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Bounded Hankel Products on the Bergman Space of the Polydisk

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Abstract. We consider the problem of determining for which square integrable functions f and g on the polydisk the densely defined Hankel product $H_f H_g^*$ is bounded on the Bergman space of the polydisk. Furthermore, we obtain similar results for the mixed Haplitz products $H_g T_{\bar{f}}$ and $T_f H_g^*$, where f and g are square integrable on the polydisk and f is analytic.

1 Introduction

Let D be the unit disk in the complex plane $\mathbb{C}$. For a fixed positive integer $n \geq 2$, the unit polydisk D^n is the cartesian product of n copies of D. The torus T^n is the cartesian product of n copies of T, where T is the unit circle, *i.e.*, the boundary of D. Observe that T^n is only a small part of the topological boundary of D^n. T^n is usual called the distinguished boundary of D^n. Let $L^p = L^p(D^n)$ denote the usual Lebesgue space with respect to the volume measure $V = V_n$ on D^n normalized so that $V_n(D^n) = 1$. The Bergman space A^2 is the space of holomorphic functions on D^n which are also in $L^2(D^n)$. For $\lambda \in D$, let φ_λ be the fractional linear transformation on D given by $\varphi_\lambda(z) = (\lambda - z)/(1 - \bar{\lambda} z)$. Each φ_λ is an automorphism on the disk, in fact, $\varphi_\lambda^{-1} = \varphi_\lambda$. For $w = (w_1, \dots, w_n) \in D^n$ the mapping φ_w on the polydisk D^n given by $\varphi_w(z) = (\varphi_{w_1}(z_1), \dots, \varphi_{w_n}(z_n))$ is an automorphism on D^n. The reproducing kernel in A^2 is given by

$$K_w(z) = \prod_{j=1}^{n} \frac{1}{(1 - \overline{w}_j z_j)^2},$$

for $z, w \in D^n$. If $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2, then $\langle h, K_w \rangle = h(w)$ for every $h \in A^2$ and $w \in D^n$. The orthogonal projection P of L^2 onto A^2 is given by

$$(Pg)(w) = \langle g, K_w \rangle = \int_{D^n} g(z) \prod_{j=1}^{n} \frac{1}{(1 - \bar{z}_j w_j)^2} dV(z),$$

for $g \in L^2$ and $w \in D^n$. Given $f \in L^\infty$, the Toeplitz operator T_f is defined on A^2 by $T_f h = P(fh)$. We have

$$(T_f h)(w) = \int_{D^n} f(z) h(z) \prod_{j=1}^{n} \frac{1}{(1 - \bar{z}_j w_j)^2} dV(z),$$

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for $h \in A^2$ and $w \in D^n$. Note that the above formula also makes sense if $f \in L^2$ and defines an analytic function on D^n.So, if $g \in A^2$, then we define $T_{\bar{g}}$ by the formula

$$(T_{\bar{g}}h)(w) = \int_{D^n} \overline{g(z)} h(z) \prod_{j=1}^{n} \frac{1}{(1-\bar{z}_j w_j)^2} dV(z),$$

for $h \in A^2$ and $w \in D^n$.

Next, we consider Hankel products. If f is bounded and $h \in A^2$, then the Hankel operator H_f is defined by the following formula:

$$\begin{aligned}(H_f h)(w) &= (I-P)(fh)(w) \\ &= \int_{D^n} \big(f(w) - f(z)\big) h(z) \prod_{j=1}^{n} \frac{1}{(1-\bar{z}_j w_j)^2} dV(z),\end{aligned}$$

for all $w \in D^n$. The latter formula is to be used to define H_f densely on A^2 if $f \in L^2$. If g is bounded and $u \in (A^2)^\perp$, then

$$H_g^* u(w) = \langle H_g^* u, K_w \rangle = \langle u, H_g K_w \rangle = \langle u, g K_w \rangle,$$

for all $w \in D^n$. Since K_w is bounded, the latter formula makes sense for all $g \in L^2$, and we use it to define the operator H_g^* densely on $(A^2)^\perp$. Note that the star no longer needs to be the adjoint (but would of course coincide with the adjoint in case the operator H_g is itself bounded).

By [1, Theorem 3.14], the set $C_c(D^n)$ of all continuous functions with compact support in D^n, is dense in $L^2(D^n)$, so certainly $C_c(D^n) \cap (A^2)^\perp$, the set of compactly supported continuous functions in $(A^2)^\perp$, is dense in $(A^2)^\perp$. If $f, g \in L^2$ and $u \in C_c(D^n) \cap (A^2)^\perp$, then $H_g^* u$ is bounded and the meaning of $H_f H_g^* u$ is clear: it is the function $H_f(H_g^* u)$. This defines the Hankel product on a dense subset of $(A^2)^\perp$, namely $C_c(D^n) \cap (A^2)^\perp$.

The mixed Haplitz operators are defined as follows. For $f \in A^2$, $g \in L^2$, and $u \in C_c(D^n) \cap (A^2)^\perp$, $T_f(H_g^* u)$ is the analytic function $f(H_g^* u)$. If $h \in H^\infty$, then $T_{\bar{g}} h \in A^2$, and we define $H_f T_{\bar{g}} h$ to be the function $H_f(T_{\bar{g}} h)$.

The general problem that we are interested in is the following: for which $f, g \in L^2(D^n)$ is the operator $H_f H_g^*$ bounded on $(A^2)^\perp$?

When $n = 1$, K. Stroethoff and D. Zheng [2] gave a necessary condition for boundedness of the Hankel product $H_f H_g^*$ and proved that this necessary condition is very close to being sufficient. In this paper we extend Stroethoff and Zheng's results on the unit disk to higher dimensional polydisks. While our method is partially adapted from [3], a substantial amount of extra work is necessary in the setting of higher dimensional polydisks.

2 Preliminaries

Suppose f and g are in L^2. Consider the operator $f \otimes g$ on L^2 by

$$(f \otimes g)h = \langle h, g \rangle f,$$

for $h \in L^2$. It is easily proved that $f \otimes g$ is bounded on L^2 with norm equal to $\|f \otimes g\| = \|f\|_2\|g\|_2$. If T and S are bounded linear operators, then $T(f \otimes g)S^* = (Tf) \otimes (Sg)$.

