, for this system, write (Ferraz-Mello, 1966)

$$\begin{aligned} DC_j &= \frac{\partial C_j}{\partial U_j} G_j - \frac{\partial C_j}{\partial S_j} H_j, \\ DC'_j &= \frac{\partial C'_j}{\partial U_j} G_j - \frac{\partial C'_j}{\partial S_j} H_j. \end{aligned} \tag{44}$$

If the partial derivatives are computed and substituted, it follows, after some algebra:

$$\begin{aligned} DC_j &= \tfrac{1}{2}\kappa_j\left[P_j(C'_j)^{-1/2}\zeta^{-\mathscr{A}_j} - Q_j(C'_j)^{1/2}\zeta^{\mathscr{A}_j}\right], \\ DC'_j &= \mathscr{D}_j\left[P_j(C'_j)^{1/2}\zeta^{-\mathscr{A}_j} + Q_j(C'_j)^{3/2}\zeta^{\mathscr{A}_j}\right] - \\ &\quad - \kappa_j\mathscr{A}'_j\left[P_j(C'_j)^{1/2}\zeta^{-\mathscr{A}_j} - Q_j(C'_j)^{3/2}\zeta^{\mathscr{A}_j}\right]\log\zeta, \end{aligned} \tag{45}$$

where

$$\mathscr{A}'_j = \mathrm{d}\mathscr{A}_j/\mathrm{d}C_j = -\kappa_j(C_j^{-2} + \tfrac{1}{2}), \tag{46}$$

and

$$\mathscr{D}_j = \kappa_j C_j (C_j^{-2} - \tfrac{1}{2}). \tag{47}$$

It is easily seen that any iterative procedure of integration (the P_j and Q_j being assumed as known Fourier's series) will lead to Poisson's secular terms. This fact is well apparent when the equation for DC'_j is modified taking into account that $\mathscr{A}'_j DC_j = D\mathscr{A}_j$:

$$D\log C'_j = \mathscr{D}_j\left[P_j(C'_j)^{-1/2}\zeta^{-\mathscr{A}_j} + Q_j(C'_j)^{1/2}\zeta^{\mathscr{A}_j}\right] + 2\mathscr{A}_j - 2D(\mathscr{A}_j \log\zeta). \tag{48}$$

These Poisson terms are of the same kind as those arising in the formulation of

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Lagrange's equations of variation of the parameters. They may be avoided by making use of the Tisserand's transformation (Tisserand, 1868). Let the parameter

$$\Gamma_j = \log C'_j + 2\mathscr{A}_j \log \zeta, \tag{49}$$

be introduced instead of C'_j. Equations (45) become:

$$\begin{aligned} DC_j &= \tfrac{1}{2}\kappa_j [P_j/\gamma_j - Q_j\gamma_j], \\ D\Gamma_j &= 2\mathscr{A}_j + \mathscr{D}_j [P_j/\gamma_j + Q_j\gamma_j], \end{aligned} \tag{50}$$

where

$$\gamma_j^2 = \exp \Gamma_j = C'_j \zeta^{2\mathscr{A}_j}. \tag{51}$$

For these equations we can get a formal quasi-periodic solution provided that the Fourier's series in the right hand side are of zero average. These solutions are formal first integrals of the motion.

8. The Integration

In order to integrate these equations, successive approximations may be used. Let be remarked that the radius vector and the longitude of the satellites are given by

$$\begin{aligned} r_j &= a_j C_j, \\ \theta_j &= \theta_{0j} + g_j \nu_3 t - \Gamma_j/2i, \end{aligned} \tag{52}$$

so that the start solution may be

$$C_j = 1, \qquad \Gamma_j = 0 \quad (\gamma_j = 1). \tag{53}$$

We must observe that $\mathscr{A}_j$ is one order lesser than the other terms of the differential Equations (50). So, at each step, C_j must be computed before Γ_j, and it must be taken for getting the new value of $\mathscr{A}_j$ in the computation of Γ_j. The order-one solution is computed thereafter.

