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On Induced Representations Distinguished by Orthogonal Groups

Cesar Valverde

Abstract. Let F be a local non-archimedean field of characteristic zero. We prove that a representation of $GL(n,F)$ obtained from irreducible parabolic induction of supercuspidal representations is distinguished by an orthogonal group only if the inducing data is distinguished by appropriate orthogonal groups. As a corollary, we get that an irreducible representation induced from supercuspidals that is distinguished by an orthogonal group is metic.

1 Introduction

Let $\mathbb{F}$ be a number field with ring of adeles $\mathbb{A}$. We will denote by $\widetilde{GL(n)}$ a twofold cover of $GL(n)$, consisting of (g,z) with $g \in GL(n)$, $z = \pm 1$. We remark that $GL(n,\mathbb{F})$ embeds in $\widetilde{GL(n,\mathbb{A})}$ via $g \mapsto (g,1)$. A function f on $\widetilde{GL(n,\mathbb{A})}$ is *genuine* if one has $f(g,z) = f(g,1)z$. We denote by $\widetilde{L}^2$ the subspace of $L^2(GL(n,\mathbb{F})\backslash\widetilde{GL(n,\mathbb{A})})$ consisting of genuine functions. A constituent of the $\widetilde{GL(n,\mathbb{A})}$ module $\widetilde{L}^2$ is called a *genuine* automorphic representation. The metaplectic correspondence is a lifting of the genuine automorphic representations of $\widetilde{GL(n,\mathbb{A})}$ to automorphic representations of $GL(n,\mathbb{A})$.

The work of Flicker and Kazhdan in [FK] suggests the following criterion:

An irreducible cuspidal automorphic representation of $GL(n,\mathbb{A})$ is a lift from $\widetilde{GL(n,\mathbb{A})}$ if and only if it is metic (*i.e.,* all of its local components are metic).

A representation π of $GL(n)$ over a local field F is defined to be *metic* if it is equivalent to a representation unitarily induced from a $\prod_i GL(r_i,F)$ module $\otimes_i \sigma_i \nu^{s_i}$, where $\sum r_i = n$, $s_i \in \mathbb{C}$, $\nu = |\det|$ and the σ_i are square integrable $GL(r_i,F)$ modules whose central characters are trivial on $\{\pm 1\}$.

Let χ be a quadratic character of $\mathbb{F}^\times\backslash\mathbb{A}^\times$ and let π be an irreducible cuspidal automorphic representation of $GL(n,\mathbb{A})$. For a diagonal matrix $D \in GL(n)$ we have the associated orthogonal group $O_D = \{g \in GL(n) | gD^tg = D\}$ and the similitude group $GO_D = \{g \in GL(n) \mid gD^tg = \lambda(g)D\}$, where $\lambda(g)$ is a scalar.

We recall that a cuspidal automorphic representation π with trivial central character is said to be (GO_D,χ)-*distinguished* if for some ϕ in the space of π we have

$$\int_{Z(\mathbb{A})GO_D(\mathbb{F})\backslash GO_D(\mathbb{A})} \phi(hg)\chi(\lambda(h))dh \not\equiv 0,$$

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where Z is the center of $GL(n)$.

A consequence of the (conjectural) relative trace formula suggested by Jacquet in [J] (also discussed in [Mao]) is that a global cuspidal representation π of $GL(n,\mathbb{A})$ is expected to be distinguished by some similitude group GO_D if and only if it is globally a lift from a genuine generic representation on $\widetilde{GL}(n,\mathbb{A})$. Thus, a global cuspidal representation π of $GL(n,\mathbb{A})$ is expected to be distinguished by some similitude group if and only if all its local components are metic.

We now move to a local setting. We let F be a local non-archimedean field of characteristic zero. We recall the local notion of distinction, namely, let π be a representation of G, let H be a closed subgroup of G, then π is said to be *H-distinguished* if the space of H-invariant linear forms on π is nontrivial.

In this paper we consider a representation obtained by parabolic induction from supercuspidals, $\pi = \boxplus\pi_i$. Assuming that $\pi = \boxplus\pi_i$ is irreducible, we show that if π is O_D-distinguished, then π is metic. More precisely, given a set of diagonal matrices $\{D_1,\dots,D_r\}$ with $D_i \in GL(M_i,F)$, we denote by $\prod D_i$ the matrix obtained by diagonally embedding $(D_1,\dots,D_r)$ in G where $n = M_1+\cdots+M_r$. For diagonal matrices D and D', we write $D \sim D'$ if there exists $g \in GL(n,F)$ with $gD^tg = D'$. We have the following theorem.

Theorem 1.1 *Let $M_1+M_2+\cdots+M_r = n$ be a partition of n. For $1 \le i \le r$, let π_i be an irreducible supercuspidal representation of $GL(M_i,F)$. Assuming the parabolically induced representation $\pi = \boxplus\pi_i$ is irreducible and π is O_D distinguished, then there exists $D' \sim D$ such that for all i, π_i is $O_{D'_i}$-distinguished, where $D' = \prod D'_i$.*

Note that if π_i is a supercuspidal representation distinguished by O_{D_i}, then its central character is trivial on $\{\pm 1\}$. Thus we have the following corollary.