Using the reproducing property , we have

$$\|K_w\|_2^2 = \langle K_w, K_w\rangle = K_w(w) = \prod_{j=1}^{n} \frac{1}{(1-|w_j|^2)^2}.$$

The functions

$$k_w(z) = \prod_{j=1}^{n} \frac{1-|w_j|^2}{(1-\bar{w}_j z_j)^2}$$

are the normalized reproducing kernels for A^2 .

The real Jacobian for the change of variable $\zeta = \varphi_w(z)$ is equal to $|k_w(z)|^2$, thus we have the change of variable formula

$$\int_{D^n} f(\varphi_w(z))dV(z) = \int_{D^n} f(z)|k_w(z)|^2 dV(z), \tag{2.1}$$

where f is an integrable function on D^n.

For $w \in D^n$, the operator U_w on L^2 is defined by $U_w f = (f \circ \varphi_w)k_w$. It is easy to see that U_w is a unitary operator that commutes with the Bergman projection. In particular, $T_f U_w = U_w T_{f\circ\varphi_w}$.

Under the decomposition $L^2 = A^2 \oplus (A^2)^\perp$, for $f \in L^\infty$, the multiplication operator M_f is represented as

$$M_f = \begin{bmatrix} T_f & H_{\bar{f}}^* \\ H_f & S_f \end{bmatrix}.$$

The operator S_f is the operator on $(A^2)^\perp$; we call S_f the dual Toeplitz operator with symbol f. Although these operators differ in many ways from Toeplitz operators, they do have the some of the same basic algebraic properties. We have $S_f^* = S_{\overline{f}}$ and $S_{\alpha f+\beta g} = \alpha S_f + \beta S_g$, for $f, g \in L^\infty$, and $\alpha, \beta \in \mathbb{C}$. Dual Toeplitz operators are studied in [4] and [6]. The identity $M_{fg} = M_f M_g$ implies the following basic algebraic relations between these operators:

$$T_{fg} = T_f T_g + H_{\bar{f}}^* H_g, \tag{2.2}$$

$$S_{fg} = S_f S_g + H_f H_{\bar{g}}^*, \tag{2.3}$$

$$H_{fg} = H_f T_g + S_f H_g. \tag{2.4}$$

Suppose $\varphi \in H^\infty$, and $\psi \in L^\infty$. If we take $f = \varphi$ and $g = \psi$ in (2.4) we get $H_{\varphi\psi} = S_\varphi H_\psi$, since $H_\varphi = 0$; on the other hand, taking $f = \psi$ and $g = \varphi$ in (2.4) gives $H_{\psi\varphi} = H_\psi T_\varphi$. Thus, if $\varphi \in H^\infty$, and $\psi \in L^\infty$, then

$$H_\psi T_\varphi = S_\varphi H_\psi, \tag{2.5}$$

and, by taking adjoints,

$$T_{\bar{\varphi}} H_{\psi}^{*} = H_{\psi}^{*} S_{\bar{\varphi}}. \tag{2.6}$$

It is easily to prove that identities (2.5) and (2.6) also hold if $\varphi \in H^{\infty}$, and $\psi \in L^2$.

In the following we write Q for the integral operator defined by

$$Q[u](w) = \int_{D^n} u(z) \prod_{j=1}^{n} \frac{1}{|1 - \bar{w}_j z_j|^2} dV(z),$$

for $u \in L^1$. The integral operator is L^p-bounded for $1 < p < \infty$. (This can be proved similarly to [7, Theorem 4.2.3] by considering the test function $\prod_{j=1}^{n}(1 - |z_j|^2)^{-1/(pq)}$.)

Let dA denote Lebesgue area measure on the unit disk D, normalized so that the measure of D equals 1. For a nonempty subset $\beta = \{\beta_1, \dots, \beta_m\}$ of $\{1, \dots, n\}$ with $\beta_1 < \cdots < \beta_m$, let μ_β be the measure on D^n defined by

$$d\mu_\beta(z) = \frac{3^{n-m}}{6^m}(1 - |z_1|^2)^2 \cdots (1 - |z_n|^2)^2 \prod_{j \in \beta}(5 - 2|z_j|^2) dA(z_1) \cdots dA(z_n),$$

for $z = (z_1, \dots, z_n)$, where m is the cardinality of α, and let $D^\beta h = D_{\beta_1} \cdots D_{\beta_m} h$, where $D_j h(z) = \partial h / \partial z_j$. Define $D^{\varnothing} h = h$. Note that

$$d\mu_{\varnothing}(z) = 3^n (1 - |z_1|^2)^2 \cdots (1 - |z_n|^2)^2 dA(z_1) \cdots dA(z_n)$$

and

$$d\mu_\beta(z) \le 3^n (1 - |z_1|^2)^2 \cdots (1 - |z_n|^2)^2 dA(z_1) \cdots dA(z_n)$$

for all subsets β of $\{1, \dots, n\}$.

We have Lemma 2.1 and Lemma 2.2 proved in [3].

Lemma 2.1 *Let $\varepsilon > 0$, $f \in A^2$ and $h \in H^{\infty}(D^n)$. If $\beta = \{\beta_1, \dots, \beta_m\}$ is a nonempty subset of $\{1, \dots, n\}$ with $\beta_1 < \cdots < \beta_m$, then*

(a) $|(T_{\bar{f}} h)(w)| \le \prod_{j=1}^{n} \frac{1}{1 - |w_j|^2} \|f \circ \varphi_w\|_2 \|h\|_2$, $w \in D^n$;

(b) $|D^\beta (T_{\bar{f}} h)(w)| \le 2^{2n} \prod_{j=1}^{n} \frac{1}{1 - |w_j|^2} \|f \circ \varphi_w\|_{2+\varepsilon} Q[|h|^\delta](w)^{1/\delta}$, $w \in D^n$, *where* $\delta = (2 + \varepsilon)/(1 + \varepsilon)$.

Lemma 2.2 *For $f, g \in A^2$ we have*

$$\int_{D^n} f(z)\overline{g(z)} dV(z) = \sum_{\beta} \int_{D^n} D^\beta f(z) \overline{D^\beta g(z)} d\mu_\beta(z),$$

where β runs over all subsets of $\{1, \dots, n\}$.

