REIDEMEISTER PROJECTIVE PLANES

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1. Introduction. By a Reidemeister plane we mean a projective plane having the property that every ternary ring coordinatizing it has associative addition. Finite Reidemeister planes have been investigated by Gleason (**2**), Lüneburg (**6**), and Kegel and Lüneburg (**4**). In the first paper, Gleason proved that if the order of the plane is a prime power, then it is Desarguesian. Lüneburg showed that this is still true if the order is not 60. In the third paper, this last restriction is removed. For infinite planes, the only result is the following theorem due to Pickert (**7**, p. 301).

THEOREM. *Let $\mathscr{P}$ be a Reidemeister plane. $\mathscr{P}$ is Desarguesian if it is coordinatized by a nearfield.*

We shall extend this result to the case of Moufang planes. Specifically, we prove that a Reidemeister plane is a Moufang plane if it is coordinatized by a Veblen-Wedderburn system satisfying

$$(1) \qquad (xy)(zx) = (x\cdot yz)x.$$

As a corollary, we have the result that a projective plane in which the small axial theorem of Pappus holds is a Moufang plane if and only if it is coordinatized by a Veblen-Wedderburn system satisfying (1).

For the definition of a projective plane and the fundamentals of the theory, the reader is referred to either Hall (**3**) or Pickert (**7**). We shall generally follow Pickert.

2. Preliminaries. In what follows we shall rely heavily on various coordinate systems of a projective plane. We shall use the coordinate scheme of Pickert (**7**). In a plane $\mathscr{P}$, let U, V, O, I be four points, no three of which are collinear. Let the pencil of lines through V, other than UV (the line through U and V), and the pencil of lines through U, other than UV, be labeled biuniquely by the symbols of a set R under the following restrictions: (i) the line γ through U and the line δ through V have the same symbol if and only if $\gamma \cap \delta$ lies on the line OI; (ii) the symbols of OV and IV are 0 and 1, respectively. If P is a point not on UV, it is labeled with the unique ordered

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pair (x, y) of elements in R if PV and PU have the symbols x and y, respectively. If $P \in UV$, $P \neq V$, it is labeled (m) if $OP \cap IV = (1, m)$. Finally, V is given the label (∞); note, however, that $\infty \notin R$.

On R we define a ternary operation F as follows: $F(m, a, b) = y$ if and only if $(a, y) = (0, b)(m) \cap (a, 0)V$. The following theorem is true (**7**, pp. 35–36).

THEOREM 1. *Let $\mathscr{P}$ be a projective plane, R a set of symbols coordinatizing $\mathscr{P}$, and F the ternary operation defined on R as above. F satisfies the following*:

(i) *$F(m, a, c) \in R$ for all $m, a, c \in R$;*

(ii) *$F(0, a, c) = F(a, 0, c) = c$ for all $a, c \in R$;*

(iii) *$F(1, a, 0) = F(a, 1, 0) = a$ for all $a \in R$;*

(iv) *if $m_1, m_2, c_1, c_2 \in R$ with $m_1 \neq m_2$, then there exists a unique $a \in R$ such that $F(m_1, a, c_1) = F(m_2, a, c_2)$;*

(v) *if $m, a, d \in R$, then there is a unique $c \in R$ such that $F(m, a, c) = d$;*

(vi) *if $a_1, a_2, y_1, y_2 \in R$ and if $a_1 \neq a_2$, $y_1 \neq y_2$, then there exist $m, c \in R$ such that $F(m, a_i, c) = y_i$, $i = 1, 2$.*

Conversely, if R is a set and F a ternary function defined on R which satisfies (i)–(vi), *then there exists a projective plane which can be coordinatized by R.*

The set R with the operation F comprise the ternary ring (R, F) of the plane $\mathscr{P}$ with respect to the quadrangle U, V, O, I. We shall call U, V, O the basis points of (R, F). In R we define addition $(+)$ and multiplication $(\cdot)$ by $a + b = F(1, a, b)$ and $a \cdot b = F(a, b, 0)$. Then the elements of R form a loop under addition whose identity is 0 and the elements of $R - \{0\}$ form a loop under multiplication with 1 as the identity. When we wish to emphasize these two operations we write $(R, F, +, \cdot)$ for the ternary ring.