Corollary 1.2 *Assume as above that $\pi = \boxplus\pi_i$ is irreducible; if π is O_D-distinguished, then π is metic.*

Clearly, a (GO_D,χ)-distinguished representation is O_D-distinguished.

In [HL], Hakim and Lansky set out to determine when a supercuspidal representation is O_D distinguished. Let $G = GL(n,F)$ with n odd and $H = O_D$. We understand by an irreducible *tamely supercuspidal* representation of G, one of the representations constructed by Howe in [H]. A result of [HL], based on work of Hakim and Murnaghan in [HM], implies that when F is of odd residual characteristic, an irreducible tamely supercuspidal representation π of G is H-distinguished if and only if

(i) the central character ω_π of π satisfies $\omega_\pi(-1) = 1$, and
(ii) O_D is quasi-split.

We remark that by [Moy], we have that for p not dividing n all supercuspidals are tamely supercuspidal where p is the characteristic of the residue field of F. Thus if in Theorem 1.1 M_i are odd and not divisible by p, the condition of π_i being $O_{D'_i}$-distinguished is equivalent to π being metic and $O_{D'_i}$ being all quasi-split.

We have a refined version of Theorem 1.1 in the case when π is a principal series representation.

Theorem 1.3 *Let D be any diagonal matrix, let $\pi = \boxplus\chi_i$ be an irreducible principal series representation of G. Then π is O_D-distinguished if and only if $\chi_i(-1) = 1$ for all χ_i.*

As a consequence, an irreducible principal series representation is distinguished by O_D if and only if it is distinguished by $O_{D'}$ for any other diagonal matrix $D' \in GL(n, F)$.

It is worth contrasting Theorem 1.3 with the result of [HL] on supercuspidal representations. We consider an irreducible tame supercuspidal representation of $GL(n, F)$ with n odd. Then such a representation is distinguished by a quasisplit orthogonal group exactly when its central character is trivial on -1. Distinction by non-quasisplit orthogonal subgroups never occurs.

The proof of Theorem 1.1 proceeds by studying the double coset decomposition of $P\backslash GL(n, F)/O_D$ (where P denotes the standard parabolic subgroup of $GL(n, F)$ associated with the partition $n = M_1 + \cdots + M_r$) and then showing that the linear form is supported on one of the double cosets. We then rule out the support consisting of any cosets except the open cosets by considerations of irreducibility and cuspidality. The form being supported on the open cosets is seen to give the appropriate condition on the inducing data.

2 Double Coset Decomposition

For F a local non-archimedean field of characteristic zero, we will denote $G = GL(n, F)$. For $g \in G$, the transpose of g will be denoted ${}^t g$, and B, N, T, W will denote, respectively, the standard Borel subgroup of G, the group of upper triangular matrices with unit diagonal, the group of diagonal matrices, and the Weyl group of G, identified with the subgroup of permutation matrices in G. The standard parabolic associated with the partition $M_1 + \cdots + M_r = n$ will be denoted by P. We denote by W_P the Weyl group corresponding to the Levi subgroup of P. Throughout the paper we use right-invariant Haar measures and we write Δ_B (resp. Δ_P) for the modulus function of B (resp. P).

We briefly review the classification of quadratic forms on a finite dimensional vector space V over F, *cf.* [MVW, Section 1.6]. An *isometry* of spaces equipped with such forms is a F-linear bijection respecting the pairing. We have that up to isometry any quadratic form is given on row vectors v_1, v_2 by $(v_1, v_2) \mapsto v_1 D\, {}^t v_2$ for some diagonal matrix D.

We denote by O_D, the group of automorphisms of V leaving the quadratic form given by D invariant, *i.e.*, $O_D = \{g \in G | g D\, {}^t g = D\}$.

The following lemma is well known.

Lemma 2.1 *Let U be an algebraic connected unipotent group over F. Let θ be an automorphism of U with $\theta^2 = 1$. If $x \in U$ verifies $x\theta(x) = 1$, then there is $u \in U$ with $x = \theta(u^{-1})u$.*

We need the following version of Bruhat decomposition.

Lemma 2.2 *Every symmetric matrix s in G admits a Bruhat decomposition $s = nwa\,{}^t n$ where $a \in T$, $n \in N$, and $w \in W$ is uniquely determined by s.*

Proof Let $s = n_1 wa\, {}^t n_2$ be a symmetric matrix, then

$$n_2^{-1} n_1 wa\, {}^t n_2\, {}^t n_1^{-1} = a\, {}^t w,$$

which implies $a\,{}^t w = wa$, ${}^t w = w^{-1} = w$, and $a = waw$. Write $n = n_2^{-1} n_1$, then we have $s = n_2 nwa\,{}^t n_2$ so we may assume $s = nwa$. We remark that if w is trivial, then na is symmetric, in which case we see that $n = 1$ and our lemma follows.

