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GROWTH OF SOLUTIONS OF WEAKLY COUPLED PARABOLIC SYSTEMS AND LAPLACE'S EQUATION

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Abstract

Let $u_i(x,t)$ be the ith component of a nonnegative solution of a weakly coupled system of second-order, linear, parabolic partial differential equations, for $x \in \mathbf{R}^n$ and $0 < t < T$. We obtain lower estimates, near $t = 0$, for the Lebesgue measure of the set of x for which $t^{\alpha/2}u_i(x,t)$ exceeds 1. Related results for Poisson integrals on a half-space are also described, some applications are given, and interesting comparisons emerge.

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1. Introduction

Let μ be a finite, positive Borel measure on $\mathbf{R}^n$. Let w be the Poisson integral of μ on the half-space $\mathbf{R}^n \times]0,\infty[$, so that

$$w(x,t) = \sigma_n \int_{\mathbf{R}^n} t\left(\|x-y\|^2 + t^2\right)^{-(n+1)/2} d\mu(y),$$

where σ_n is the reciprocal of

$$\int_{\mathbf{R}^n} \left(\|y\|^2 + 1\right)^{-(n+1)/2} dy.$$

Then w is a harmonic function and, if μ is singular with respect to Lebesgue measure m on $\mathbf{R}^n$, then $w(x,t) \to \infty$ as $t \to 0$ for μ-almost every $x \in \mathbf{R}^n$. Recently, Ahern [1] obtained lower estimates for $m(\{x: w(x,t) > 1\})$ near $t = 0$.

In this paper we first extend his results to obtain estimates for $m(\{x\colon t^\alpha w(x,t) > 1\})$, given a condition on μ which implies that $t^\alpha w(x,t) \to \infty$ as $t \to 0$ for μ-almost all x in some Borel set A; here $0 \leqslant \alpha < n$. We then present the corresponding estimates for solutions of certain weakly coupled systems of linear, second-order, parabolic partial differential equations. Of particular interest is the difference between two of the estimates. In the harmonic case, the lower estimate depends essentially on α, while $\mu(A)$ appears as a multiplicative factor; in the parabolic case, α appears only as part of a multiplicative constant, and $\mu(A)$ is completely absent. Some applications of the estimates are also given, which differ significantly in form due to the difference in the estimates.

2. The harmonic case

For any $x, y \in \mathbf{R}^n$, we put $d(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$ and $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$, so that

$$d(x,y) \leqslant \|x - y\| \leqslant n^{1/2} d(x,y).$$

The open cube $\{y \in \mathbf{R}^n\colon d(x,y) < r\}$, with centre x and edge length $2r$, is denoted by $Q(x,r)$.

Throughout this section, μ denotes a positive Borel measure on $\mathbf{R}^n$ such that

$$\int_{\mathbf{R}^n} (1 + \|x\|^2)^{-(n+1)/2}\, d\mu(x) < \infty.$$

This condition is necessary and sufficient for the Poisson integral w of μ to be harmonic on $\mathbf{R}^n \times]0, \infty[$, by Theorem 6 of [6].

Suppose that $0 \leqslant \alpha < n$, and that there is a Borel set A such that

$$\mu(\overline{Q}(x,r))r^{\alpha-n} \to \infty \quad \text{as } r \to 0 \tag{1}$$

μ-a.e. on A. It then follows easily from the corollary to Theorem 2 of [2] that

$$t^\alpha w(x,t) \to \infty \quad \text{as } t \to 0$$

μ-a.e. on A. It is therefore reasonable to seek a lower estimate for $m(\{x\colon t^\alpha w(x,t) > 1\})$ near $t = 0$. Note that, if $\alpha = 0$, then (1) holds μ-a.e. on $\mathbf{R}^n$ if and only if μ is singular with respect to m. However, if $0 < \alpha < n$ and (1) holds μ-a.e. on $\mathbf{R}^n$, then μ is singular with respect to $(n-\alpha)$-dimensional Hausdorff measure (by Lemma 4 of [8]), but not conversely (see pages 19–21 of [8]). If α is a positive integer and μ is supported by a smooth surface of dimension $n - \alpha$, then $\mu(\overline{Q}(x,r))r^{\beta-n} \to \infty$ as $r \to 0$, μ-a.e. on $\mathbf{R}^n$, for every $\beta < \alpha$.

We use C, with various subscripts, to denote a positive constant which depends only on the subscripts; its value may vary from line to line. If C is replaced by a Greek letter, the constant always has the same value.

Our first result is an easy modification of Lemma 1 of [1].