LEMMA 1. *Let $\mathscr{P}$ be a projective plane with distinguished non-collinear points A, B, C. Suppose that every ternary ring $(R, F, +, \cdot)$ of $\mathscr{P}$ with these points as basis points, $U = A$, $V = B$, $O = C$, has the property*

$$F(a, b, a) = ab + a \tag{2}$$

for all $a, b \in R$. If $(R', F', +, \cdot)$ is a particular ternary ring of $\mathscr{P}$ with basis points U, V, O, then

$$F'(a, b, c) = ab + c$$

for all $a, b, c \in R$.

Proof. (x, y) will represent the coordinates of a point of $\mathscr{P}$ with respect to $(R', F', +, \cdot)$. Pick $f, g \in R'$ with $f \neq 0$ and $g \neq 0$. We define a new ternary ring $(R', F'', \oplus, \otimes)$ of $\mathscr{P}$ as follows: Representing the new coordinates by $[a, b]$, the points of $\mathscr{P}$ are coordinatized under the scheme

$$\begin{array}{ll} U' \leftrightarrow U, & [a, b] \leftrightarrow (aL_g^{-1}, b), \\ V' \leftrightarrow V, & [a] \leftrightarrow (aR_f^{-1}), \\ O' \leftrightarrow O, & \end{array}$$

where $L_g^{-1}\colon R' \to R'$ and $R_f^{-1}\colon R' \to R'$ are given by $g \cdot aL_g^{-1} = a$ and $aR_f^{-1} \cdot f = a$. L_g^{-1} and R_f^{-1} are well-defined and onto since $(R' - \{0\}, \cdot)$ is a loop. The element $e = gf$ is the multiplicative identity of $(R', F'', \oplus, \otimes)$ and 0 is the additive identity. Furthermore, $F''(a, b, c) = F'(aR_f^{-1}, bL_g^{-1}, c)$, $a \otimes b = aR_f^{-1} \cdot bL_g^{-1}$, and $a \oplus b = F'(g, aL_g^{-1}, b)$. By hypothesis, $F''(a, b, a) = a \otimes b \oplus a$. Hence,

$$F'(aR_f^{-1}, bL_g^{-1}, a) = F'(g, (aR_f^{-1} \cdot bL_g^{-1})L_g^{-1}, a)$$

for all $a, b \in R'$. Setting $a = g$ yields, for all $b \in R'$,

$$F'(gR_f^{-1}, bL_g^{-1}, g) = gR_f^{-1} \cdot bL_g^{-1} + g.$$

If $c, d, g \in R'$ with $c \neq 0$ and $g \neq 0$, pick $f = gL_c^{-1}$. Then

$$F'(c, d, g) = F'(gR_f^{-1}, (gd)L_g^{-1}, g) = gR_f^{-1} \cdot (gd)L_g^{-1} + g = gR_f^{-1} \cdot d + g = c \cdot d + g.$$

If $c = 0$ or $g = 0$, this last statement is still true. Hence, $F'(a, b, c) = ab + c$ for all $a, b, c \in R'$.

Definition 1. Let $\mathscr{P}$ be a projective plane. A collineation of $\mathscr{P}$ is a mapping of $\mathscr{P}$ onto $\mathscr{P}$ which preserves collinearity. $\mathscr{P}$ is (U, γ)-transitive ($U \in \mathscr{P}$, γ is a line of $\mathscr{P}$) if for any two points P, Q with $PU = QU$, $P, Q \neq U$, and $P, Q \notin \gamma$, there exists a collineation τ of $\mathscr{P}$ which fixes the point U and the line γ pointwise and for which $\tau(P) = Q$. $\mathscr{P}$ is (γ, γ)-transitive for some line γ if $\mathscr{P}$ is (U, γ)-transitive for all $U \in \gamma$.