Suppose w is nontrivial. We denote by N_w the unipotent subgroup $N_w = N \cap w^{-1}({}^t N)w$. We have $nwa = aw\,{}^t n$ so that $n \in N_w$ and $n^{-1} = wa\,{}^t n^{-1} wa^{-1}$. Define an involution on N_w by

$$\theta(x) = aw\,{}^t x^{-1} a^{-1} w.$$

Then $n^{-1}\theta(n^{-1}) = 1$, so by the previous lemma there exists $u \in N_w$ with $n^{-1} = \theta(u^{-1})u$. Thus $n = u^{-1}\theta(u)$ and $s = nwa = u^{-1} aw\,{}^t u^{-1} a^{-1} wwa = u^{-1} wa\,{}^t u^{-1}$ as desired. ∎

The following lemma is clear.

Lemma 2.3 *Let S denote the set of invertible symmetric matrices. The map $m\colon G/O_D \to S$ defined by $g \mapsto m(g) = gD\,{}^t g$ is a continuous map to its image.*

We remark that for any class A of $P\backslash G/O_D$, we have that $m(A) = \{pwa\,{}^t p \mid p \in P\}$ for some $w \in W$, $a \in T$.

Lemma 2.4 *Let $g \in G$ with $gD\,{}^t g = wa$ for some $w \in W$, $a \in T$. Then the coset PgO_D is open if and only if $w \in W_P$.*

Proof We use an argument from [BlD, Section 2.4]. We denote by $\mathrm{Lie}(H)$ the Lie algebra of a group H. We denote by θ the involution on G given by $\theta(h) = D\,{}^t h^{-1} D^{-1}$ for $h \in G$. We denote by P_g the subgroup $g^{-1}Pg$. Assuming that $gD\,{}^t g = wa$ with $w \in W_P$, we have that $g\theta(P_g)g^{-1} = gg^{-1} wa\,{}^t P a^{-1} w^{-1} g g^{-1} = {}^t P$. In other words, $\theta(P_g)$ is opposite to P_g. Hence for $x \in \mathrm{Lie}(G)$ there exist $y \in \mathrm{Lie}(P_g)$, $z \in \mathrm{Lie}(\theta(P_g))$ with $x = y + z$. We write $x = y - \theta(z) + \theta(z) + z$ and remark that $y - \theta(z) \in \mathrm{Lie}(P_g)$ and $\theta(z) + z \in \mathrm{Lie}(O_D)$. Thus $\mathrm{Lie}(G) = \mathrm{Lie}(P_g) + \mathrm{Lie}(O_D)$. Hence $g^{-1}PgO_D$ contains a neighborhood of the identity, thus PgO_D is open. On the other hand, if $w \notin W_P$, the above calculation shows that $\theta(P_g)$ is not opposite to P_g. By [BlD, Lemme 2.4], all open cosets are of the form PyO_D, where $\theta(P_y)$ is opposite to P_y. Thus PgO_D is not open when $m(g) = wa$ with $w \notin W_P$, $a \in T$. ∎

Lemma 2.5 *The classes $A_1, A_2, \dots, A_m$ of $P\backslash G/O_D$ can be ordered in such a way that A_1 is closed in G and for $2 \le i \le m$, A_i is closed in $G - \bigcup_{k=1}^{i-1} A_k$. Moreover, there exists $1 \le l \le m-1$ such that A_i is open if and only if $l+1 \le i \le m$.*

Proof By [BlD, Lemme 3.1] we have a sequence of $P \times O_D$-invariant open subsets

$$U_0 = \varnothing \subset U_1 \subset \cdots \subset U_m = G,$$

where for $0 \le i \le m-1$, $U_{i+1}\backslash U_i$ is a $P \times O_D$ orbit. Moreover, we choose this sequence so that there exists $1 \le l \le m-1$ with $U_{i+1}\backslash U_i$ open if and only if $0 \le i \le m-l-1$. For $1 \le j \le m$, we take $A_j = U_{m-j+1}\backslash U_{m-j}$. ∎

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3 A Key Proposition

Let H be a closed subgroup of G, X a locally closed subspace of G, and $D(X)$ the space of smooth complex valued functions on X with compact support. If τ is a smooth representation of H in a complex vector space V, then $D(H\backslash X, \tau, V)$ (or $(D(H\backslash X, \tau))$) will denote the space of smooth V-valued functions f on X with compact support modulo H that satisfy $f(hx) = \tau(h)f(x)$.

The following lemma is found in [Mat].

Lemma 3.1 *The map from $D(X) \otimes V$ to $D(H\backslash X, \tau, V)$ defined by*

$$f \otimes v \mapsto \Big(x \mapsto \int_H f(hx)\tau(h^{-1})v dh\Big)$$

is surjective. As a corollary, if Y is an H-stable closed subset of X, then the restriction map from $D(H\backslash X, \tau, V)$ to $D(H\backslash Y, \tau, V)$ is surjective.

We remark that the following argument is standard, *cf.* [Mat] and [BH, Section 3.4]. Our main contribution in the proof of Proposition 3.2 is the calculation of certain modulus functions which imply that the support of the invariant functional lies in the open cosets.