LEMMA 1. *Suppose that* $0 \leqslant \alpha < n$, *and that* (1) *holds* μ-*a.e. on a Borel set* A *with* $\mu(A) < \infty$. *Then, if* $\varepsilon > 0$, *there is a Borel set* E *such that* $\mu(A \setminus E) < \varepsilon$ *and* (1) *holds uniformly on* E.

In Lemma 1, we use a general Borel set A instead of $\mathbf{R}^n$ partly because, for a given μ, (1) may hold with different values of α on different sets. See also the proof of Theorem 3 below.

We recall Lemmas 2 and 3 of [1] for ease of reference.

LEMMA 2. *Suppose that* $d(x, z) \leqslant r$, *that* $0 < t \leqslant 2r$, *and that* w *is the Poisson integral of* μ. *Then there is a positive constant* γ_n *such that*

$$w(x,t) \geqslant \gamma_n t r^{-n-1} \mu(\overline{Q}(z,r)). \tag{2}$$

LEMMA 3. *If* $\mathscr{F}$ *is a finite collection of open cubes, then there is a subcollection* $\mathscr{G}$ *of disjoint cubes such that, for each* $Q \in \mathscr{F}$, *there is* $Q(x,r) \in \mathscr{G}$ *with* $Q \subseteq Q(x, 3r)$.

Given any Borel set A, the modulus of continuity of μ over A is defined by

$$\omega_A(r) = \sup\{\mu(\overline{Q}(x,r)) \colon x \in A\}.$$

If $0 \leqslant \alpha < n$, and if γ_n is the same as in (2), we put

$$\delta_{\alpha,A}(t) = \inf\{r \colon \gamma_n t^{\alpha+1} r^{-n-1} \omega_A(r) \leqslant 1\}$$

for all $t > 0$.

With these notations, Theorem 1 of [1] can be extended in the following way.

THEOREM 1. *Suppose that* $0 \leqslant \alpha < n$, *that* (1) *holds* μ-*a.e. on a Borel set* A *with* $\mu(A) < \infty$, *and that* w *is the Poisson integral of* μ. *Then there is* $t_0 > 0$ *such that*

$$m(\{x \colon t^\alpha w(x,t) > 1\}) \geqslant C_n \mu(A) t^{\alpha+1} / \delta_{\alpha,A}(t)$$

whenever $0 < t \leqslant t_0$.

The proof of Theorem 1 differs from that of Ahern's result only in that $\mathbf{R}^n$ is replaced by A and appropriate items in his argument are multiplied by t^α; his auxiliary function $\delta(x,t)$ is replaced by $\delta_\alpha(x,t) = \inf\{r \colon \gamma_n t^{\alpha+1} r^{-n-1} \mu(\overline{Q}(x,r)) \leqslant 1\}$.

In the extension of the corollary to Theorem 1 of [1], the replacement of $\mathbf{R}^n$ by A causes some difficulty, and so full details of the proof are given.

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THEOREM 2. *Under the hypotheses of Theorem* 1, *there is a positive* t_0 *such that*

$$m(\{x\colon t^\alpha w(x,t) > 1\}) \geqslant C_n\big(\mu(A)t^{\alpha+1}\big)^{n/(n+1)}$$

whenever $0 < t \leqslant t_0$.

PROOF. Since μ is regular, we can find a compact subset of K of A such that $\mu(K) \geqslant \mu(A)/2$, and an open superset V of K such that $\mu(V) \leqslant 2\mu(A)$ (ignoring the trivial case where $\mu(A) = 0$). Now let ν be a point mass at $x_0 \in A$ with $\nu(\mathbf{R}^n) = 2\mu(A)$. Then, inserting extra subscripts to distinguish between the two measures, we have $\omega_{A,\nu}(r) = 2\mu(A)$ for all $r > 0$, and hence $\omega_{K,\mu}(r) \leqslant \mu(V) \leqslant 2\mu(A) = \omega_{A,\nu}(r)$ for all $r < r_0$, where r_0 is the distance between K and $\mathbf{R}^n \setminus V$ in the d-metric. Next, $\delta = \delta_{\alpha,A,\nu}(t)$ satisfies

$$\gamma_n t^{\alpha+1}\delta^{-n-1}2\mu(A) = 1,$$

so that

$$\delta = \big(2\gamma_n\mu(A)t^{\alpha+1}\big)^{1/(n+1)}.$$

Therefore $\delta \to 0$ as $t \to 0$, so that we can find $t_1 > 0$ such that $\delta < r_0$ whenever $0 < t \leqslant t_1$. Hence there are values of r less than r_0 such that

$$\gamma_n t^{\alpha+1}r^{-n-1}\omega_{K,\mu}(r) \leqslant \gamma_n t^{\alpha+1}r^{-n-1}\omega_{A,\nu}(r) \leqslant 1.$$