Lemma 2.3 *Let $\varepsilon > 0$, $u \in (A^2)^{\perp}$, and $f \in L^2$. If $\beta = \{\beta_1, \dots, \beta_m\}$ is a nonempty subset of $\{1, \dots, n\}$ with $\beta_1 < \cdots < \beta_m$, then*

(a) $|(H_f^* u)(w)| \le \prod_{j=1}^n \frac{1}{1-|w_j|^2} \|f\circ\varphi_w - P(f\circ\varphi_w)\|_2 \|u\|_2, w \in D^n$;

(b) $|D^\beta (H_f^* u)(w)| \le 2^{2n} \prod_{j=1}^n \frac{1}{1-|w_j|^2} \|f\circ\varphi_w - P(f\circ\varphi_w)\|_{2+\varepsilon} Q[|u|^\delta](w)^{1/\delta}, w \in D^n$, *where* $\delta = (2+\epsilon)/(1+\epsilon)$.

Proof (a) By [5, Proposition 1], $H_f k_w = (f - P(f\circ\varphi_w)\circ\varphi_w)k_w$, we have

$$H_f^* u(w) = \prod_{j=1}^n \frac{1}{1-|w_j|^2}\langle u, H_f k_w\rangle = \prod_{j=1}^n \frac{1}{1-|w_j|^2}\langle u, (f - P(f\circ\varphi_w)\circ\varphi_w)k_w\rangle.$$

By change of variable formula (2.1) we obtain

$$\|(f - P(f\circ\varphi_w)\circ\varphi_w)k_w\|_2 = \|f\circ\varphi_w - P(f\circ\varphi_w)\|_2.$$

Applying the Cauchy–Schwarz inequality we get

$$|\langle u, (f - P(f\circ\varphi_w)\circ\varphi_w)k_w\rangle| \le \|u\|_2 \|f\circ\varphi_w - P(f\circ\varphi_w)\|_2.$$

(b) We will first prove the estimate for $\beta = \{1,\dots,n\}$. For $u \in (A^2)^\perp$, we have

$$\begin{aligned}(H_f^* u)(w) &= \langle H_f^* u, K_w\rangle = \langle u, H_f K_w\rangle = \langle u, fK_w\rangle \\ &= \int_{D^n} u(z)\overline{f(z)} \prod_{j=1}^n \frac{1}{(1-\bar z_j w_j)^2} dV(z).\end{aligned}$$

Thus

$$\frac{\partial^n (H_f^* u)(w)}{\partial w_1 \dots \partial w_n} = 2^n \int_{D^n} u(z)\overline{f(z)} \prod_{j=1}^n \frac{\bar z_j}{(1-\bar z_j w_j)^3} dV(z).$$

Let $G_w = P(f\circ\varphi_w)\circ\varphi_w$, then the function $z \to G_w(z)\prod_{j=1}^n \frac{z_j}{(1-\overline{w}_j z_j)^3}$ is in A^2, and since $u \in (A^2)^\perp$ we have

$$\int_{D^n} u(z)\overline{G_w(z)} \prod_{j=1}^n \frac{\bar z_j}{(1-\bar z_j w_j)^3} dV(z) = 0.$$

Thus

$$\frac{\partial^n (H_f^* u)(w)}{\partial w_1 \cdots \partial w_n} = 2^n \int_{D^n} u(z)\overline{(f(z) - G_w(z))} \prod_{j=1}^n \frac{\bar z_j}{(1-\bar z_j w_j)^3} dV(z).$$

Let $\varepsilon > 0$, applying Hölder's inequality we get

$$\begin{aligned}\Big|\frac{\partial^n (H_f^* u)(w)}{\partial w_1 \cdots \partial w_n}\Big| &\le 2^n \int_{D^n} |f(z) - G_w(z)||u(z)| \prod_{j=1}^n \frac{|1 - w_j \bar z_j|}{|1 - w_j \bar z_j|^4} dV(z) \\ &\le 2^n \Big[\int_{D^n} \frac{|f(z) - G_w(z)|^{2+\varepsilon}}{\prod_{j=1}^n |1 - w_j \bar z_j|^4} dV(z)\Big]^{1/(2+\varepsilon)} \Big[\int_{D^n} \frac{|u(z)|^\delta}{\prod_{j=1}^n |1 - w_j \bar z_j|^{4-\delta}} dV(z)\Big]^{1/\delta} \\ &= 2^n \frac{\|f\circ\varphi_w - P(f\circ\varphi_w)\|_{2+\varepsilon}}{\prod_{j=1}^n (1 - |w_j|^2)} \\ &\quad \times \Big(\int_{D^n} |u(z)|^\delta \prod_{j=1}^n \frac{(1 - |w_j|^2)^{\varepsilon/(1+\varepsilon)}}{|1 - w_j \bar z_j|^2 |1 - w_j \bar z_j|^{\varepsilon/(1+\varepsilon)}} dV(z)\Big)^{1/\delta}.\end{aligned}$$

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For each j we have $\frac{(1-|w_j|^2)}{|1-w_j\bar{z}_j|} \le 2$, so

$$\Big(\prod_{j=1}^{n} \frac{(1-|w_j|^2)}{|1-w_j\bar{z}_j|}\Big)^{\varepsilon/(1+\varepsilon)} \le (2^n)^{\varepsilon/(1+\varepsilon)} \le 2^n.$$

Hence we have

$$\Big|\frac{\partial^n (H_f^* u)(w)}{\partial w_1\cdots\partial w_n}\Big| \le 2^{2n} \frac{\|f\circ\varphi_w - P(f\circ\varphi_w)\|_{2+\varepsilon}}{\prod_{j=1}^{n}(1-|w_j|^2)} \Big(\int_{D^n} \frac{|u(z)|^\delta}{\prod_{j=1}^{n}|1-w_j\bar{z}_j|^2} dV(z)\Big)^{1/\delta},$$

as desired.