Recall that the space of $\pi = \boxplus \pi_i$ is $D(P\backslash G, \Delta_P^{1/2}\rho)$, where ρ denotes the representation $\pi_1 \otimes \cdots \otimes \pi_r$ of $\prod GL(M_i, F)$ extended trivially to the unipotent radical of P. Lemmas 2.5 and 3.1 imply that we have the following exact sequence of smooth O_D modules:

$$0 \to D\big(P\backslash(G - A_1), \Delta_P^{1/2}\rho\big) \to D(P\backslash G, \Delta_P^{1/2}\rho) \to D(P\backslash A_1, \Delta_P^{1/2}\rho) \to 0.$$

Hence if π is O_D-distinguished, then there is a nonzero O_D-invariant linear form on either $D(P\backslash A_1, \Delta_P^{1/2}\rho)$, or on $D(P\backslash(G - A_1), \Delta_P^{1/2}\rho)$. In the second case we have the exact sequence

$$0 \to D\big(P\backslash G-(A_1\cup A_2), \Delta_P^{1/2}\rho\big) \to D\big(P\backslash(G-A_1), \Delta_P^{1/2}\rho\big) \to D(P\backslash A_2, \Delta_P^{1/2}\rho) \to 0.$$

Repeating this process, we deduce the existence of a nonzero O_D-invariant linear form on one of the spaces $D(P\backslash A_i, \Delta_P^{1/2}\rho)$.

We have an isomorphism of O_D-modules between $D(P\backslash A_i, \Delta_P^{1/2}\rho)$ and $D(g^{-1}Pg\cap O_D\backslash O_D, \Delta'_P\rho')$ given by $f \mapsto [x \mapsto f(gx)]$, where $\Delta'_P(x) = \Delta_P^{1/2}(gxg^{-1})$, $\rho'(x) = \rho(gxg^{-1})$, and g is a representative of A_i with $m(g) = wa$ and $w \in W, a \in T$. Such a representative exists by the remark following Lemma 2.3.

Lemma 3.1 gives a surjection

$$D(O_D) \otimes V \to D\big(g^{-1}Pg \cap O_D\backslash O_D, \Delta'_P\rho'\big)$$

defined by

$$(f \otimes v) \mapsto \Big[x \mapsto \widetilde{(f \otimes v)}(x) = \int_H \Delta'_P(h^{-1})f(hx)\rho'(h^{-1})v dh\Big], \tag{3.1}$$

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where $h \in H = g^{-1}Pg \cap O_D$ and dh denotes a right invariant Haar measure.

We point out that if λ denotes the action of the group O_D on $D(O_D)$ by left translation, namely $\lambda_\sigma f(x) = f(\sigma^{-1}x)$, then the above surjection is seen to satisfy, for $k \in H$

$$\begin{aligned} \widetilde{\lambda_k f \otimes v}(x) &= \int_H \Delta'_P(h^{-1}) f(k^{-1}hx)\rho'(h^{-1})v\,dh \\ &= \Delta'_P(k^{-1})\Delta_H(k) \widetilde{f \otimes \rho'(k^{-1})v}(x). \end{aligned} \tag{3.2}$$

If $D(g^{-1}Pg \cap O_D \backslash O_D, \Delta'_P\rho')$ admits an O_D-invariant form L', then we get an O_D-invariant linear form on $D(O_D) \otimes V$. We also have that $(D(O_D) \otimes V)^{*O_D} \cong D(O_D)^{*O_D} \otimes V^*$, and we remark that $\mathrm{Hom}_{O_D}(D(O_D), \mathbb{C})$ is a one dimensional space spanned by the right Haar integral. Thus the form L' is given by

$$L'(f \otimes v) = \int_{\sigma \in O_D} f(\sigma)d\sigma L(v)$$

for some $L \in V^*$. The existence of the O_D-invariant form on

$$D(g^{-1}Pg \cap O_D \backslash O_D, \Delta'_P\rho')$$

implies that the right Haar integral factors through the quotient map (3.1). Equation (3.2) implies that the kernel of the induced form L' contains the functions of the form $\lambda_h(f) \otimes v - \Delta'_P(h^{-1})\Delta_H(h) f \otimes \rho'(h^{-1})v$ for $h \in H$ so that we have

$$\int_{O_D} f(h^{-1}\sigma)d\sigma L(v) = \Delta'_P(h^{-1})\Delta_H(h) \int_{O_D} f(\sigma)d\sigma L\big(\rho'(h^{-1})v\big).$$

After a change of variables $\sigma \mapsto h\sigma$ this implies

$$\int_{O_D} f(\sigma)d\sigma L(v) = \Delta_P^{-1/2}(p)\Delta_{P\cap gO_Dg^{-1}}(p) \int_{O_D} f(\sigma)d\sigma L\big(\rho(p^{-1})v\big)$$

for all $f \in D(O_D)$, $p \in P \cap gO_Dg^{-1}$, $v \in V$, which implies

$$L(v) = \Delta_P^{-1/2}(p)\Delta_{P\cap gO_Dg^{-1}}(p) L\big(\rho(p^{-1})v\big) \tag{3.3}$$

for all $v \in V$ and $p \in P \cap gO_Dg^{-1}$.