It follows that, for $t \leqslant t_1$, we have

$$\delta_{\alpha,K,\mu}(t) = \inf\big\{r < r_0\colon \gamma_n t^{\alpha+1}r^{-n-1}\omega_{K,\mu}(r) \leqslant 1\big\} \leqslant \delta_{\alpha,A,\nu}(t).$$

Theorem 1 now implies that, for all sufficiently small t,

$$\begin{aligned} m(\{x\colon t^\alpha w(x,t) > 1\}) &\geqslant C_n\mu(K)t^{\alpha+1}/\delta_{\alpha,K,\mu}(t) \\ &\geqslant C_n\mu(A)t^{\alpha+1}/\delta_{\alpha,A,\nu}(t) \\ &= C_n\big(\mu(A)t^{\alpha+1}\big)^{n/(n+1)}. \end{aligned}$$

By considering the case where μ is a point mass at the origin, it is easy to verify, by direct calculation, that $t^{(\alpha+1)n/(n+1)}$ can be the exact rate of decrease of $m(\{x : t^\alpha w(x,t) > 1\})$ as $t \to 0$. The advantage of Theorem 2 over Theorem 1 is that it does not mention $\delta_{\alpha,A}$, which may be defined only implicitly.

One consequence of the use of an arbitrary Borel set A, rather than $\mathbf{R}^n$, in the above results, is the following condition for absolute continuity.

THEOREM 3. *If w is the Poisson integral of* μ, *and*

$$\liminf_{t\to 0} t^{-n/(n+1)}m(\{x\colon w(x,t) > 1\}) = 0,$$

then μ is absolutely continuous with respect to m.

PROOF. By Theorem 2, if (1) holds (with $\alpha = 0$) μ-a.e. on a Borel set A, then

$$\liminf_{t\to 0} t^{-n/(n+1)} m(\{x: w(x,t) > 1\}) \geqslant C_n \mu(A)^{n/(n+1)}.$$

Therefore our hypothesis implies that (1) can hold only on sets of μ-measure zero.

By the Lebesgue Decomposition Theorem, we can write $\mu = \mu_a + \mu_s$, where μ_a and μ_s are non-negative measures, μ_a is absolutely continuous and μ_s is singular (with respect to m). By Lemma 6.1 of [4], we have

$$\mu_s(\overline{Q}(x,r)) r^{-n} \to \infty \quad \text{as } r \to 0$$

μ_s-a.e. on $\mathbf{R}^n$. Therefore (1) holds (with $\alpha = 0$) at every point of a set E with $\mu_s(\mathbf{R}^n \setminus E) = 0$. But, from above, $\mu(E) = 0$, so that $\mu_s(E) = 0$ and hence μ_s is null.

As an application of Theorem 3, we give a variant of the well-known result that, if v is continuous on $\mathbf{R}^n \times [0, \infty[$, non-negative and harmonic on $\mathbf{R}^n \times]0, \infty[$, and $v(\cdot, 0) = 0$, then $v(x,t) \equiv Ct$ for some constant $C \geqslant 0$.

THEOREM 4. *Let v be non-negative and harmonic on $\mathbf{R}^n \times]0, \infty[$. If*

$$\liminf_{t\to 0} v(x,t) = 0$$

m-a.e. on $\mathbf{R}^n$, and

$$\liminf_{t\to 0} t^{-n/(n+1)} m(\{x: v(x,t) > 1\}) = 0,$$

then there is a constant C such that $v(x,t) = Ct$ throughout $\mathbf{R}^n \times]0, \infty[$.

PROOF. Since v is non-negative and harmonic, we can write

$$v(x,t) = Ct + w(x,t)$$

for all $(x,t) \in \mathbf{R}^n \times]0, \infty[$, where C is a non-negative constant and w is the Poisson integral of a non-negative measure μ (see [5]). Since $w \leqslant v$, we can apply Theorem 3 to w and deduce that μ is absolutely continuous with respect to m. In view of the Fatou theorem [7, Theorem 1] and of Lebesgue's theorem on differentiation of measures, $v(x, 0+)$ exists m-a.e. and $d\mu(x) = v(x, 0+)\,dx$. It follows that $w = 0$.