Now we consider that $\beta = (\beta_1,\dots,\beta_m)$, where $\beta_1 < \cdots < \beta_m$. For $w \in D^n$, $u \in (A^2)^\perp$ and $f \in L^2$ we have

$$\begin{aligned} D^\beta(H_f^* u)(w) &= 2^m \int_{D^n} \prod_{l\in\beta} \frac{\bar{z}_l}{1-w_l\bar{z}_l} \overline{f(z)} u(z) \prod_{j=1}^{n} \frac{1}{(1-w_j\bar{z}_j)^2} dV(z) \\ &= 2^m \int_{D^n} \prod_{l\in\beta} \frac{\bar{z}_l}{(1-w_l\bar{z}_l)} \overline{(f(z)-G_w(z))} u(z) \prod_{j=1}^{n} \frac{1}{(1-w_j\bar{z}_j)^2} dV(z). \end{aligned}$$

Since

$$\prod_{l\in\beta} \frac{1}{|1-w_l\bar{z}_l|} = \prod_{i=1}^{n} \frac{1}{|1-w_i\bar{z}_i|} \times \prod_{j\in\{1,\cdots,n\}\setminus\beta} |1-w_j\bar{z}_j| \le \frac{2^{n-m}}{\prod_{j=1}^{n}|1-w_j\bar{z}_j|},$$

we get

$$\begin{aligned} |D^\beta(H_f^* u)(w)| &\le 2^m \int_{D^n} \prod_{l\in\beta} \Big|\frac{\bar{z}_l}{|1-w_l\bar{z}_l|}\Big| \Big|\overline{(f(z)-G_w(z))}\Big| |u(z)| \prod_{j=1}^{n} \frac{1}{|1-w_j\bar{z}_j|^2} dV(z) \\ &\le 2^n \int_{D^n} \Big|\overline{(f(z)-G_w(z))}\Big| |u(z)| \prod_{j=1}^{n} \frac{1}{|1-w_j\bar{z}_j|^3} dV(z), \end{aligned}$$

and the stated inequality follows from the proof of the first part of the lemma. ■

For any multi-index $\alpha = (\alpha_1,\dots,\alpha_n)$, where each α_k is a nonnegative integer, we will write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad C_{2,\alpha} = (-1)^{|\alpha|} \binom{2}{\alpha_1} \cdots \binom{2}{\alpha_n}.$$

We will also write $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$, for $z = (z_1,\dots,z_n) \in D^n$.

Lemma 2.4 *On A^2, we have*

$$k_w \otimes k_w = \sum_{|\alpha|=0}^{2n} C_{2,\alpha} T_{\varphi_w^\alpha} T_{\overline{\varphi_w^\alpha}}$$

for all $w \in D^n$.

Proof For $f \in A^2$, by the mean value property, we have

$$f(0) = (1 \otimes 1)f = \int_{D^n} f(w)dV(w) = \int_{D^n} K_w(z)^{-1}K_w(z)f(w)dV(w).$$

Since

$$K_w(z)^{-1} = \prod_{i=1}^{n}(1 - \overline{w_i}z_i)^2 = \sum_{|\alpha|=0}^{2n} C_{2,\alpha}\overline{w}^{\alpha}z^{\alpha}$$

and

$$T_{\overline{w}^{\alpha}} f(z) = \int_{D^n} \overline{w}^{\alpha}K_w(z)f(w)dV(w),$$

we have

$$f(0) = (1 \otimes 1)f = \sum_{|\alpha|=0}^{2n} C_{2,\alpha}T_{z^{\alpha}}T_{\overline{z}^{\alpha}}f.$$

It follows that

$$(1 \otimes 1) = \sum_{|\alpha|=0}^{2n} C_{2,\alpha}T_{z^{\alpha}}T_{\overline{z}^{\alpha}}.$$

Note that if $U_w 1 = k_w$, we obtain

$$k_w \otimes k_w = (U_w 1) \otimes (U_w 1) = U_w(1 \otimes 1)U_w = \sum_{|\alpha|=0}^{2n} C_{2,\alpha}T_{\varphi_w^{\alpha}}T_{\overline{\varphi}_w^{\alpha}}. \qquad \blacksquare$$

3 Bounded Hankel Products and Haplitz Products

In this section we give conditions for boundedness of Hankel products. The following result gives a necessary condition for the products $H_f H_g^*$ to be bounded.

Theorem 3.1 *Let f and g be in L^2. If $H_f H_g^*$ is bounded, then*

$$\sup_{w \in D^n} \|f \circ \varphi_w - P(f \circ \varphi_w)\|_2 \|g \circ \varphi_w - P(g \circ \varphi_w)\|_2 < \infty.$$

Proof Using the fact that $\varphi_w \in H^{\infty}$, we have $H_f T_{\varphi_w} = S_{\varphi_w} H_f$ and $T_{\overline{\varphi_w}} H_g^* = H_g^* S_{\overline{\varphi_w}}$, and by Lemma 2.4 we have

$$\begin{aligned} H_f(k_w \otimes k_w)H_g^* &= \sum_{|\alpha|=0}^{2n} C_{2,\alpha} H_f T_{\varphi_w{}^{\alpha}} T_{\overline{\varphi_w{}^{\alpha}}} H_g^* \\ &= \sum_{|\alpha|=0}^{2n} C_{2,\alpha} S_{\varphi_w{}^{\alpha}} (H_f H_g^*) S_{\overline{\varphi_w{}^{\alpha}}}. \end{aligned}$$

The estimate $\|S_{\varphi_w^\alpha}\| \leq 1$ implies that

$$\|H_f(k_w \otimes k_w)H_g^*\| \leq \Big(\sum_{|\alpha|=0}^{2n} C_{2,\alpha}\Big)\|H_fH_g^*\|.$$

Hence there exists a finite positive number N such that

$$\begin{aligned}\|f\circ\varphi_w - P(f\circ\varphi_w)\|_2\|g\circ\varphi_w - P(g\circ\varphi_w)\|_2 &= \|H_fk_w\|_2\|H_gk_w\|_2\\ &= \|(H_fk_w)\otimes H_g(k_w)\|\\ &= \|H_f(k_w\otimes k_w)H_g^*\|\\ &\leq N\|H_fH_g^*\|. \qquad \blacksquare\end{aligned}$$

We have not been able to prove the converse of the above theorem. We do however have the following result, which supports [2, Conjecture 8.2(i)].

Theorem 3.2 *Let f and g be in L^2. If there is a positive constant ε such that ,*

$$\sup_{w\in D^n}\|f\circ\varphi_w - P(f\circ\varphi_w)\|_{2+\varepsilon}\|g\circ\varphi_w - P(g\circ\varphi_w)\|_{2+\varepsilon} < \infty,$$

then the product $H_fH_g^*$ is bounded.