The following theorems give the connection between the ternary rings of $\mathscr{P}$ and (U, γ)-transitivity. For proofs, see either Hall (**3**) or Pickert (**7**).

THEOREM 2. *Let $\mathscr{P}$ be a projective plane and $(R, F, +, \cdot)$ a ternary ring of $\mathscr{P}$ with basis points U, V, O. $\mathscr{P}$ is (V, UV)-transitive if and only if*

(i) $F(a, b, c) = ab + c$ *for all* $a, b, c \in R$;

(ii) R *forms a group under* $+$.

THEOREM 3. *Let $\mathscr{P}$ be a projective plane and $(R, F, +, \cdot)$ a ternary ring of $\mathscr{P}$ with basis points U, V, O. $\mathscr{P}$ is (UV, UV)-transitive if and only if $(R, F, +, \cdot)$ is a Veblen-Wedderburn system; that is, $(R, F, +, \cdot)$ satisfies* (i) *and* (ii) *of Theorem* 2 *and also*

(iii) $a(b + c) = ab + ac$ *for all* $a, b, c \in R$.

THEOREM 4. *A projective plane $\mathscr{P}$ is coordinatized by an alternative division ring if and only if $\mathscr{P}$ is (γ, γ)- and (δ, δ)-transitive for two distinct lines γ and δ of $\mathscr{P}$.*

Definition 2. A plane which fulfills the conditions of Theorem 4 is called a Moufang plane.

3. Reidemeister planes. In this section we prove the following theorem.

THEOREM 5. *Let $\mathcal{P}$ be a Reidemeister plane and $(R, F, +, \cdot)$ a ternary ring coordinatizing $\mathcal{P}$ with basis points U, V, O. If $(R, F, +, \cdot)$ is a Veblen-Wedderburn system satisfying*

$$(xy)(zx) = (x\cdot yz)x \tag{3}$$

for all $x, y, z \in R$, then $\mathcal{P}$ is a Moufang plane.

The proof will be by means of a sequence of lemmas. The first lemma is a well-known result in loop theory (**1**, pp. 115–117).

LEMMA 2. *Let M be a loop satisfying* (3). *The following identities hold for all $x, y, z \in M$:*
(i) $(xy\cdot z)y = x(yz\cdot y)$;
(ii) $x(y\cdot xz) = (x\cdot yx)z$;
(iii) $xx\cdot y = x\cdot xy$, $xy\cdot x = x\cdot yx$, and $x\cdot yy = xy\cdot y$.
For each element $x \in M$ there exists a unique element $x^{-1} \in M$ such that $x\cdot x^{-1} = x^{-1}\cdot x = 1$, the identity of M. Moreover, if x, y, z are elements of M such that $xy\cdot z = x\cdot yz$, then x, y, and z generate an associative subloop of M.

LEMMA 3. *In $(R, F, +, \cdot)$ of Theorem 5 the following hold:*
(i) $b(c\cdot ab^{-1}) = (bc\cdot a)b^{-1}$ *for all $a, b, c \in R$ with $b \neq 0$;*
(ii) $a(-b) = (-a)b = -ab$ *for all $a, b \in R$, where $-a$ is defined by $a + (-a) = 0$;*
(iii) *b, a, and $(b + a)^{-1}$ associate under multiplication for all $a, b \in R$ with $b + a \neq 0$.*

Proof. (i) follows from

$$b(c\cdot ab^{-1})\cdot b = bc\cdot(ab^{-1}\cdot b) = bc\cdot a.$$

From $0 = a[b + (-b)] = ab + a(-b)$, we have $a(-b) = -ab$. In particular, $(-1)^2 = 1$. Thus $(-1)[a + (-1)a] = (-1)a + a$. Hence, either $(-1)a = -a$ or $-1 = 1$. In the first case we have, letting $b = -1$ in (i):

$$(-1)[c\cdot a(-1)] = (-c)a\cdot(-1) = (-1)\cdot(-c)a;$$

thus $c(-a) = (-c)a = -ca$. If $-1 = 1$, then $-c = c$ for all c and our proof is complete.