Proposition 3.2 *Assume that g is a representative of A_i with $m(g) = wa$, $w \in W$, $a \in T$. Assume the existence of a nonzero $L \in V^*$ satisfying equation (3.3) for all $v \in V$ and $p \in P \cap gO_Dg^{-1}$. Then $w \in W_P$.*

Proof As a simple example, we now do the relevant calculations for the principal series case first. Here $\rho = \otimes_{i=1}^n \chi_i$, where the χ_i are characters of $F^\times$.

The existence of L as in the statement of the proposition implies equation (3.3), which in the present case is

$$\chi(b)\Delta_B^{1/2}(b)\Delta_{B\cap gO_Dg^{-1}}(b^{-1}) = 1 \tag{3.4}$$

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for $b \in B \cap gO_D g^{-1}$.

Assume now that there is a smallest l not fixed by w, we consider d where d is a diagonal matrix with x and x^{-1} in the (l,l) and $(w(l),w(l))$ coordinates respectively and 1 elsewhere. We remark that $d \in B \cap gO_D g^{-1}$.

We need to compute $\Delta_{B\cap gO_D g^{-1}}(d)$. For this we compute the tangent space of $N \cap gO_D g^{-1}$ by taking ϵ with $\epsilon^2 = 0$ and looking for elements X with $I + \epsilon X \in N \cap gO_D g^{-1}$, *i.e.*, we look for X strictly upper triangular with $(I+\epsilon X)wa\,{}^t(I+\epsilon X) = wa$. We have $X = -aw\,{}^tXwa^{-1}$, *i.e.*, $X_{ij} = -a_i X_{w(j)w(i)} a_j^{-1}$, where we write a_i for $a_{(i,i)}$.

Claim. The first $l-1$ rows, l columns of X are zero.

The first row is clearly zero. If the first $k-1$ rows are zero, then the first k columns are zero (recall that X is upper triangular). If $w(k) = k$, then we have $X_{kj} = a_{kk} X_{w(j)w(k)} a_{jj} = 0$ proving our claim by induction.

Claim. The number of nonzero free entries in the l-th row of X is $w(l) - l - 1$.

For $j > l$ we have $X_{lj} = 0$ if and only if $w(j) = l$ (in which case we have $X_{lj} = X_{l,w(l)} = -a_l X_{l,w(l)} a_{w(l)}^{-1} = 0$ using $a_{w(l)} = a_l$, which follows from wa being symmetric) or $w(j) > w(l)$. There is exactly one instance in which the first case occurs and $n - w(l)$ for the second case; this proves our claim.

The previous two claims prove the following one.

Claim. $\Delta_{B\cap gO_D g^{-1}}(d) = |x|^{w(l)-l-1}$.

Also note that $\Delta_B^{1/2}(d) = |x|^{w(l)-l}$. By equation (3.4), this implies

$$|x|^{w(l)-l}|x|^{1+l-w(l)}\chi_l(x) = \chi_{w(l)}(x),$$

i.e., $|x|\chi_l(x) = \chi_{w(l)}(x)$, thus contradicting the irreducibility of π. This proves that $w = 1$.

In the general case, let us assume the existence of a nonzero linear functional $L \in V^*$ satisfying equation (3.3). We will prove that $w \in W_P$.

We take a partition $R_1 \cup R_2 \cup \cdots \cup R_r = \{1,2,\ldots,n\}$, where $|R_i| = M_i$; we call R_i a block. We denote by α_i, β_i the initial and final elements of R_i respectively. We choose the partition so that R_i consists of consecutive positive integers with $\alpha_1 = 1$ and $\alpha_{i+1} = \beta_i + 1$. The group G acts on $\oplus_{i=1}^n Fe_i$, and the group $GL(M_i,F)$ acts on $\oplus_{j\in R_i} Fe_j$.

Lemma 3.3 *The element w permutes the blocks R_i.*

Proof Let R_i have the property that at least two elements of R_i map to different blocks and let R_i be minimal in the sense that R_j for $j < i$ does not have this property. After conjugation by a suitable Weyl element we may assume that for any two elements $p < q \in R_i$ either $w(p), w(q) \in R_j$, or $w(p) > w(q)$ whenever $w(p), w(q) \in R_l, R_k$ respectively with $l \neq k$.

Let l denote the smallest member of R_i such that $w(l) \in R_j$ with the property that for $x \in R_i, x < l$ we have $w(x) \in R_j$ and $w(l+1) \in R_h$ with $j \neq h$. We further partition R_i as a disjoint union $R_i = R_i^1 \cup R_i^2$, where $R_i^1 = \{\alpha_i, \ldots, l\}$ and $R_i^2 = \{l+1, \ldots, \beta_i\}$.