We have, first,

$$DC_j = \tfrac{1}{2}\kappa_j (P_j - Q_j),$$

and then,

$$C_j = 1 + \tfrac{1}{2}\kappa_j D^{-1}(P_j - Q_j). \tag{54}$$

The choice of the constant of integration is provided by the fact that at each approximation, if P_j and Q_j are given by Fourier series, then the mean value of C_j must be equal to one. This value of C_j allows to compute the first-order approximation for $\mathscr{A}_j$:

$$\mathscr{A}_j = -\tfrac{3}{4}\kappa_j^2 D^{-1}(P_j - Q_j). \tag{55}$$

Then we have

$$D\Gamma_j = \tfrac{1}{2}\kappa_j (P_j + Q_j) - \tfrac{3}{2}\kappa_j^2 D^{-1}(P_j - Q_j),$$

and

$$\Gamma_j = \tfrac{1}{2}\kappa_j D^{-1}(P_j + Q_j) - \tfrac{3}{2}\kappa_j^2 D^{-2}(P_j - Q_j). \tag{56}$$

Here, the arbitrary constant is chosen such that the mean value of Γ_j be equal to zero.

In both cases the operator D^{-1} has the meaning of the primitive function of the trigonometric function involved, in the ordinary sense, i.e., without integration constant.

The complete integration involves also Equations (33) where the new parameters C_j and Γ_j are to be introduced through

$$(1 + U_j) = C_j\gamma_j; \qquad (1 + S_j) = C_j/\gamma_j. \tag{57}$$

The integration procedure may be chosen among the usual techniques. Nevertheless, this choice must be made with some care. The technique should not generate Poisson's secular terms, and it must allow for an adequate study of the free oscillations (see Section 5). If we wish only a low-order solution, approximations like those above computed may be used for eliminating C_j and γ_j from the equations for P_j and Q_j. The resulting equations are integrodifferential equations.

For example, to get the second-order solutions, it is enough to take the first order approximations given by Equations (54) and (55) for C_j and Γ_j. The resulting equations are, to the second order,

$$\begin{aligned} DP_j + \kappa_j P_j = \\ = \frac{\lambda_j - \kappa_j^2}{\kappa_j}\,&[1 + \tfrac{1}{4}\kappa_j D^{-1}(P_j + Q_j) - \tfrac{3}{4}\kappa_j^2 D^{-2}(P_j - Q_j) - \kappa_j D^{-1}(P_j - Q_j)] + \\ &+ \frac{1}{\kappa_j}\mathscr{R}_j - A_j^I[\tfrac{7}{8}P_j - \tfrac{5}{8}Q_j + \tfrac{7}{8}\kappa_j D^{-1}(P_j - Q_j)] + \\ &+ \frac{1}{4\kappa_j}\chi_j[1 + \tfrac{1}{2}\kappa_j D^{-1}(P_j - Q_j) + \tfrac{1}{4}\kappa_j D^{-1}(P_j + Q_j) - \tfrac{3}{4}\kappa_j^2 D^{-2}(P_j - Q_j)], \end{aligned} \tag{58}$$

and their conjugates. In these equations A_j^I represent the first-order part of A_j. From Equations (33) we see that

$$DA_j^I = -\tfrac{1}{2}\chi_j,$$

and then

$$A_j^I = -\tfrac{1}{2}D^{-1}\chi_j, \tag{59}$$

where the constant of integration is taken as zero, since $A_j^I = 0$ for the undisturbed motion.

For the integration Krasinsky's method may be used since the equations are linear. In this order of approximation, Krasinsky's method reduces to the derivation of a

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transformation of coordinates having the form

$$P_j = P_j^* + b_j + \sum_i [c_{ji}P_i^* + d_{ji}Q_i^* + e_{ji}D^{-1}P_i^* + f_{ji}D^{-1}Q_i^* + g_{ji}D^{-2}P_i^* + h_{ji}D^{-2}Q_i^*],$$
$$Q_j = Q_j^* + b_j' + \sum_i [c_{ji}'Q_i^* + d_{ji}'P_i^* - e_{ji}'D^{-1}Q_i^* - f_{ji}'D^{-1}P_i^* + g_{ji}'D^{-2}Q_i^* + h_{ji}'D^{-2}P_i^*], \quad (60)$$

where $b_j, \ldots, h_{ji}'$ are quasi-periodic functions of the time through ζ, all of them being of the first order. This transformation is built in such a way that the resulting system, which will remain linear, has constant coefficients. It has the same nature as Euler's differential equations. These equations are not homogeneous: they have a constant independent term.