For (iii) we have, if $b + a \neq 0$, $a \neq 0$, $b \neq 0$,

$$b^{-1}\cdot a^{-1}(b + a) = b^{-1}\cdot a^{-1}b + b^{-1}\cdot a^{-1}a = b^{-1}a^{-1}\cdot b + b^{-1}a^{-1}\cdot a = b^{-1}a^{-1}\cdot(b + a),$$

by (i). Thus b^{-1}, a^{-1}, and $b + a$ associate, and by Lemma 2, (iii) is true.

LEMMA 4. *$\mathcal{P}$ is (O, OU)-transitive.*

Proof. We remark here that by Theorem 3, any ternary ring of $\mathcal{P}$ with basis points U, V, O is a Veblen-Wedderburn system. $(R, F, +, \cdot)$ satisfying

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(3) implies that the three Bol configuration theorems hold in the (U, O, V)-net of lines (**7**, pp. 50–59). This, in turn, implies that any ternary ring with basis points U, V, O also satisfies (3). Thus, Lemmas 2 and 3 hold for any ternary ring of $\mathscr{P}$ with basis points U, V, O.

Let $(R', F', \oplus, \otimes)$ be a ternary ring of $\mathscr{P}$ with basis points $U' = U$, $V' = O$, $O' = V$. We denote the coordinates of points of $\mathscr{P}$ with respect to (R', F') by $[a, b]$ and $[a]$. We define a new ternary ring $(R', F'', +, \cdot)$ of $\mathscr{P}$. Although we will use the same symbols for addition and multiplication in (R', F'') as we do for the operations in (R, F), no confusion will arise since (R, F) does not directly appear in the proof.

The new coordinates, denoted by (a, b) and (a), are as follows:

$$\begin{array}{rlrll} (1,1) \leftrightarrow [1,1], & & (a,a) \leftrightarrow [1,a], & a \neq 0, \\ (0,0) \leftrightarrow 0, & & (a,0) \leftrightarrow [a], & a \neq 0, \\ (\infty) \leftrightarrow V, & & (a) \leftrightarrow [a,0], & a \neq 0, \\ (0) \leftrightarrow U, & & (0,a) \leftrightarrow [0,a], & a \neq 0. \end{array}$$

Then the basis points of (R', F'') are U, V, O, and hence $F''(a, b, c) = ab + c$ and $R' - \{0\}$ is a loop under "$\cdot$" satisfying (3).

Comparing $(R', F', \oplus, \otimes)$ and $(R', F'', +, \cdot)$ we see that 0 is the identity for both $\oplus$ and $+$, 1 the identity for $\otimes$ and $\cdot$; furthermore, if $-c$ is defined by $c + (-c) = 0$, $F'(a, b, -ba) = 0$, $a \otimes b = ba$, and

$$F'(a, b, c) = (-c)a^{-1}\cdot d + c = bd,$$

d defined by the second equality, for all non-zero a, b, $c \in R$ with $c \neq -ba$. We also have that

$$c \oplus (-c) = 0, \qquad 0 \oplus b = b,$$

and

$$b \oplus c = bd = (-c)d + c, \qquad b \neq 0,\ c \neq 0,\ -b,$$

where d is defined by the second equality.