Proof Let $u, v \in C_c(D^n)\cap(A^2)^\perp$. It follows from the definitions of H_g^*u and H_f^*v and Fubini's Theorem that we have $\langle H_fH_g^*u, v\rangle = \langle H_g^*u, H_f^*v\rangle$. By Lemma 2.2, $\langle H_fH_g^*u, v\rangle = \langle H_g^*u, H_f^*v\rangle = \sum_\beta I_\beta$, where

$$I_\beta = \int_{D^n} D^\beta(H_g^*u)(z)\overline{D^\beta(H_f^*v)(z)}d\mu_\beta(z)$$

and β runs over all subsets of $\{1, 2, \ldots, n\}$.

We will estimate I_β for all β. It follows from Lemma 2.3(a) that

$$\begin{aligned}|(H_g^*u)(z)\overline{(H_f^*v)(z)}| \leq \prod_{j=1}^n \frac{1}{(1-|z_j|^2)^2}\|f\circ\varphi_z - P(f\circ\varphi_z)\|_2\|g\circ\varphi_z\\ - P(g\circ\varphi_z)\|_2\|u\|_2\|v\|_2,\end{aligned}$$

thus

$$\begin{aligned}|I_\varnothing| &\leq \int_{D^n}|(H_g^*u)(z)\overline{(H_f^*v)(z)}|d\mu_\varnothing(z)\\ &\leq 3^n\sup_{z\in D^n}\|f\circ\varphi_z - P(f\circ\varphi_z)\|_2\|g\circ\varphi_z - P(g\circ\varphi_z)\|_2\|u\|_2\|v\|_2.\end{aligned}$$

Using Lemma 2.3(b) we have

$$|D^{\beta}(H_g^* u)(z)\overline{D^{\beta}(H_f^* v)(z)}| \leq 4^{2n} \prod_{j=1}^{n} \frac{1}{(1-|z_j|^2)^2} \|f \circ \varphi_z - P(f \circ \varphi_z)\|_{2+\varepsilon}$$
$$\|g \circ \varphi_z - P(g \circ \varphi_z)\|_{2+\varepsilon} Q[|u|^{\delta}](z)^{1/\delta} Q[|v|^{\delta}](z)^{1/\delta}.$$

If

$$\sup_{w \in D^n} \|f \circ \varphi_w - P(f \circ \varphi_w)\|_{2+\varepsilon} \|g \circ \varphi_w - P(g \circ \varphi_w)\|_{2+\varepsilon} \leq M$$

for all $w \in D^n$, then the above inequality implies

$$\begin{aligned} |I_\beta| &\leq \int_{D^n} |D^{\beta}(H_g^* u)(z)\overline{D^{\beta}(H_f^* v)(z)}| d\mu_\beta(z) \\ &\leq 3^n 4^{2n} M \int_{D^n} Q[|u|^{\delta}](z)^{1/\delta} Q[|v|^{\delta}](z)^{1/\delta} dV(z). \end{aligned}$$

Since $p = 2/\delta > 1$ and the operator Q is L^p-bounded, there exists a constant C such that for all $h \in L^p$,

$$\int_{D^n} |(Qh)(z)|^p dV(z) \leq C^p \int_{D^n} |h(z)|^p dV(z).$$

In particular ,

$$\int_{D^n} Q[|u|^{\delta}]^p(z) dV(z) \leq C^p \|u\|_2^2,$$

and a similar inequality holds for the function v. By the Cauchy–Schwarz inequality,

$$\begin{aligned} &\int_{D^n} Q[|u|^{\delta}](z)^{1/\delta} Q[|v|^{\delta}](z)^{1/\delta} dV(z) \\ &\qquad \leq \Big(\int_{D^n} Q[|u|^{\delta}](z)^{2/\delta} dV(z) \Big)^{1/2} \Big(\int_{D^n} Q[|v|^{\delta}](z)^{2/\delta} dV(z) \Big)^{1/2} \\ &\qquad \leq \big(C^p \|u\|_2^2 \big)^{1/2} \big(C^p \|v\|_2^2 \big)^{1/2} = C^{2/\delta} \|u\|_2 \|v\|_2. \end{aligned}$$

Thus

$$|I_\beta| \leq \int_{D^n} |D^{\beta}(H_g^* u)(z)\overline{D^{\beta}(H_f^* v)(z)}| d\mu_\beta(z) \leq 3^n 4^{2n} M C^{2/\delta} \|u\|_2 \|v\|_2$$

for every subset β of $\{1, \dots, n\}$. We conclude that there exists a finite constant C' such that

$$|\langle H_f H_g^* u, v \rangle| = |\langle H_g^* u, H_f^* v \rangle| \leq C' \|u\|_2 \|v\|_2.$$

So we prove that the product $H_f H_g^*$ is bounded. ■

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Analogous to the necessary condition for boundedness of Hankel products, the following result gives a necessary condition for the boundedness of the mixed Haplitz products.

Theorem 3.3 *Let $f \in A^2$ and $g \in L^2$. If $T_f H_g^*$ or $H_g T_{\bar f}$ is bounded, then*

$$\sup_{w\in D^n} \|f \circ \varphi_w\|_2 \|g \circ \varphi_w - P(g \circ \varphi_w\|_2 < \infty.$$

Proof Using the fact that f is analytic and $\varphi_w \in H^\infty$, we have $T_f T_{\varphi_w} = T_{\varphi_w} T_f$ and $T_{\overline{\varphi_w}} H_g^* = H_g^* S_{\overline{\varphi_w}}$, and by Lemma 2.4 we have

$$\begin{aligned} T_f(k_w \otimes k_w)H_g^* &= \sum_{|\alpha|=0}^{2n} C_{2,\alpha} T_f T_{\varphi_w{}^\alpha} T_{\overline{\varphi_w{}^\alpha}} H_g^* \\ &= \sum_{|\alpha|=0}^{2n} C_{2,\alpha} T_{\varphi_w{}^\alpha} (T_f H_g^*) S_{\overline{\varphi_w{}^\alpha}}. \end{aligned}$$

The estimates $\|S_{\varphi_w^\alpha}\| \le 1$ and $\|T_{\varphi_w^\alpha}\| \le 1$ imply that

$$\|T_f(k_w \otimes k_w)H_g^*\| \le \Big(\sum_{|\alpha|=0}^{2n} C_{2,\alpha}\Big) \|T_f H_g^*\|.$$

Thus there exists a finite positive number N such that

$$\begin{aligned} \|f \circ \varphi_w\|_2 \|g \circ \varphi_w - P(g \circ \varphi_w\|_2 &= \|T_f k_w\|_2 \|H_g k_w\|_2 \\ &= \|(T_f k_w) \otimes H_g(k_w)\| \\ &= \|T_f(k_w \otimes k_w)H_g^*\| \le N \|T_f H_g^*\|. \end{aligned}$$

The second result can be proved similarly. ■

We have not been able to prove the converse of the above theorem, but we have the following result.