It must be emphasized that a necessary condition for success in getting constant coefficient equations through Equations (60) as defined, is that $m \neq 0$. Indeed all variable coefficients and terms in Equations (58) depend on the time through $\zeta^{(I|g)}$, where $I \in \mathbf{Z}^4$, and $I_1 + I_2 + I_3 + I_4 = 0$; since g_1, g_2 and g_3 are integers and since $m \neq 0$, i.e., $\kappa_j \neq g_j$, the only possibility of having $(I\,|\,g)$ too close to κ_j is for higher-order resonances involving g_4.

As a final remark to this section let us mention that an exact set of equations could be found instead of Equations (58) if the variables $\mathscr{P}_j = P_j/\gamma_j$ and $\mathscr{Q}_j = Q_j\gamma_j$ were used instead of P_j and Q_j. Nevertheless, this change corresponds to modifying the functional relations introduced in Section 4 and lead to nonlinear second-order terms. So, the main characteristics of the result obtained above – the linearity of the second-order equations – would be lost.

9. The Constant Perturbation. The Libration

After the integration of Equations (58) is performed, the results are to be introduced in Equations (50). If C_j are replaced by the first-order quantities

$$\varepsilon_j = C_j - 1,$$

these equations write

$$D\varepsilon_j = \tfrac{1}{2}\kappa_j(P_j - Q_j) - \tfrac{1}{4}\kappa_j(P_j + Q_j)\,\Gamma_j,$$
$$D\Gamma_j = -\tfrac{3}{2}\kappa_j\varepsilon_j + \kappa_j\varepsilon_j^2 + \tfrac{1}{2}\kappa_j(P_j + Q_j) - \tfrac{3}{2}\kappa_j(P_j + Q_j)\,\varepsilon_j - \tfrac{1}{4}\kappa_j(P_j - Q_j)\,\Gamma_j. \quad (62)$$

P_j and Q_j are, now, known functions of ζ, involving the four circulatory frequencies g_j and the four oscillatory frequencies ϖ_j introduced by the integration of the constant coefficient equations for the P_j^* and Q_j^*.

Once again Krasinsky's method may be used in order to eliminate periodic coefficients. This aim is achieved notwithstanding the fact that this set of equations has the nonlinear term $\kappa_j\varepsilon_j^2$; indeed, the coefficient of ε_j^2 is constant and we are interested only in second-order equations. As in Section 8, Krasinsky's method reduces itself to the derivation of the transformation of coordinates,

$$\varepsilon_j = (1 + \beta_j)\,\varepsilon_j^* + \delta_j + \eta_j\Gamma_j^*,$$
$$\Gamma_j = (1 + \beta_j')\,\Gamma_j^* + \delta_j' + \eta_j'\varepsilon_j^*, \quad (63)$$

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where $\beta_j, \ldots, \eta'_j$ are quasi-periodic functions of first order. The transformation is such that the resulting system has constant coefficients:

$$\begin{aligned} D\varepsilon_j^* &= T_2' \Gamma_j^*, \\ D\Gamma_j^* &= T_0 - \tfrac{3}{2}\kappa_j \varepsilon_j^* + T_1 \varepsilon_j^* + \kappa_j \varepsilon_j^{*2}. \end{aligned} \tag{64}$$

The solution of this system to the second order shows a shift for Γ_j^* proportional to T_0. This fact contradicts the working hypothesis after which ν_j is already the observed mean motion and this phase shift may not exist. So, the constants T_0 must be made equal to zero, which allows to determinate the normalization factors a_j, i.e., the mean distances from the satellites to the planet (these constant terms gave the so-called constant perturbation). On the other hand, the first-order parts of T_1 and T_2' are proportional to the first-order part of T_0. Equations (64) reduce so, to

$$\begin{aligned} D\varepsilon_j^* &= 0, \\ D\Gamma_j^* &= -\tfrac{3}{2}\kappa_j \varepsilon_j^* + \kappa_j \varepsilon_j^{*2}, \end{aligned} \tag{65}$$

for which the trivial solution, $\varepsilon_j^* = 0$ and $\Gamma_j^* = 0$, is chosen.