3. The parabolic analogue of Theorem 1

We now consider the case of a weakly coupled parabolic system of second-order, linear, partial differential equations:

$$(3) \qquad \frac{\partial u_k}{\partial t} = \sum_{i,j=1}^{n} a_{ij}^k(x,t) \frac{\partial^2 u_k}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^k(x,t) \frac{\partial u_k}{\partial x_i} + \sum_{l=1}^{N} c_l^k(x,t) u_l$$

for $k = 1, \ldots, N$ and $(x, t) \in \mathbf{R}^n \times]0, T[$. We require the following properties of the solutions of (3). Conditions on the coefficients which guarantee these properties are given in [3] or [4], where further details and references can be found.

(i) There exists a fundamental solution (or matrix) $\{\Gamma_{ij}(x, t; y, s)\}_{N\times N}$, defined and non-negative for all (x, t), $(y, s) \in \mathbf{R}^n \times [0, T]$ such that $t > s$, which satisfies

$$\kappa(t-s)^{-n/2}\exp(-\lambda\|x-y\|^2/4(t-s)) \leqslant \Gamma_{ii}(x, t; y, s)$$

for $i = 1, \ldots, N$, where κ and λ are positive constants. (See Section 4 of [4].)

(ii) A function $u = (u_1, \ldots, u_N)$ is a non-negative solution of (3) if and only if there are non-negative Borel measures $\mu_1, \ldots, \mu_N$ on $\mathbf{R}^n$ such that

$$u_i(x, t) = \int_{\mathbf{R}^n} \sum_{j=1}^{N} \Gamma_{ij}(x, t; y, 0)\, d\mu_j(y) \tag{4}$$

for $i = 1, \ldots, N$ and $(x, t) \in \mathbf{R}^n \times]0, T[$, where the integrals are finite. (See Theorem 1 of [3] and Theorem 5.1 of [4].) In the sequel, we shall always assume that our measures are such that the integrals in (4) are finite.

(iii) If $0 \leqslant \alpha < n$, (4) holds, and

$$\mu_i(\overline{Q}(x, r))r^{\alpha-n} \to \infty \quad \text{as } r \to 0 \tag{5}$$

for some i and some $x \in \mathbf{R}^n$, then

$$t^{\alpha/2}u_i(x, t) \to \infty \quad \text{as } t \to 0.$$

(This follows easily from Corollary 6.1 of [4].)

We now develop analogues of Lemma 2 and Theorem 1. It is important for our applications in Section 5 that we keep track of the values of our various constants, and that these should be the best that our methods allow. We let χ denote an arbitrary, fixed number greater than 1.

LEMMA 4. *Suppose that* $d(x, z) \leqslant r$, *that* $0 < t \leqslant \chi r^2$, *and that* (4) *holds. Then*

$$u_i(x, t) \geqslant \pi_{n,\kappa}\exp(-\lambda n r^2/t)\mu_i(\overline{Q}(z, r))r^{-n}, \tag{6}$$

where λ *and* κ *are the same as in* (i), *and where* $\pi_{n,\kappa} = \kappa\chi^{-n/2}$.

PROOF. If $d(x, z) \leqslant r$ and $y \in \overline{Q}(z, r)$, then

$$\|x-y\| \leqslant n^{1/2}d(x, y) \leqslant 2n^{1/2}r.$$

It follows from (i) and (4) that, if $0 < t \leqslant \chi r^2$, then

$$\begin{aligned} u_i(x,t) &\geqslant \int_{\mathbf{R}^n} \Gamma_{ii}(x,t;y,0)\, d\mu_i(y) \\ &\geqslant \kappa t^{-n/2} \int_{\overline{Q}(z,r)} \exp\left(-\lambda \|x-y\|^2/4t\right) d\mu_i(y) \\ &\geqslant \kappa\left(\chi r^2\right)^{-n/2} \int_{\overline{Q}(z,r)} \exp\left(-\lambda n r^2/t\right) d\mu_i(y) \\ &= \pi_{n,\kappa} \exp\left(-\lambda n r^2/t\right) \mu_i\left(\overline{Q}(z,r)\right) r^{-n}. \end{aligned}$$

For each integer i ($1 \leqslant i \leqslant N$) and Borel set A, we denote the modulus of continuity of μ_i over A by $\omega_{A,i}$.

Let $\pi_{n,\kappa}$ be the same as in (6). If $0 \leqslant \alpha < n$, and if A is a Borel set, we put

$$\rho_A(t) = \rho_{\alpha,A,i}(t) = \inf\left\{r\colon \pi_{n,\kappa} t^{\alpha/2} \exp\left(-\lambda n r^2/t\right) \omega_{A,i}(r) r^{-n} \leqslant 1\right\}$$

for all $t \in]0, T[$.