We consider the matrix X given by

$$X_{i',j'} = \begin{cases} U_{i'j'} & \text{for } \alpha_i \le i' \le l, l+1 \le j' \le \beta_i, \\ -a^{-1}_{w(j')} U_{w(j'),w(i')} a_{w(i')} & \text{for } \alpha_i \le w(j') \le l, l+1 \le w(i') \le \beta_i, \\ 0 & \text{otherwise,} \end{cases}$$

where the $U_{i'j'}$ for $\alpha_i \le i' \le l, l+1 \le j' \le \beta_i$ are free variables.

Let U be the unipotent radical of $P = MU$, where $M = \prod GL(M_i, F)$. Let U_i denote the unipotent radical of $GL(M_i, F)$ corresponding to the partition $R_i = R_i^1 \cup R_i^2$. We identify U_i with the unipotent radical of the parabolic subgroup of M with Levi subgroup equal to

$$\prod_{j=1}^{i-1} GL(M_j, F) \times GL\big(|R_i^1|, F\big) \times GL\big(|R_i^2|, F\big) \prod_{k=i+1}^{r} GL(M_k, F),$$

and we let $U^i = UU_i$. As in the principal series case, we have that X is by construction in the tangent space of $U^i \cap gO_D g^{-1}$. We consider $\exp(X)$ and remark that $\exp(X) \subset U^i \cap gO_D g^{-1}$. We write $\exp(X) = u'u$ with $u' \in U$, $u \in U_i$ and remark that u exhausts U_i as the free variables $U_{i'j'}$ vary over F.

From equation (3.3) we obtain that

$$L\big(\rho(\exp(X))v\big) = L\big(\rho(u'u)v\big) = L(v) \tag{3.5}$$

for all $v \in V$. It follows from equation (3.3) that L is U-invariant. Hence from equation (3.5) and the U-invariance of L we have that

$$L\big(\rho(u)v\big) = L(v)$$

for all $u \in U_i'$ and $v \in V$. Because ρ is supercuspidal, V is spanned by vectors of the form $\rho(u)v - v$, where $u \in U_i'$ and $v \in V$. Therefore $L = 0$, which contradicts an assumption of Proposition 3.2. This proves the lemma. ■

Lemma 3.4 *The element w maps each block R_i to itself.*

Proof We know by the previous claim that w maps blocks to blocks. Let R_i be the smallest block mapped to a different block R_j; we remark that we have $M_i = M_j$.

We consider the block diagonal matrix p consisting of h, ${}^t h^{-1}$ as the R_i, R_j diagonal blocks and the identity for the other blocks. Then this matrix satisfies $p \in P \cap gO_D g^{-1}$. We have for such p that

$$L(v) = \Delta_P^{1/2}(p)\Delta^{-1}_{P\cap gO_D g^{-1}}(p) L\big(\rho(p)v\big),$$

which is equation (3.3).

We remark that if w permutes the blocks R_i and R_j, then after acting by a suitable parabolic element, we may assume $wa = w'a'$ with $w'(\alpha_i + l) = \alpha_j + l$ for $0 \le l \le M_i$ and $a_l' = a_k' = 1$ for $l, k \in R_i, R_j$ respectively. We now assume that wa is of the form $w'a'$.

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An easy computation gives

$$\Delta_P^{1/2}(p) = |\det(h)^{2(M_{i+1}+\cdots+M_{j-1})+M_i+M_j}|^{1/2} = |\det(h)^{M_i+\cdots+M_{j-1}}|,$$

where we used $M_i = M_j$. To obtain $\Delta^{-1}_{P\cap gO_Dg^{-1}}(p)$, we compute the tangent space of $U \cap gO_Dg^{-1}$ as in the principal series case by taking X with $I+\epsilon X \in U \cap gO_Dg^{-1}$ and $\epsilon^2 = 0$. Elements X in this tangent space are seen to verify $X_{i'j'} = -a'_{i'}X_{w(j')w(i')}a'^{-1}_{j'}$.

We write $X = (X_{kl})$ with X_{kl} an $M_k \times M_l$ matrix. The following two claims are proved as in the principal series case, and we omit their proofs.

Claim. If $k < i$, then $X_{kl} = 0$. If $l \le i$, then $X_{kl} = 0$.

Claim. There are exactly $j-i-1$ free nonzero blocks of the form X_{il}. The only other possibly nonzero block of the form X_{il} is $X_{iw(i)}$.

Let Λ denote the set of l with the property that X_{il} is nonzero and free.

Claim.

$$\sum_{l\in\Lambda} M_l = M_{i+1} + \cdots + M_{j-1}. \tag{3.6}$$

We assume that l satisfies $i < l < w(i)$. If $l \notin \Lambda$, then each element in $X_{iw(l)}$ is up to a factor equal to an element in $X_{lw(i)}$. Our assumption that $i < l < w(i)$ implies that $w(l) \in \Lambda$. On the other hand, since w permutes blocks, we have that $M_l = M_{w(l)}$ and thus our claim is proved.

Claim. The block $X_{i,w(i)}$ is a square matrix $(Y_{p,q})$ for $1 \le p, q \le M_i$ with $Y_{p,p} = 0$, $Y_{q,p} = -Y_{p,q}$, and the $Y_{p,q}$ are free variables.