This completes the integration of the central problem to the second-order, when libration is disregarded. Indeed in the calculations shown above there was no question to investigate the nature of the constant terms involved. They could be of an essential nature – i.e. true constant terms – or of an accidental nature since $g_1 - 3g_2 + 2g_3 = 0$.

If we suppose that the relation does not exactly hold,

$$g_1 - 3g_2 + 2g_3 = G,$$

Equations (64) must be slightly modified in order to avoid the integration of terms in ζ^G in the computation of the $\beta_j, \ldots, \eta'_j$ for $j = 1, 2, 3$. The resulting system is

$$\begin{aligned} D\varepsilon_j^* &= T_0' + T_2' \Gamma_j^*, \\ D\Gamma_j^* &= T_0 - \tfrac{3}{2}\kappa_j \varepsilon_j^* + T_1 \varepsilon_j^* + T_2 \Gamma_j^* + \kappa_j \varepsilon_j^{*2}, \end{aligned} \tag{66}$$

where T_0, T_1 and T_2' are the same constants as before, plus functions of $(\zeta^{kG} + \zeta^{-kG} - 2)$; T_0' and T_2 are functions of $(\zeta^{kG} - \zeta^{-kG})$. The constants may be eliminated in the same way as before. The discussion of the remaining system will be the subject of a separate paper.

10. The Complete Second-Order Theory

Let us first show that the equations for the space variables are, at the second order, completely independent of the solutions for the planar parameters. Let the technique discussed in Section 3 be adopted. From Equations (19) it follows:

$$\begin{aligned} \kappa_j(DK_j + \kappa_j K_j) &= \quad (1 + U_j) \cdot D^2 Z_j - Z_j[(D + \kappa_j)^2\, U_j + \kappa_j^2], \\ \kappa_j(DH_j - \kappa_j H_j) &= -(1 + S_j) \cdot D^2 Z_j + Z_j[(D - \kappa_j)^2\, S_j + \kappa_j^2]. \end{aligned} \tag{67}$$

When substituting Equations (9) into the right-hand sides of these equations, the

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Keplerian parts will disappear and it results,

$$\begin{aligned} \kappa_j(DK_j+\kappa_j K_j) &= \quad (1+U_j)\,\mathscr{V}_j - Z_j\,\mathscr{R}_j, \\ \kappa_j(DH_j-\kappa_j H_j) &= -(1+S_j)\,\mathscr{V}_j + Z_j\mathscr{T}_j, \end{aligned} \tag{68}$$

It is easily seen that the right-hand sides are of the second order. So the Z_j may be substituted by the first-order approximations,

$$Z_j = -\tfrac{1}{2}(K_j+H_j). \tag{69}$$

So, the second-order equations are

$$\begin{aligned} \kappa_j(DK_j+\kappa_j K_j) &= \quad \mathscr{V}_j + \tfrac{1}{2}(K_j+H_j)\,\mathscr{R}_j, \\ \kappa_j(DH_j-\kappa_j H_j) &= -\mathscr{V}_j - \tfrac{1}{2}(K_j+H_j)\,\mathscr{T}_j, \end{aligned} \tag{70}$$

where the u_i, s_i, r_i and r_{ji} in $\mathscr{R}_j$, $\mathscr{T}_j$ and $\mathscr{V}_j$ may be taken as for the circular zeroth-order approximation (Equations (6)). So Equations (70) are homogeneous linear with quasi-periodic coefficients, not involving the U_j, S_j, P_j and Q_j. Again Krasinsky's method may be used. The transformation

$$\begin{aligned} H_j &= (1+\varrho_j)\,H_j^* + \tau_j K_j^*, \\ K_j &= (1+\varrho_j')\,K_j^* + \tau_j' H_j^* \end{aligned} \tag{71}$$

may be derived in such a way that the coefficients in the equations for H_j^* and K_j^* are constants. The necessary condition to succeed in getting constant coefficient equations through Equations (71) is the same as for Equations (60) and it is fulfilled.