THEOREM 5. *Suppose that* $0 \leqslant \alpha < n$, *that* u_i *is given by* (4), *and that* (5) *holds* μ_i*-a.e. on a Borel set* A *with* $\mu_i(A) < \infty$. *Then there exists* $t_0 > 0$ *such that*

$$m\left(\left\{x\colon t^{\alpha/2} u_i(x,t) > 1\right\}\right) \geqslant \theta_{n,\kappa} \mu_i(A) t^{\alpha/2} \exp\left(-\lambda n \rho_{\alpha,A,i}(t)^2/t\right)$$

whenever $0 < t \leqslant t_0$, *where* $\theta_{n,\kappa} = (2/3)^n \chi^{-(n+2)/2}\kappa$.

PROOF. It follows from Lemma 1, and the regularity of μ_i, that there is a compact set K such that $\mu_i(K) \geqslant \mu_i(A)/\chi$ and

$$\mu_i\left(\overline{Q}(x,r)\right) r^{\alpha-n} \to \infty \quad \text{as } r \to 0$$

uniformly for x in K. Therefore we can find $t_0 > 0$ such that, whenever $0 < r \leqslant t_0^{1/2}$, we have

$$\pi_{n,\kappa} \mu_i\left(\overline{Q}(x,r)\right) r^{\alpha-n} \geqslant e^{\lambda n} \quad \text{for all } x \in K. \tag{7}$$

Fix $t \in]0, t_0]$. For every $x \in K$, put

$$\rho(x,t) = \rho_{\alpha,i}(x,t) = \inf\left\{r\colon \pi_{n,\kappa} t^{\alpha/2} \exp\left(-\lambda n r^2/t\right) \mu_i\left(\overline{Q}(x,r)\right) r^{-n} \leqslant 1\right\}.$$

Then

$$\rho(x,t) \leqslant \rho_A(t) \tag{8}$$

for all $x \in K$. Furthermore, given x, there is a sequence $\{r_k\}$ which decreases to $\rho(x,t)$ and which is such that

$$\pi_{n,\kappa} t^{\alpha/2} \exp\left(-\lambda n r_k^2/t\right) \mu_i\left(\overline{Q}(x,r_k)\right) r_k^{-n} \leqslant 1$$

for all k. Making $k \to \infty$, we deduce that

$$\pi_{n,\kappa} t^{\alpha/2} \exp\left(-\lambda n \rho(x,t)^2/t\right) \mu_i\left(\overline{Q}(x,\rho(x,t))\right) \rho(x,t)^{-n} \leqslant 1. \tag{9}$$

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Next, if $0 < r < t^{1/2}$, then it follows from (7) that

$$\pi_{n,\kappa} t^{\alpha/2} \exp(-\lambda n r^2/t) \mu_i(\overline{Q}(x,r)) r^{-n} \geqslant t^{\alpha/2} \exp(-\lambda n r^2/t) r^{-\alpha} e^{\lambda n} > 1.$$

Therefore, for all $x \in K$, we have

(10) $$t^{1/2} \leqslant \rho(x,t).$$

Since K is compact, and since the family

$$\{Q(x, \rho(x,t)/3) \colon x \in K\}$$

of open cubes covers K, we can select a finite subfamily $\mathscr{F}$ which also covers K. By Lemma 3, $\mathscr{F}$ has a subfamily

(11) $$\{Q(x_l, \rho(x_l,t)/3) \colon l = 1, \ldots, q\}$$

of disjoint cubes such that

$$K \subseteq \bigcup_{l=1}^{q} Q(x_l, \rho(x_l,t)).$$

Therefore

$$\mu_i(A) \leqslant \chi \mu_i(K) \leqslant \chi \sum_{l=1}^{q} \mu_i(Q(x_l, \rho(x_l,t))).$$

It now follows from (9) and (8) that

$$\pi_{n,\kappa} t^{\alpha/2} \mu_i(A) \leqslant \chi \sum_{l=1}^{q} \exp\left(\lambda n \rho(x_l,t)^2/t\right) \rho(x_l,t)^n$$
$$\leqslant \chi \exp\left(\lambda n \rho_A(t)^2/t\right) \sum_{l=1}^{q} \rho(x_l,t)^n.$$