This claim is a consequence of the equation $X_{i'j'} = -a'_{i'}X_{w(j')w(i')}a'^{-1}_{j'}$ and of the normalization $wa = w'a'$ described above.

We remark that the last claim implies that there are $\frac{n(n-1)}{2}$ free variables in the block $X_{i,w(i)}$. This remark combined with equation (3.6) and the claim preceding equation (3.6) prove the following claim.

Claim. $\Delta^{-1}_{P\cap gO_Dg^{-1}}(p) = |\det(h)^{-M_i-\cdots-M_{j-1}+1}|$.

From equation (3.3), we obtain $L(v) = |\det(h)|L(\rho(p)v)$. Then

$$L(v_1,\dots,v_i,\dots,v_j,\dots,v_r) = \\ |\det(h)|L\big(v_1\dots,\pi_i(h)v_i,\dots,\pi_j({}^th^{-1})v_j,\dots,v_r\big) \tag{3.7}$$

where for $1 \le l \le r$, $v_l \in V_l$, where V_l is the space of π_l. Since L is nonzero, for $1 \le l \le r$, there exist $x_l \in V_l$ with $L(x_1,\dots x_r) \ne 0$. We define a nonzero linear form L_{ij} on $V_i \otimes V_j$ by

$$L_{ij}(v_i \otimes v_j) = L(x_1,\dots,x_{i-1},v_i,\dots,x_{j-1},v_j,\dots,x_r).$$

We may think of L_{ij} as defined on the space of $|\det(\cdot)|\pi_i \otimes \pi_j^\vee$, where $\pi_j^\vee(h) = \pi_j({}^th^{-1})$. By equation (3.7) we have that L_{ij} is invariant under the diagonal embedding of $GL(M_i,F)$ into $GL(M_i,F) \times GL(M_i,F)$. It follows that $|\det(\cdot)|\pi_i$ is equivalent to the contragredient π_j of $\pi_j^\vee$. By [Zel, Theorem 4.2], this contradicts the irreducibility of π. ■

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By Lemma 3.4, w restricts to a permutation within the blocks. Thus we obtain in the general parabolic case that $w \in W_P$; this gives Proposition 3.2. ∎

4 Proof of the Theorems

Proof of Theorem 1.1 Recall that M is the Levi subgroup of the standard parabolic subgroup P, and that $\otimes_i \pi_i$ is a representation of M with π_i irreducible, supercuspidal for $1 \leq i \leq r$. The representation $\pi = \boxplus \pi_i$ is assumed to be irreducible. Let $\mathcal{O}$ denote the union of the (P, O_D) open double cosets in G, and let $J = \{\phi \in \pi \mid \phi \text{ is supported in } \mathcal{O}\}$. We remark that by Lemma 2.5 we have $\mathcal{O} = A_{l+1} \cup \cdots \cup A_m$. For a vector space X with the action of a group H, denote the dual of X by X^* and let X^{*H} denote the elements in X^* invariant under H.

By Lemma 2.5, we have, for $1 \leq i \leq l$, the exact sequence

$$(4.1)\quad 0 \to D\big(P\backslash G - \cup_{k=0}^{i} A_k, \Delta_P^{1/2}\rho\big) \to D\big(P\backslash G - \cup_{k=0}^{i-1} A_k, \Delta_P^{1/2}\rho\big) \to D(P\backslash A_i, \Delta_P^{1/2}\rho) \to 0,$$

where A_0 denotes the empty set.

Taking duals we obtain the exact sequence

$$0 \to D\big(P\backslash A_i, \Delta_P^{1/2}\rho\big)^{*O_D} \to D\big(P\backslash G - \cup_{k=0}^{i-1} A_k, \Delta_P^{1/2}\rho\big)^{*O_D} \to D\big(P\backslash G - \cup_{k=0}^{i} A_k, \Delta_P^{1/2}\rho\big)^{*O_D}.$$

By Lemma 2.4 and Proposition 3.2 we have that $D(P\backslash A_i, \Delta_P^{1/2}\rho)^{*O_D} = 0$ for $1 \leq i \leq l$. Thus, for $1 \leq i \leq l$, we get

$$(4.2)\qquad 0 \to D\big(P\backslash G - \cup_{k=0}^{i-1} A_k, \Delta_P^{1/2}\rho\big)^{*O_D} \to D\big(P\backslash G - \cup_{k=0}^{i} A_k, \Delta_P^{1/2}\rho\big)^{*O_D}.$$

Using equation (4.2) iteratively, we obtain that

$$0 \to (\boxplus \pi_i)^{*\,O_D} = D(P\backslash G, \Delta_P^{1/2}\rho)^{*\,O_D} \to D(P\backslash \mathcal{O}, \Delta_P^{1/2}\rho)^{*O_D} = J^{*\,O_D}.$$

By [BlD, Théorème 2.8], we have that

$$J^{*\,O_D} \cong \bigoplus_{x \in S} (V_{\rho'})^{*\,M^x \cap O_D},$$

where $M^x = x^{-1}Mx$, ρ' is the representation given by $\rho'(g) = \rho(x^{-1}gx)$ on the space V and S is a set of representatives for the double cosets A_i with the property that for $x \in S$ we have $m(x) = wa$ with $w \in W_P$. We remark that by changing representative x to px for a suitable element p in the Levi of P, we may assume that $m(x)$ is diagonal for $x \in S$.