For the planar variables the same technique employed in solving the central problem is adopted. The functional relations (24) are adopted, but, for the sake of simplicity, all space terms are collected in W_{2j} and W_{2j}^*. So, A_j is taken as for the central problem (Equation (31)), and for W_{2j} and W_{2j}^* we take

$$\begin{aligned} W_{2j} &= \quad \tfrac{1}{4}(3K_j+H_j)\cdot DZ_j - \tfrac{1}{4}\kappa_j Z_j^2, \\ W_{2j}^* &= -\tfrac{1}{4}(3H_j+K_j)\cdot DZ_j - \tfrac{1}{4}\kappa_j Z_j^2. \end{aligned} \tag{72}$$

The functional relations then write

$$DU_j = \kappa_j P_j + \kappa_j(C_j^{-1}-1)(1+U_j) + \tfrac{1}{2}A_j(1+U_j) + W_{2j}(1+U_j),$$

and their conjugates. The resulting equations for the P_j and Q_j are the same as for the central problem (Equations (33)) and they are solved in the same way. The second-order space terms came through

$$W_{3j} = DW_{2j} + 2\kappa_j W_{2j} - \tfrac{3}{4}\kappa_j(W_{2j} - W_{2j}^*) + \tfrac{3}{2}\kappa_j^2 Z_j^2,$$

and their conjugates. But, it is easily seen, by using Equation (69), and the first-order relations $DK_j = -\kappa_j K_j$, $DH_j = \kappa_j H_j$, $D^2 Z_j = \kappa_j^2 Z_j$, and

$$DZ_j = \tfrac{1}{2}\kappa_j(K_j - H_j), \tag{74}$$

that W_{3j} is indeed a third-order quantity.

At last, the equations for ε_j and Γ_j must be considered. The new equations for C_j and C'_j are those given by Equations (45) to which we add the space contributions

$$\delta(DC_j) = \frac{\partial C_j}{\partial U_j} W_{2j}(1+U_j) - \frac{\partial C_j}{\partial S_j} W^*_{2j}(1+S_j),$$

$$\delta(DC'_j) = \frac{\partial C'_j}{\partial U_j} W_{2j}(1+U_j) - \frac{\partial C'_j}{\partial S_j} W^*_{2j}(1+S_j),$$

i.e., after some algebra,

$$\delta(DC_j) = \tfrac{1}{4}\kappa_j(K_j+H_j)(K_j-H_j),$$

$$\delta(DC') = \tfrac{1}{4}\kappa_j C'_j(K_j-H_j)^2 - \tfrac{1}{8}\kappa_j C'_j(K_j+H_j)^2 - \\ - \mathscr{A}'_j C'_j[\tfrac{1}{2}\kappa_j(K_j+H_j)(K_j-H_j)]\log\zeta,$$

and for Γ_j, defined by Equation (49), the new equation is Equation (50) with the additive space term,

$$\delta(D\Gamma_j) = \tfrac{1}{4}\kappa_j(K_j-H_j)^2 - \tfrac{1}{8}\kappa_j(K_j+H_j)^2.$$

The integration of these equations is made exactly as it has been discussed in Section 9.

11. Conclusion

Notwithstanding the fact that this theory has been derived with special regard to the problem of the motion of the Galilean satellites of Jupiter, it may be useful in the study of other problems of planetary kind, in which the motions are close to circularity and coplanarity, and, in which, quasi-resonances lead to strong perturbations. Nevertheless, usual resonant problems must be excluded: the Galilean resonance is a too particular kind of resonance and does not involve the same kind of difficulties as, e.g., Hecubian resonance (see Sagnier, 1973b).

The main characteristic of the theory is that it allows to keep the main frequencies fixed from the earlier stages, and so, to have a purely trigonometric solution. Also the distances are to be fixed from the earlier stages; but the observational data for distances are not so good, and these distances are to be modified after computing the constant perturbation (Section 9).

In practice the Laplace coefficients in the development of the disturbing function are to be taken numerically. The algebra of the series may be, then, performed, by using a computer. The work is made iteratively by getting better distances at each step. It must be remarked that this algebra is not too involved and does not require too powerful computers.

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DISCUSSION

K. Ziolkowski: Did you try to use your theory to other satellite systems or to planets?

S. Ferraz-Mello: Yes, I am trying to use my theory to an asteroid, Hestia.

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