Therefore, because the cubes in the family (11) are disjoint, we obtain

(12) $$(2/3)^n \chi^{-1} \pi_{n,\kappa} t^{\alpha/2} \mu_i(A) \exp\left(-\lambda n \rho_A(t)^2/t\right)$$
$$\leqslant \sum_{l=1}^{q} (2\rho(x_l,t)/3)^n$$
$$= \sum_{l=1}^{q} m(Q(x_l, \rho(x_l,t)/3))$$
$$\leqslant m\left(\bigcup_{l=1}^{q} Q(x_l, \rho(x_l,t))\right).$$

If y belongs to the set whose measure is the last term in (12), then $y \in Q(z, \rho(z,t))$ for some $z \in K$. Since $d(y,z) < \rho(z,t)$, and since $(t/\chi)^{1/2} < \rho(z,t)$ by (10), we can choose r such that $t < \chi r^2$ and $d(y,z) < r < \rho(z,t)$. Therefore, by Lemma 4,

$$t^{\alpha/2} u_i(y,t) \geqslant \pi_{n,\kappa} t^{\alpha/2} \exp(-\lambda n r^2/t) \mu_i(\overline{Q}(z,r)) r^{-n} > 1$$

since $r < \rho(z, t)$. Therefore (12) implies that

$$\theta_{n,\kappa} t^{\alpha/2} \mu_i(A) \exp\left(-\lambda n \rho_A(t)^2/t\right) \leqslant m\left(\{y\colon t^{\alpha/2} u_i(y,t) > 1\}\right),$$

as required.

4. The parabolic version of Theorem 2

Theorem 5 is analogous to Theorem 1, and a comparison of the two yields no real surprises. The situation is different, however, for Theorem 2 and its analogue, Theorem 6 below. In Theorem 2, the parameter α appears as a power of t, while $\mu(A)$ appears as a multiplicative factor; in Theorem 6, α appears only as part of a multiplicative constant, and $\mu_i(A)$ is completely absent. We must therefore assume that $\mu_i(A) > 0$.

THEOREM 6. *Suppose that* $0 \leqslant \alpha < n$, *that* u_i *is given by* (4), *and that* (5) *holds* μ_i*-a.e. on a Borel set* A *with* $0 < \mu_i(A) < \infty$. *Then we can find* $t_0 > 0$ *such that*

$$m\left(\{x\colon t^{\alpha/2} u_i(x,t) > 1\}\right) \geqslant \Delta_{n,\lambda}(n-\alpha)^{n/2} |t \log t|^{n/2}$$

whenever $0 < t \leqslant t_0$, *where* $\Delta_{n,\lambda} = 2^{n/2} 3^{-n} \chi^{-n-3} (\lambda n)^{-n/2}$.

PROOF. Let ν be a point mass at $x_0 \in A$ with $\nu(A) = \chi \mu_i(A)$, so that $\omega_{A,\nu}(r) = \chi \mu_i(A)$ for all $r > 0$. The method for proving Theorem 6 is basically similar to that for Theorem 2, and so we want to find $\rho = \rho_{\alpha,A,\nu}(t)$ which satisfies

$$\chi \mu_i(A) \pi_{n,\kappa} t^{\alpha/2} \exp(-\lambda n \rho^2/t) \rho^{-n} = 1.$$

Since ρ is defined only implicitly, we shall have to manage with an approximation to ρ for small t.

Consider the function σ defined by

$$\sigma(\tau) = \left\{-(\tau/2) \log\left(-a\tau^{(n-\alpha)/n} \log \tau\right)\right\}^{1/2}$$

for all small positive τ, where $\tau = t/\lambda$, and where

$$a = (n-\alpha)\left\{2n\left(\chi \mu_i(A) \pi_{n,\kappa} \lambda^{\alpha/2}\right)^{2/n}\right\}^{-1}.$$

As $t \to 0$, we have

$$\begin{aligned} & t^{\alpha/2} \exp(-\lambda n \sigma^2/t) \sigma^{-n} \\ &\quad = (\lambda\tau)^{\alpha/2} \exp(-n\sigma^2/\tau) \sigma^{-n} \\ &\quad = (\lambda\tau)^{\alpha/2} \left(-a\tau^{(n-\alpha)/n} \log \tau\right)^{n/2} \left\{-(\tau/2) \log\left(-a\tau^{(n-\alpha)/n} \log \tau\right)\right\}^{-n/2} \\ &\quad = \lambda^{\alpha/2} \left\{2a \log \tau / \log\left(-a\tau^{(n-\alpha)/n} \log \tau\right)\right\}^{n/2} \\ &\quad \to \lambda^{\alpha/2} \{2an/(n-\alpha)\}^{n/2} \\ &\quad = \left(\chi \mu_i(A) \pi_{n,\kappa}\right)^{-1}; \end{aligned}$$