Theorem 3.4 *Let $f \in A^2$ and $g \in L^2$. If there is a positive constant $\varepsilon > 0$ such that:*

$$\sup_{w\in D^n} \|f \circ \varphi_w\|_{2+\varepsilon} \|g \circ \varphi_w - P(g \circ \varphi_w)\|_{2+\varepsilon} < \infty,$$

then the products $T_f H_g^*$ and $H_g T_{\bar f}$ are bounded.

Proof Let $u \in C_c(D^n) \cap (A^2)^\perp$ and $h \in H^\infty$. It follows from Lemmas 2.1 and 2.3 that

$$|(H_g^* u)(z)\overline{(T_{\bar f} h)(z)}| \le \prod_{j=1}^n \frac{1}{(1-|z_j|^2)^2} \|f \circ \varphi_w\|_2 \|g \circ \varphi_z - P(g \circ \varphi_z)\|_2 \|u\|_2 \|h\|_2,$$

thus

$$\int_{D^n} |(H_g^* u)(z)\overline{(T_{\bar f}h)(z)}| d\mu_{\varnothing}(z) \le 3^n \sup_{z\in D^n} \|f\circ\varphi_w\|_2 \|g\circ\varphi_z - P(g\circ\varphi_z)\|_2 \|u\|_2\|h\|_2.$$

Using Lemmas 2.1 and 2.3 again, we have

$$|D^\beta(H_g^* u)(z)\overline{D^\beta(T_{\bar f}h)(z)}| \le 4^{2n}\prod_{j=1}^n \frac{1}{(1-|z_j|^2)^2}\|f\circ\varphi_w\|_{2+\varepsilon}\|g\circ\varphi_z - P(g\circ\varphi_z)\|_{2+\varepsilon} Q[|u|^\delta](z)^{1/\delta} Q[|h|^\delta](z)^{1/\delta}.$$

If

$$\sup_{w\in D^n} \|f\circ\varphi_w\|_{2+\varepsilon}\|g\circ\varphi_w - P(g\circ\varphi_w)\|_{2+\varepsilon} \le M,$$

then the above inequality implies

$$\int_{D^n} |D^\beta(H_g^* u)(z)\overline{D^\beta(T_{\bar f}h)(z)}| d\mu_\beta(z) \le 3^n 4^{2n} M \int_{D^n} Q[|u|^\delta](z)^{1/\delta} Q[|h|^\delta](z)^{1/\delta} dV(z).$$

Analogous to the proof of Theorem 3.2, we have

$$\int_{D^n} |D^\beta(H_g^* u)(z)|\overline{D^\beta(T_{\bar f}h)(z)} d\mu_\beta(z) \le 3^n 4^{2n} M C^{2/\delta}\|u\|_2\|h\|_2,$$

for every subset β of $\{1,\dots,n\}$. Applying Lemma 2.2, we conclude that there is a finite constant N such that

$$|\langle T_f H_g^* u, h\rangle| = |\langle H_g^* u, T_{\bar f}h\rangle| \le N\|u\|_2\|h\|_2.$$

So we prove that the products $T_f H_g^*$ is bounded.

The second result can be proved similarly. ■

4 Compact Hankel Products and Haplitz Products

In this section we discuss conditions for compactness of Hankel products and Haplitz products. The following lemma gives necessary conditions for compactness of operators on A^2, operators on $(A^2)^\perp$, or operators between these spaces.

Lemma 4.1 *If $A\colon A^2 \to A^2$, $B\colon A^2 \to (A^2)^\perp$, $C\colon (A^2)^\perp \to A^2$, $D\colon (A^2)^\perp \to (A^2)^\perp$ are compact operators, then for each $1 \le j \le n$*

$$\|A - T_{\varphi_{w_j}} A T_{\overline{\varphi}_{w_j}}\| \to 0, \quad \|B - S_{\varphi_{w_j}} B T_{\overline{\varphi}_{w_j}}\| \to 0,$$
$$\|C - T_{\varphi_{w_j}} C S_{\overline{\varphi}_{w_j}}\| \to 0, \quad \|D - S_{\varphi_{w_j}} D S_{\overline{\varphi}_{w_j}}\| \to 0,$$

as $|w_j| \to 1^-$. *Thus*

$$\Big\| \sum_{|\alpha|=0}^{n} C_{2,\alpha} T_{\varphi_w^\alpha} A T_{\overline{\varphi}_w^\alpha} \Big\| \to 0, \quad \Big\| \sum_{|\alpha|=0}^{n} C_{2,\alpha} S_{\varphi_w^\alpha} B T_{\overline{\varphi}_w^\alpha} \Big\| \to 0,$$

$$\Big\| \sum_{|\alpha|=0}^{n} C_{2,\alpha} T_{\varphi_w^\alpha} C S_{\overline{\varphi}_w^\alpha} \Big\| \to 0, \quad \Big\| \sum_{|\alpha|=0}^{n} C_{2,\alpha} S_{\varphi_w^\alpha} D S_{\overline{\varphi}_w^\alpha} \Big\| \to 0,$$

as $w = (w_1, \dots, w_n) \to T^n$.

Proof If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $S\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a compact operator, then, since operators of finite rank are dense in the set of compact operators, given $\epsilon > 0$ there exist $f_1, \dots, f_n \in \mathcal{H}_2$ and $g_1, \dots, g_n \in \mathcal{H}_1$ so that

$$\|S - \sum_{i=1}^{n} f_i \otimes g_i\| < \epsilon.$$

Thus the above statements follow once we prove them for operators of rank one.

If $f \in L^2$ as $|w_j| \to 1^-$, then for every $z_j \in D$ we have

$$w_j - \varphi_{w_j}(z_j) = (1 - |w_j|^2) z_j / (1 - \bar{w}_j z_j) \to 0,$$

so by the Lebesgue Dominated Convergence Theorem, $\|w_j f - \varphi_{w_j} f\|_2 \to 0$ as $|w_j| \to 1^-$. It follows that $\|\zeta f - \varphi_{w_j} f\|_2 \to 0$ if $w_j \in D$ tends to $\zeta \in \partial D$.