Thus, if the representation π is distinguished, then for some $x \in S$, we have that $(V_{\rho'})^{*\,M^x \cap O_D}$ is nontrivial. Since

$$(V_{\rho'})^{*\,M^x \cap O_D} \cong (V_\rho)^{*\,M \cap x O_D x^{-1}} \quad \text{and} \quad M \cap x O_D x^{-1} \cong \prod_i O_{D_i'},$$

with $D' = m(x) \in T$, we obtain that π_i is $O_{D_i'}$ distinguished for $1 \leq i \leq r$. ∎

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Proof of Theorem 1.3 In the case of principal series representation $\pi = \boxplus\chi_i$, we have $O_{D_i'} = \{\pm 1\}$. It is clear that χ_i is $O_{D_i'}$-distinguished if and only if $\chi_i(-1) = 1$. Thus Theorem 1.1 implies one direction of the theorem.

For the other direction, let $\chi_i(-1) = 1$. Then J^{*O_D} is nontrivial. We prove that $J^{*O_D} \cong \pi^{*O_D}$ when $\pi = \boxplus\chi_i$ is irreducible. This implies that π is O_D-distinguished.

We argue as in the proof of Theorem 1.1 and use some results in [BlD]. To simplify the notations, we let $D'(X) = D(B\backslash X, \Delta_B^{1/2}\rho)$ where $\rho = \otimes\chi_i$. The exact sequence (4.1) gives, for $1 \le j \le l$,

(4.3)
$$H_0\big(O_D, D'(A_j)\big) \to H_0\big(O_D, D'(G - \cup_{k=0}^{j-1} A_k)\big) \to H_0\big(O_D, D'(G - \cup_{k=0}^{j} A_k)\big) \\ \to H_1\big(O_D, D'(A_j)\big).$$

Here $H_*(O_D, *)$ are homology groups ([BlD, Définition 1.2]). In particular for X an O_D module, $H_0(O_D, X)$ is the quotient of X by the span of $hx - x$ with $h \in O_D$ and $x \in X$. Thus $H_0(O_D, X)^* \cong X^{*O_D}$.

Lemma 4.1 *If A_j is not open, then $H_*(O_D, D'(A_j)) = 0$.*

Proof By Lemma 2.4 we may assume $A_j = BxO_D$ for some representative x with $xD^tx = wa$, $w \ne 1$. Then

$$D'(A_j) \cong D(O_D \cap x^{-1}Bx\backslash O_D, \rho_x) \cong ind_{O_D\cap x^{-1}Bx}^{O_D}\rho_x,$$

where $\rho_x(g) = \Delta_B^{1/2}\rho(xgx^{-1})$. By Shapiro's Lemma ([BlD, Proposition 1.15]),

$$H_*(O_D, D'(A_j)) \cong H_*(O_D \cap x^{-1}Bx, \rho_x\Delta^{-1}_{O_D\cap x^{-1}Bx}).$$

(The statements of Lemme 1.14 and Proposition 1.15 in [BlD] need to be modified with extra modulus factors.)

Note that $\rho_x\Delta^{-1}_{O_D\cap x^{-1}Bx}$ is trivial on $O_D \cap x^{-1}Nx$. From [BlD, Lemme 2.2], we have

$$H_*(O_D, D'(A_j)) \cong H_*(O_D \cap x^{-1}Tx, \rho_x\Delta^{-1}_{O_D\cap x^{-1}Bx}).$$

Now $O_D \cap x^{-1}Tx$ is an abelian group. The fact $w \ne 1$ and the irreducibility of π means equation (3.4) does not hold; thus $\rho_x\Delta^{-1}_{O_D\cap x^{-1}Bx}$ is a nontrivial one dimensional representation of $O_D \cap x^{-1}Tx$. It is then clear that

$$H_*(O_D \cap x^{-1}Tx, \rho_x\Delta^{-1}_{O_D\cap x^{-1}Bx}) = 0,$$

(see [BlD, Corollary 1.9]). ∎

From Lemma 4.1 and equation (4.3), we get

$$H_0\big(O_D, D'(G - \cup_{k=0}^{j-1} A_k)\big) \cong H_0\big(O_D, D'(G - \cup_{k=0}^{j} A_k)\big).$$

Applying the argument iteratively we get

$$H_0(O_D, \boxplus\chi_i) \cong H_0(O_D, J),$$

thus $(\boxplus\chi_i)^{*O_D} \cong J^{*O_D}$. Since J^{*O_D} is nontrivial, we get that $\boxplus\chi_i$ is O_D-distinguished. ∎

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Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102-1811
e-mail: cvalverd@rutgers.edu