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thus

$$\chi\mu_i(A)\pi_{n,\kappa}t^{\alpha/2}\exp(-\lambda n\sigma^2/t)\sigma^{-n}\to 1$$

as $t\to 0$. Put

$$f(r,t)=\chi\mu_i(A)\pi_{n,\kappa}t^{\alpha/2}\exp(-\lambda nr^2/t)r^{-n}$$

for all $r>0$ and $t>0$, and put $\zeta(t)=\sigma(t/\lambda)$. Then

$$f(\zeta(t),t)\to 1 \quad \text{as } t\to 0,$$

and

$$f(\rho(t),t)=1 \quad \text{for all sufficiently small } t,$$

so that

(13) $$f(\zeta(t),t)\sim f(\rho(t),t) \quad \text{as } t\to 0.$$

We assert that

(14) $$\zeta(t)\sim\rho(t) \quad \text{as } t\to 0.$$

If (14) is false, then there is $\eta>0$ and a null sequence $\{t_k\}$ such that either

$$\zeta(t_k)>(1+\eta)\rho(t_k) \quad \text{or} \quad \rho(t_k)>(1+\eta)\zeta(t_k)$$

for all k. We suppose that the former is the case; the proof for the latter is similar. For each fixed t, the function $f(\cdot,t)$ is strictly decreasing, so that

$$f(\zeta(t_k),t_k)<f((1+\eta)\rho(t_k),t_k)$$

for all k. Therefore

$$\frac{f(\zeta(t_k),t_k)}{f(\rho(t_k),t_k)}<(1+\eta)^{-n}\exp\left\{-\lambda n\rho(t_k)^2(\eta^2+2\eta)/t_k\right\}$$
$$<(1+\eta)^{-n}$$

for all k, contrary to (13). Thus (14) is proved, so that

(15) $$\rho(t)\sim\left\{-(t/2\lambda)\log\left(-a(t/\lambda)^{(n-\alpha)/n}\log(t/\lambda)\right)\right\}^{1/2}$$

as $t\to 0$.

We can now complete the proof. Since μ_i is regular, we can find a compact subset K of A such that $\mu_i(K)\geqslant\mu_i(A)/\chi$, and an open superset V of K such that $\mu_i(V)\leqslant\chi\mu_i(A)$. With ν as above, we have

$$\omega_{K,i}(r)\leqslant\mu_i(V)\leqslant\chi\mu_i(A)=\omega_{A,\nu}(r)$$

for all $r<r_0$, where r_0 is the distance between K and $\mathbf{R}^n\setminus V$ in the d-metric. In view of (15), $\rho\to 0$ as $t\to 0$, so that we can find $t_1>0$ such that $\rho<r_0$ whenever $0<t\leqslant t_1$. Hence there are values of r less than r_0 such that

$$\pi_{n,\kappa}t^{\alpha/2}\exp(-\lambda nr^2/t)\omega_{K,i}(r)r^{-n}\leqslant\pi_{n,\kappa}t^{\alpha/2}\exp(-\lambda nr^2/t)\omega_{A,\nu}(r)r^{-n}$$
$$\leqslant 1.$$

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It now follows that

$$\rho_{\alpha,K,i}(t) = \inf\{r < r_0 \colon \pi_{n,\kappa} t^{\alpha/2} \exp(-\lambda n r^2/t)\,\omega_{K,i}(r) r^{-n} \leqslant 1\}$$
$$\leqslant \rho_{\alpha,A,\nu}(t) = \rho(t).$$

Therefore, by Theorem 5,

$$m(\{x \colon t^{\alpha/2} u_i(x,t) > 1\}) \geqslant \theta_{n,\kappa} \mu_i(K) t^{\alpha/2} \exp(-\lambda n \rho_{\alpha,K,i}(t)^2/t)$$
$$\geqslant \theta_{n,\kappa} \chi^{-1} \mu_i(A) t^{\alpha/2} \exp(-\lambda n \rho(t)^2/t).$$

Since $f(\rho(t), t) = 1$ for all sufficiently small t, it follows that

$$m(\{x \colon t^{\alpha/2} u_i(x,t) > 1\}) \geqslant \mathcal{C}_{n,\kappa} \chi^{-2} \pi_{n,\kappa}^{-1} \rho(t)^n = (2/3)^n \chi^{-3} \rho(t)^n.$$

In view of (15) we have, as $t \to 0$,

$$\frac{\rho(t)^2}{|t \log t|} \geqslant \frac{-(t/2\lambda) \log(-a(t/\lambda)^{(n-\alpha)/n} \log(t/\lambda))}{-\chi t \log t}$$
$$\to (n-\alpha)/2\chi\lambda n.$$

Therefore, for all sufficiently small t, we have

$$\rho(t)^2 \geqslant (n-\alpha)|t \log t|/2\chi^2 \lambda n,$$

and hence

$$m(\{x \colon t^{\alpha/2} u_i(x,t) > 1\}) \geqslant \Delta_{n,\lambda} (n-\alpha)^{n/2} |t \log t|^{n/2},$$

as required.