If $f \in A^2$, we apply P to obtain $\|\zeta f - T_{\varphi_{w_j}} f\|_2 = \|\zeta f - P(\varphi_{w_j} f)\|_2 \to 0$, as w_j in D tends to $\zeta \in \partial D$. If $f, g \in A^2$, then writing

$$\begin{aligned} \|f \otimes g - T_{\varphi_{w_j}}(f \otimes g) T_{\overline{\varphi}_{w_j}}\| &= \|(\zeta f) \otimes (\zeta g) - (T_{\varphi_{w_j}} f) \otimes (T_{\varphi_{w_j}} g)\| \\ &\le \|(\zeta f - T_{\varphi_{w_j}} f) \otimes (\zeta g)\| + \|(T_{\varphi_{w_j}} f) \otimes (\zeta g - T_{\varphi_{w_j}} g)\| \\ &\le \|\zeta f - T_{\varphi_{w_j}} f\|_2 \|g\|_2 + \|f\|_2 \|\zeta g - T_{\varphi_{w_j}} g\|_2, \end{aligned}$$

we see that $\|f \otimes g - T_{\varphi_{w_j}}(f \otimes g) T_{\overline{\varphi}_{w_j}}\| \to 0$ as w_j in D tends to $\zeta \in \partial D$. This proves the statement for operator A.

Suppose $f \in (A^2)^\perp$, then $(I - P)(\zeta f) = \zeta f$, so that

$$\|\zeta f - S_{\varphi_{w_j}} f\|_2 = \|(I - P)(\zeta f - \varphi_{w_j} f)\|_2 \to 0,$$

as w_j in D tends to $\zeta \in \partial D$. If $f, g \in (A^2)^\perp$, then writing

$$\begin{aligned} \|f \otimes g - S_{\varphi_{w_j}}(f \otimes g) S_{\overline{\varphi}_{w_j}}\| &= \|(\zeta f) \otimes (\zeta g) - (S_{\varphi_{w_j}} f) \otimes (S_{\varphi_{w_j}} g)\| \\ &\le \|(\zeta f - S_{\varphi_{w_j}} f) \otimes (\zeta g)\| + \|(S_{\varphi_{w_j}} f) \otimes (\zeta g - S_{\varphi_{w_j}} g)\| \\ &\le \|\zeta f - S_{\varphi_{w_j}} f\|_2 \|g\|_2 + \|f\|_2 \|\zeta g - S_{\varphi_{w_j}} g\|_2, \end{aligned}$$

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we see that $\|f \otimes g - S_{\varphi_{w_j}}(f \otimes g)S_{\overline{\varphi}_{w_j}}\| \to 0$ as w_j in D tends to $\zeta \in \partial D$. This proves the statement for operator D.

If $f \in A^2$ and $g \in (A^2)^{\perp}$, and w_j in D tends to $\zeta \in \partial D$, then $\|\zeta f - T_{\varphi_{w_j}} f\|_2 \to 0$, and $\|\zeta g - S_{\varphi_{w_j}} g\|_2 \to 0$, imply that $\|f \otimes g - T_{\varphi_{w_j}} f \otimes g S_{\overline{\varphi}_{w_j}}\| \to 0$ as $|w_j| \to 1^-$. This proves the statement for operator C.

This statement for operator B is proved similarly.

Note that

$$\begin{aligned}
&\sum_{|\alpha|=0}^{n} C_{2,\alpha} T_{\varphi_w^\alpha} A T_{\overline{\varphi}_w^\alpha} \\
&\quad = \sum_{\alpha_1,\dots,\alpha_n=0}^{2} (-1)^{|\alpha|} \binom{2}{\alpha_1} \cdots \binom{2}{\alpha_n} T_{\varphi_{w_1}^{\alpha_1}} \cdots T_{\varphi_{w_n}^{\alpha_n}} A T_{\overline{\varphi}_{w_1}^{\alpha_1}} \cdots T_{\overline{\varphi}_{w_n}^{\alpha_n}} \\
&\quad = \sum_{\alpha_2,\cdots,\alpha_n=0}^{2} A_\alpha T_{\varphi_{w_2}^{\alpha_2}} \cdots T_{\varphi_{w_n}^{\alpha_n}} \big(A - 2T_{\varphi_{w_1}} A T_{\overline{\varphi}_{w_1}} + T_{\varphi_{w_1}^2} A T_{\overline{\varphi}_{w_1}^2}\big) T_{\overline{\varphi}_{w_2}^{\alpha_2}} \cdots T_{\overline{\varphi}_{w_n}^{\alpha_n}},
\end{aligned}$$

where $A_\alpha = (-1)^{\alpha_2+\cdots+\alpha_n} \binom{2}{\alpha_2} \cdots \binom{2}{\alpha_n}$.

Since

$$\begin{aligned}
\|A - 2T_{\varphi_{w_1}} A T_{\overline{\varphi}_{w_1}} + T_{\varphi_{w_1}^2} A T_{\overline{\varphi}_{w_1}^2}\| &= \|(A - T_{\varphi_{w_1}} A T_{\overline{\varphi}_{w_1}}) - T_{\varphi_{w_1}}(A - T_{\varphi_{w_1}} A T_{\overline{\varphi}_{w_1}}) T_{\overline{\varphi}_{w_1}}\| \\
&\le 2\|A - T_{\varphi_{w_1}} A T_{\overline{\varphi}_{w_1}}\| \to 0
\end{aligned}$$

as $|w_1| \to 1^-$, we get the desired result for the operator A.

The other statements are proved similarly. ■

Let $0 < s < 1$, we write $D_s = D^n \backslash sD^n$, where $sD^n = \{sz : z \in D^n\}$ is a compact subset of D^n. The following theorem provides support for [2, Conjecture 8.2(ii)].