We wish to show that $F'(a, b, a) = a \otimes b \oplus a$ in order to apply Lemma 1. If $a = 0$, $b = 0$, or $b = -1$, we have nothing to prove. Thus, we assume that a and b are non-zero and $b \neq -1$. Then

$$F'(a, b, a) = bd' = -d' + a, \tag{4}$$

$$a \otimes b \oplus a = ba \oplus a = ba\cdot d = -ad + a, \tag{5}$$

where d' (d) is uniquely defined by the second equality of (4) ((5)). It is sufficient to show that $ba\cdot d = b\cdot ad$ since this implies that $d' = ad$. Note that $(R', F'', +, \cdot)$ is a Veblen-Wedderburn system satisfying (3), and hence Lemmas 2 and 3 apply.

First, let b and c be any elements in R' with $b \neq -c$. Then

$$b \oplus c = -ce + c = be,$$

$$(b \oplus c) \oplus (-c) = be \oplus (-c) = be\cdot e' = ce' - c.$$

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Since $\oplus$ is associative, we have that $b = be \cdot e'$ and this implies that $e' = e^{-1}$ and

$$(6) \qquad -ce + c = be = (ce^{-1} - c)e.$$

The last equality holds for all non-zero c and e in R'; for given c and e, pick $b = (-ce + c)e^{-1}$.

Then

$$ba \cdot d = -ad + a = (ad^{-1} - a)d,$$

since $d \neq 0$. Hence $ba = ad^{-1} - a$ or $d = (ba + a)^{-1}a$. Thus

$$ba \cdot d = ba \cdot (ba + a)^{-1}a = [ba \cdot (ba + a)^{-1}]a = b \cdot [a \cdot (ba + a)^{-1}a] = b \cdot ad.$$

Therefore, $F'(a, b, a) = a \otimes b \oplus a$ for all $a, b \in R'$. Since $(R', F', \oplus, \otimes)$ is an arbitrary ternary ring of $\mathscr{P}$ with basis points U, O, V, we have, by Lemma 1, that $F'(a, b, c) = a \otimes b \oplus c$ for all $a, b, c \in R'$. Also, $(R', \oplus)$ is a group by hypothesis. Thus, by Theorem 2, $\mathscr{P}$ is (O, OU)-transitive.

We can now prove Theorem 5. $\mathscr{P}$ is (UV, UV)- and (O, OU)-transitive. Hence, it is (γ, γ)-transitive for all lines passing through U (**7**, p. 66). By Theorem 4, $\mathscr{P}$ is a Moufang plane.

Theorem 5 has an application to the class of planes in which the small axial theorem of Pappus holds. (For the definition of this configuration theorem, see Pickert (**7**, p. 153).) Lüneburg (**5**) investigated these planes and showed that in the finite case such planes are Desarguesian. We can prove the following corollary.

COROLLARY. *If $\mathscr{P}$ is a projective plane satisfying the small axial theorem of Pappus, then $\mathscr{P}$ is a Moufang plane if and only if it is coordinatized by a Veblen-Wedderburn system satisfying* (3).

Proof. The necessity is well known. If $\mathscr{P}$ satisfies the small axial theorem of Pappus, then every ternary ring coordinatizing it has commutative and associative addition (**7**, p. 154). Hence, $\mathscr{P}$ is a Reidemeister plane and Theorem 5 applies.

REFERENCES

1. R. H. Bruck, *A survey of binary systems* (Springer-Verlag, Berlin, 1958).
2. A. M. Gleason, *Finite Fano planes*, Amer. J. Math. *78* (1956), 797–807.
3. Marshall Hall Jr., *The theory of groups* (Macmillan, New York, 1959).
4. Otto H. Kegel and Heinz Lüneburg, *Über die Reidemeisterbedingung*. II, Arch. Math. *14* (1963), 7–10.
5. Heinz Lüneburg, *Über die beiden kleinen Satze von Pappos*, Arch. Math. *11* (1960), 339–341.
6. ——— *Über die kleine Reidemeisterbedingung*, Arch. Math. *12* (1961), 382–384.
7. Günter Pickert, *Projektive Ebenen* (Springer-Verlag, Berlin, 1955).

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