The result of Theorem 6 is sharp for every value of α, in the sense that $|t \log t|^{n/2}$ can be the exact rate of decrease of $m(\{x \colon t^{\alpha/2} u_i(x,t) > 1\})$ as $t \to 0$, at least for the case of the heat equation ($N = 1$). This can be verified by direct calculation by taking μ_1 to be a point mass at the origin (since the lower estimate for Γ_{ii} in (i) is then an identity). However, it is extremely unlikely that the constant $\Delta_{n,\lambda}$ is the best possible, and this is slightly unfortunate when we come to consider applications in the next section.

5. Applications in the parabolic case

The theorems of this section are parabolic analogues of Theorems 3 and 4. However, the precise forms of the parabolic results are significantly different to those of their harmonic counterparts, due to the absence of $\mu_i(A)$ from the conclusion of Theorem 6.

THEOREM 7. *If u_i is given by* (4), *and if*

$$\liminf_{t\to 0} |t\log t|^{-n/2} m(\{x: u_i(x,t) > 1\}) < (2/9\lambda)^{n/2}, \tag{16}$$

then μ_i is absolutely continuous with respect to m.

PROOF. Condition (16) implies that there is a null sequence $\{t_k\}$ such that the sequence $\{|t_k \log t_k|^{-n/2} m(\{x: u_i(x,t_k) > 1\})\}$ converges to a limit $\phi < (2/9\lambda)^{n/2}$. Choose $\chi_0 > 1$ such that $\phi < \chi_0^{-n-3}(2/9\lambda)^{n/2}$. Then, for any $t_0 > 0$, we can find k such that $t_k < t_0$, and

$$m(\{x: u_i(x,t_k) > 1\}) < \chi_0^{-n-3}(2/9\lambda)^{n/2}|t_k \log t_k|^{n/2}.$$

Therefore Theorem 6 (with $\alpha = 0$ and $\chi = \chi_0$) implies that (5) does not hold on any Borel set A with $0 < \mu_i(A) < \infty$. Hence (5) can hold (with $\alpha = 0$) only on sets of μ_i-measure zero, and the proof can be completed by the argument in the last paragraph of the proof of Theorem 3.

The principal difference between Theorem 7 and its harmonic counterpart is that the lower limit in (16) can be positive. The constawnt $(2/9\lambda)^{n/2}$ is derived from the constants in the earlier theorems, and this is why it was important to keep them as large as possible; but it is most unlikely to be the best possible. If we reconsider the case of the heat equation, with μ_1 a point mass, which showed that the rate of decrease $|t\log t|^{n/2}$ is exact, we get

$$\lim_{t\to 0} |t\log t|^{-n/2} m(\{x: u_1(x,t) > 1\}) = v_n(2n/\lambda)^{n/2},$$

where $v_n = \pi^{n/2}/\Gamma((n+2)/2)$ is the volume of the unit ball in $\mathbf{R}^n$. This gives an upper bound for the best constant.

THEOREM 8. *Let $u = (u_1, \ldots, u_N)$ be a non-negative solution of* (3) *on* $\mathbf{R}^n \times]0,T[$. *If, for every i, we have*

$$\liminf_{t\to 0} u_i(x,t) = 0 \tag{17}$$

m-a.e. on $\mathbf{R}^n$, and (16) *holds, then $u = 0$ on* $\mathbf{R}^n \times]0,T[$.

PROOF. By (ii), each u_i is given by (4). Since (16) holds for all i, Theorem 7 implies that each μ_i is absolutely continuous with respect to m. Since (17) holds for all i, the Fatou theorem [4, Theorem 3.1] implies that each μ_i is null.

Added in proof

The constants in the parabolic case can be improved by using balls instead of cubes throughout. Hence, in particular, Theorem 7 holds with $(2/9\lambda)^{n/2}$ replaced by $v_n(n/18\lambda)^{n/2}$.

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