Theorem 4.2 *Let f and g be in L^2. Then $H_f H_g^*$ is compact if and only if*

$$\lim_{w \to T^n} \|f \circ \varphi_w - P(f \circ \varphi_w)\|_2 \|g \circ \varphi_w - P(g \circ \varphi_w)\|_2 = 0.$$

Proof By Lemma 2.4,

$$\begin{aligned}
H_f(k_w \otimes k_w) H_g^* &= H_f\Big(\sum_{|\alpha|=0}^{2n} C_{2,\alpha} T_{\varphi_w^\alpha} T_{\overline{\varphi_w^\alpha}}\Big) H_g^* \\
&= \sum_{|\alpha|=0}^{2n} C_{2,\alpha} H_f\big(T_{\varphi_w^\alpha} T_{\overline{\varphi_w^\alpha}}\big) H_g^* \\
&= \sum_{|\alpha|=0}^{2n} C_{2,\alpha} S_{\varphi_w^\alpha}\big(H_f H_g^*\big) S_{\overline{\varphi_w^\alpha}}.
\end{aligned}$$

Note that

$$\begin{aligned}\|H_f(k_w\otimes k_w)H_g^*\| &= \|(H_fk_w)\otimes(H_gk_w)\| = \|H_fk_w\|_2\|H_gk_w\|_2\\ &= \|f\circ\varphi_w - P(f\circ\varphi_w)\|_2\|g\circ\varphi_w - P(g\circ\varphi_w)\|_2.\end{aligned}$$

Then if $H_fH_g^*$ is compact, by Lemma 4.1,

$$\Big\|\sum_{|\alpha|=0}^{n} C_{2,\alpha}S_{\varphi_w^\alpha}(H_fH_g^*)S_{\overline{\varphi_w^\alpha}}\Big\| \to 0,$$

as $w=(w_1,\dots,w_n)\to T^n$. We get the desired result.

Conversely, let $u,v\in C_c(D^n)\cap(A^2)^\perp$. As in the proof of Theorem 3.2, we have

$$\langle H_fH_g^*u,v\rangle = \langle H_g^*u, H_f^*v\rangle = \sum_\beta I^\beta,$$

where β runs over all subsets of $\{1,\dots,n\}$ and

$$I^\beta = \int_{D^n} D^\beta(H_g^*u)(z)\overline{D^\beta(H_f^*v)(z)}\,d\mu_\beta(z).$$

For $0<s<1$, we write $I^\beta = I^\beta_{s,1}+I^\beta_{s,2}$, where

$$I^\beta_{s,1} = \int_{D_s} D^\beta(H_g^*u)(z)\overline{D^\beta(H_f^*v)(z)}\,d\mu_\beta(z).$$

It is easy to see that there exist compact operators K_s^β on $(A^2)^\perp$ such that $\langle K_s^\beta u,v\rangle = I^\beta_{s,2}$. The operator

$$K^s = \sum_\beta K_s^\beta$$

is compact, and

$$\langle (H_fH_g^* - K^s)u,v\rangle = \sum_\beta I^\beta_{s,1}.$$

Using Lemma 2.3 to estimate each of the terms $I^\beta_{s,1}$, from the proof of Theorem 3.2 there exists a constant C' such that

$$\begin{aligned}|\langle (H_fH_g^*-K^s)u,v\rangle| \le C' \sup_{z\in D_s}&\|f\circ\varphi_z - P(f\circ\varphi_z)\|_{2+\varepsilon}\\ &\times\|g\circ\varphi_z - P(g\circ\varphi_z)\|_{2+\varepsilon}\|u\|_2\|v\|_2.\end{aligned}$$

Since P is $L^{2+2\varepsilon}$-bounded, there exists a constant C_ε such that

$$\|f\circ\varphi_z - P(f\circ\varphi_z)\|_{2+\varepsilon} \le C_\varepsilon\|f\|_\infty^{(1+\varepsilon)/(2+\varepsilon)}\|f\circ\varphi_z - P(f\circ\varphi_z)\|_2^{1/(2+\varepsilon)}.$$

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A similar inequality holds for $\|g \circ \varphi_z - P(g \circ \varphi_z)\|_{2+\varepsilon}$, thus there is a constant M such that

$$|\langle (H_f H_g^* - K^s)u, v\rangle| \le M \sup_{z\in D_s} \|f \circ \varphi_z - P(f \circ \varphi_z)\|_2^{1/(2+\varepsilon)} \times \|g \circ \varphi_z - P(g \circ \varphi_z)\|_2^{1/(2+\varepsilon)} \|u\|_2 \|v\|_2,$$

from which we conclude that

$$\|H_f H_g^* - K^s\| \le M \sup_{z\in D^n} \|f \circ \varphi_z - P(f \circ \varphi_z)\|_2^{1/(2+\varepsilon)} \times \|g \circ \varphi_z - P(g \circ \varphi_z)\|_2^{1/(2+\varepsilon)}.$$

Since as $s \to 1^-, w \in D_s$ tends to T^n, and by the assumption of the theorem, we conclude that as $s \to 1^-$, $K^s \to H_f H_g^*$ in operator norm. Hence we obtain that the operator $H_f H_g^*$ is compact. ∎

Analogous to Theorem 4.2 we have the following result for the mixed Haplitz products.

Theorem 4.3 *Let $f \in H^\infty$ and $g \in L^2$. Then $T_f H_g^*$ is compact if and only if $H_g T_{\bar f}$ is compact if and only if*

$$\lim_{w\to T^n} \|f \circ \varphi_w\|_2 \|g \circ \varphi_w - P(g \circ \varphi_w)\|_2 = 0.$$

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References

[1] W. Rudin, *Real and complex analysis.* McGraw-Hill Book Co., New York, 1966.
[2] K. Stroethoff and D. Zheng, *Products of Hankel and Toeplitz operators on the Bergman space.* J. Funct. Anal. **169**(1999), no. 1, 289–313.
[3] ———, *Bounded Toeplitz products on the Bergman space of the polydisk.* J. Math. Anal. Appl. **278**(2003), no. 1, 125–135.
[4] ———, *Algebraic and spectral properties of dual Toeplitz operators.* Trans. Amer. Math. Soc. **354**(2002), no. 6, 2495–2520.
[5] ———, *Toeplitz and Hankel operators on Bergman spaces.* Trans. Amer. Math. Soc. **329**(1992), no. 2, 773–794.
[6] Y. Lu, *Commuting dual Toeplitz operators with pluriharmonic symbols.* J. Math. Anal. Appl. **302**(2005), no. 1, 149–156.
[7] K. H. Zhu, *Operator theory in function spaces.* Monographs and Textbooks in Pure and Applied Mathematics 139, Marcel Dekker, New York, 1990.

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